Timelike entanglement entropy: AdS/BCFT and gravitational anomalies

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Overview

- Timelike entanglement entropy in AdS_3/CFT_2
- Timelike entanglement entropy in AdS/BCFT
- Entanglement entropy in CFT_2 with gravitational anomalies
- TEE in CFT_2 with gravitational anomaly
- Holographic TEE in AdS_3/CFT_2 with gravitational anomaly

Timelike entanglement entropy

- Previous work on entanglement entropy has focused on cases where the subsystem in question is a spacelike separated region.
- So, EE and other entanglement measures primarily capture quantum correlations between spacelike-separated regions.
- Recently, *timelike entanglement entropy* has been introduced to include timelike scenarios where the subsystem is a timelike one. [Doi, Harper, Mollabashi, Takayanagi and Taki '22 '23]
- It is defined as the analytic continuation of entanglement entropy of a spacelike subsystem to a timelike subsystem.
- Also, entanglement measures for timelike separated subsystems known as *entanglement in time* has been introduced recently which can be interpreted as the quantum correlations between two subsystems A and B, separated in time. [Milekhin, Adamska and Preskill '25]
- The entanglement in time coincides with the TEE for relativistic quantum field theories.

- The entanglement entropy for a generic space-like interval A in a CFT_2 can be expressed in terms of twist field correlators

$$S_A = \lim_{n\to 1} \frac{1}{1-n} \log \langle \sigma_n(z_1)\bar{\sigma}_n(z_2)\rangle \,.$$

- Consider a generic space-like interval $A \equiv [(t_1, x_1), (t_2, x_2)]$ in a CFT_2 having timelike and space-like width as $t_{12} = T_0$ and $x_{12} = X_0$.
- Then, the entanglement entropy S_A of an interval A may be obtained as [Calabrese, Cardy '04]

$$S_A = \frac{c}{3} \log \frac{\sqrt{X_0^2 - T_0^2}}{\epsilon}.$$

- The timelike entanglement entropy S_A^T for a purely timelike interval A is obtained by analytically continuing the space-like interval to a timelike interval and taking $X_0 = 0$ as [Doi, Harper, Mollabashi, Takayanagi and Taki '22 '23]

$$S_A^T = \frac{c}{3} \log \left(\frac{T_0}{\epsilon}\right) + \frac{i\pi c}{6}.$$

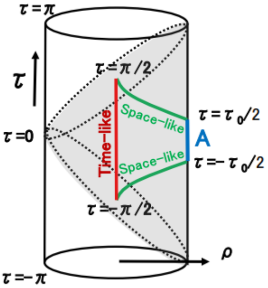
- It takes complex values.

Holographic TEE in AdS_3/CFT_2

- Consider the Poincaré patch of a AdS_3 spacetime whose metric is given by

$$ds^2 = \frac{dz^2 - dt^2 + dx^2}{z^2}.$$

- The holographic TEE is given by three geodesics where two space-like geodesics connects the endpoints of A and null infinities, and a timelike geodesic which connects the endpoints of two space-like geodesics. [Doi, Harper, Mollabashi, Takayanagi and Taki '22 '23]



Boundary conformal field theory (BCFT)

- A boundary conformal field theory is a conformal field theory defined on a manifold with boundaries such that a part of conformal symmetry is preserved by the boundaries.
- The two-point function of scalar operators in a BCFT behaves kinematically like a CFT four-point function

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|4x_\perp y_\perp|^\Delta}\mathcal{G}(\xi)\ .$$

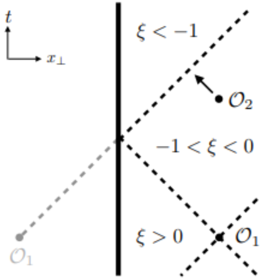
- When the two operators approach each other and away from the boundary, the BCFT two-point function in the bulk limit $\xi \to 0$ behaves like [Mazáč, Rastelli and Zhou '19]

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta}} + \dots, \quad \mathcal{G}(\xi) \sim \xi^{-\Delta}.$$

- Similarly, the two point BCFT correlator in the boundary limit $\xi \to \infty$ when the operators are much closer to the boundary than to each other, the correlator behaves like [Mazáč, Rastelli and Zhou '19]

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{\mathcal{A}^2}{|4x_\perp y_\perp|^\Delta} + \dots, \quad \mathcal{G}(\xi) \sim \mathcal{A}^2.$$

- When the operators $\mathcal{O}(x)$ and $\mathcal{O}(y)$ are timelike separated in the Lorentzian signature, there exists another singularity in the limit $\xi = -1$, known as Regge limit of the BCFT.
- In this limit, the operator $\mathcal{O}(y)$ approaches the light-cone of the mirror reflection of $\mathcal{O}(x)$ with the boundary behaving like the mirror.

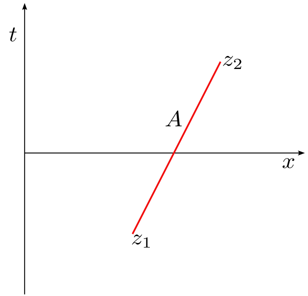


- The two point correlator behaves at the Regge limit $\xi \to -1$ as [Mazáč, Rastelli and Zhou '19]

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|4x_\perp y_\perp|^\Delta}\frac{1}{(1+\xi)^\Delta} + \dots , \quad \mathcal{G}(\xi) \sim (\xi+1)^{-\Delta}.$$

TEE in BCFT_2

- Consider the configuration of an interval $A \equiv [z_1, z_2] := [t_1, x_1), (t_2, x_2)]$ at zero temperature in a BCFT.



- The entanglement entropy for a generic space-like interval A in a BCFT_2 may be expressed in terms of twist field correlators using the replica technique as [Calabrese, Cardy '04]

$$S_A = \lim_{n \to 1} \frac{1}{1-n} \log \langle \sigma_n(z_1) \bar{\sigma}_n(z_2) \rangle \,.$$

I. Bulk limit

- Consider first the case that the interval is far away from the boundary such that $\xi \to 0$. In this case, the dominant contribution of the two point function comes from the bulk channel OPE.
- The two point twist correlator in the bulk limit may be written in the following form

$$\langle \sigma_n(z_1)\bar{\sigma}_n(z_2)\rangle = \frac{\epsilon^{2\Delta_n}}{|z_1 - z_2|^{2\Delta_n}}.$$

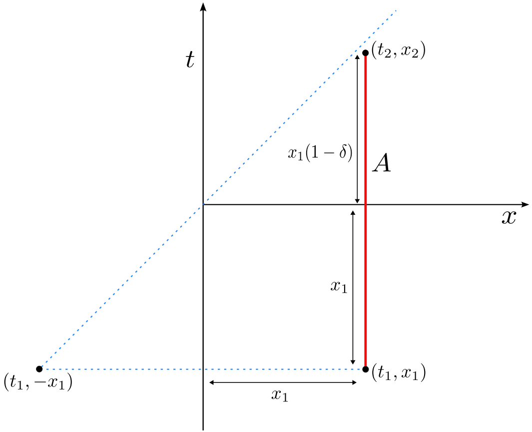
- For a purely timelike interval with $x_1 = x_2$ and $t_{12} = T_0$, the timelike entanglement entropy S_A^T in the bulk limit is given by

$$S_A^T = \frac{c}{3}\log\frac{T_0}{\epsilon} + \frac{i\pi c}{6} := S^B.$$

- We observe that the timelike entanglement entropy is complex in this phase and resembles the usual CFT_2 result.

II. Regge limit

- As T_0 increases, it will eventually reach the region $T_0 \approx 2x_1$.
- Let us parameterize the time interval by $T_0 = 2x_1(1-\delta/2)$. Then the purely timelike interval is described by
  $A[(t_1, x_1), (t_2, x_2)] \equiv [(-x_1, x_1), (x_1(1-\delta), x_1)]$.



- In the leading order of small δ, the cross ratio ξ is given by

$$\xi = \frac{-(t_2 - t_1)^2 + (x_2 - x_1)^2}{4x_1 x_2} = -(1-\delta).$$

- Using the two point function

$$\langle \sigma_n(z_1)\bar{\sigma}_n(z_2)\rangle = \frac{1}{|2x_1|^{2\Delta_n}} \frac{1}{(1+\xi)^{\Delta_n}}.$$

- We obtain the timelike entanglement entropy for the time-interval A in the Regge limit as

$$S_A^T = \begin{cases} \frac{c}{3}\log\left(\frac{2x_1}{\epsilon}\sqrt{2-\frac{T_0}{x_1}}\right), & \text{for } T_0 \to 2x_1^-, \quad \text{i.e.}\ \xi \to -1^+, \\ \frac{c}{3}\log\left(\frac{2x_1}{\epsilon}\sqrt{\frac{T_0}{x_1}-2}\right) + \frac{i\pi c}{6}, & \text{for } T_0 \to 2x_1^+, \quad \text{i.e.}\ \xi \to -1^-, \end{cases} \quad := S^R.$$

- Here $\xi = -1$ is a branch point and S^R picks up an imaginary part as T_0 crosses the value of $2x_1$.

III. Boundary limit

- In the boundary limit, the two point twist correlator may be expressed as

$$\langle \sigma_n(z_1)\bar{\sigma}_n(z_2)\rangle = \frac{g_b^{2(1-n)}\epsilon^{2\Delta_n}}{|4x_1x_2|^{\Delta_n}}.$$

- This gives the entanglement entropy for a generic interval A as

$$S_A = \frac{c}{6}\log\frac{2x_1}{\epsilon} + \frac{c}{6}\log\frac{2x_2}{\epsilon} + 2\log g_b.$$

- The last term describes the boundary entropy which depends on the boundary condition.
- Taking $x_1 = x_2$ for a pure timelike interval, we obtain the timelike entanglement entropy as

$$S_A^T = \frac{c}{3}\log\frac{2x_1}{\epsilon} + 2\log g_b := S^b.$$

- We see that the timelike entanglement entropy is real in this phase and includes boundary entropy which is expected in the boundary phase.

- The behaviour of the real part of TEE is shown in the following figure.

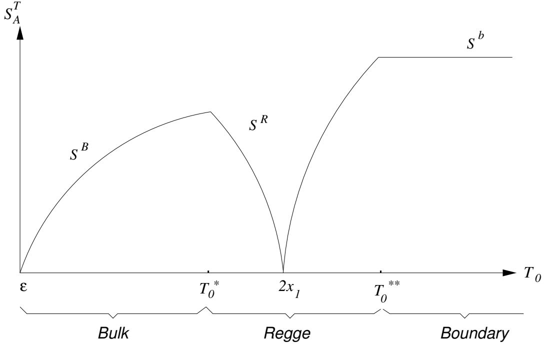


Figure: Phases of the timelike entanglement entropy S_A^T in BCFT.

- It is interesting that in contrast to the usual two phases of standard entanglement entropy in BCFT_2, we have a new phase of timelike entanglement entropy which arises from the light-cone singularities.
- This is possible only for a timelike interval.

AdS/BCFT duality

- The holographic bulk dual of a BCFT_d defined on the half space $x \geq 0$ is given by an AdS_{d+1} geometry truncated by an end-of-the-world (EOW) brane with Neumann boundary conditions. [Takayanagi'11], [Fujita, Takayanagi and Tonni '11]
- The gravitational action of the bulk manifold is given by

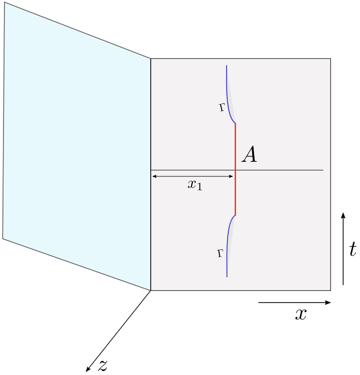
$$I = \frac{1}{16\pi G_N} \int_N d^{d+1}x \sqrt{|g|}(R - 2\Lambda) + \frac{1}{8\pi G_N} \int_Q d^d y \sqrt{|h|}(K - T).$$

- Here K is the extrinsic curvature, T is the tension of end-of-the-world (EOW) brane Q and h_{ij} is the induced metric on Q.
- The location of the EOW brane is determined by a boundary condition of the bulk gravity, which can be a Neumann boundary condition (NBC) as originally proposed.
- One may also impose alternative boundary conditions for AdS/BCFT such as the conformal boundary condition [Chu, Guo and Miao '17] or a Dirichlet boundary condition [Miao '18].

Holographic TEE in $AdS_3/BCFT_2$

- Consider a pure timelike interval A in the dual $BCFT_2$ at zero temperature at a fixed distance $x = x_1$ from the boundary.
- We have three possible choices of RT surface for this configuration depending on the size and distance of interval from the boundary.

Phase I: Bulk phase



- In this phase, the interval is far away from the boundary such that the RT surface consists of two space-like geodesics and one timelike geodesic similar to the CFT_2 case.
- So, the holographic timelike entanglement entropy for an interval having length T_0 along the time direction in this phase may be expressed as

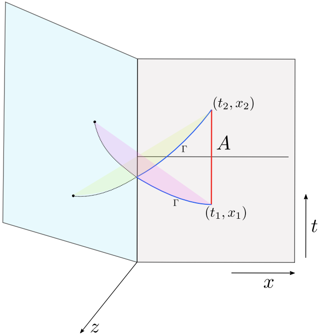
$$S_A^T = \frac{c}{3}\log\frac{T_0}{\epsilon} + \frac{i\pi c}{6}.$$

- The above result matches exactly with the corresponding dual BCFT_2 result.

Phase II: Regge phase

- For this phase, the end points of the purely timelike interval on the boundary can be parametrized as $A[(t_1, x_1), (t_2, x_2)] \equiv [(-x_1, x_1), (x_1(1-\delta), x_1)]$, with $\delta = 2 - T_0/x_1$.
- The geodesics lie along the $t = \pm mx + c$ plane. The plane passing through the points (t_2, x_2) and mirror image of (t_1, x_1), i.e $(t_1, -x_1)$, is described by

$$t = mx + c, \quad m = 1 - \frac{\delta}{2}.$$



- We obtain the length of the geodesic as follows

$$L = \log \frac{2x_1\sqrt{\delta}}{\epsilon}.$$

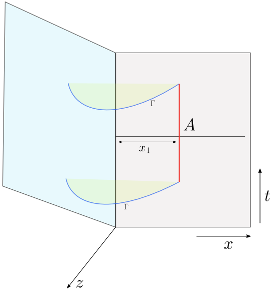
- The holographic TEE is given by the sum of these two geodesics after using the RT formula as

$$S_A^T = \frac{c}{3} \log \left(\frac{2x_1}{\epsilon} \sqrt{2 - \frac{T_0}{x_1}} \right).$$

- This agrees precisely with the corresponding dual field theory result.

Phase III: Boundary phase

- Finally we consider the boundary phase where the interval is closer to the boundary. In this phase, the RT surface end on the brane as shown below.



- So, the holographic timelike entanglement entropy for a timelike interval A may obtained using the RT formula as

$$S_A^T = \frac{c}{3}\log\frac{2x_1}{\epsilon} + \frac{c}{3}\rho_0.$$

- It matches with the dual field theory results.

CFTs with gravitational anomaly

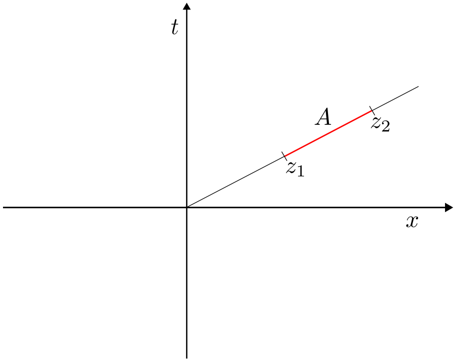
- Various types of anomalies in QFT has been studied and reproduced successively in AdS/CFT e.g scale anomaly [Henningson, Skenderis '98], R-symmetry chiral anomaly [Witten, Freedman, Klebanov....] , $1/N^2$ correction to chiral anomaly [Bilal, Chu '99]...
- Here we consider another type of anomaly i.e. gravitational anomaly [Gaume, Witten '84].
- CFT_2 with gravitational anomaly are described by two copies of the Virasoro algebra with two unequal central charges c_L and c_R.
- The anomaly appears as a gravitational anomaly where the stress tensor is symmetric but not conserved.
- This non-conservation is captured by the anomalous divergence of the stress tensor as [Kraus, Larsen '05]

$$\nabla_\mu T^{\mu\nu} = \frac{c_L - c_R}{96\pi} g^{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha \partial_\rho \Gamma^\rho_{\mu\beta}.$$

- From a bulk perspective, gravitational anomalies arise because the gravitational Chern-Simons term is only invariant under diffeomorphisms up to a boundary term that leads to a non-zero divergence of the dual field theory stress tensor.

Zero temperature

- Consider a boosted interval described by $A \equiv [z_1, z_2] = [(x_1, t_1), (x_2, t_2)]$ and $B = A^c$ represent the rest of the system.



- The entanglement entropy for a generic spacelike interval A is given by

$$S_A = \lim_{n \to 1} \frac{1}{1-n} \log \left\langle \Phi_n(z_1, \bar{z}_1) \Phi_{-n}(z_2, \bar{z}_2) \right\rangle .$$

- The scaling dimension and spin of the twist fields are then given by

$$\Delta_n = \frac{c_L + c_R}{24}\left(n - \frac{1}{n}\right)\ ,\ \ s_n = \frac{c_L - c_R}{24}\left(n - \frac{1}{n}\right).$$

- On using the two point twist correlator, one can obtain the entanglement entropy for a generic spacelike interval A as [Castro, Detournay, Iqbal and Perlmutter '14]

$$S_A = \frac{c_L + c_R}{6}\log\left(\frac{R}{\epsilon}\right) - \frac{c_L - c_R}{6}\kappa.$$

- Here κ is the boost parameter and related to rotation angle as $\theta = i\kappa$.
- The length R and boost κ for the boosted interval A is given in terms of (t, x)-coordinates as

$$R = \sqrt{x_{12}^2 - t_{12}^2}\,, \qquad \kappa = \tanh^{-1}\left(\frac{t_{12}}{x_{12}}\right).$$

- The second term in the EE arises due to contribution from the gravitational anomaly.
- In the absence of an anomaly ($c_L = c_R$), the entanglement entropy simplifies to the familiar entanglement entropy of a single interval. [Calabrese, Cardy '04]

TEE in CFT_2 with gravitational anomaly

- The TEE for a timelike interval A in CFT_2 in the presence of gravitational anomaly is obtained by analytically continuing the spacelike interval to the timelike interval case where $t_{12}^2 - x_{12}^2 > 0$,

$$S_A^T = \frac{c_L + c_R}{6} \log \frac{\sqrt{t_{12}^2 - x_{12}^2}}{\epsilon} + \frac{c_L + c_R}{12} \log(-1) - \frac{c_L - c_R}{6} \tanh^{-1} \left(\frac{t_{12}}{x_{12}} \right).$$

- The timelike entanglement entropy in the presence of anomaly for a pure timelike interval is given by

$$S_A^T = \frac{c_L + c_R}{6} \log \frac{T}{\epsilon} + \frac{c_R}{6} i\pi.$$

- This result reduces to the familiar timelike entanglement entropy in non-anomalous CFT i.e $c_L = c_R$. [Doi, Harper, Mollabashi, Takayanagi and Taki '22]
- We observe an asymmetric dependence of the TEE on the central charges, with the real part depends on both central charges and the imaginary part depends only on one of the two central charges.
- This property is an useful feature that one may exploit to detect the presence of gravitational anomaly in a chiral CFT.

- For CFTs with gravitational anomaly, the bulk dual geometry is described by topologically massive gravity (TMG) in a bulk AdS_3 spacetime. [Kraus, Larsen '05]
- The bulk action consists of the standard Einstein-Hilbert action and an additional gravitational Chern-Simons (CS) term as

$$S_{\mathrm{TMG}} = \frac{1}{16\pi G_N^{(3)}} \int d^3x \sqrt{-g} \left(R + \frac{2}{\ell^2} \right) + \frac{1}{32\pi G_N^{(3)} \mu} \int d^3x \sqrt{-g} \epsilon^{\lambda\mu\nu} \Gamma^{\rho}_{\lambda\sigma} \left(\partial_\mu \Gamma^{\sigma}_{\rho\nu} + \frac{2}{3} \Gamma^{\sigma}_{\mu\tau} \Gamma^{\tau}_{\nu\rho} \right) .$$

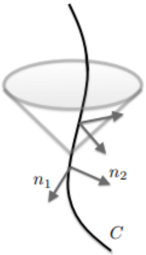
- The asymptotic symmetry analysis of TMG in AdS_3 reveals that the algebra of asymptotic Killing vector modes corresponds to two copies of the Virasoro algebra with left and right moving central charges.
- This results in a Brown-Henneaux type relation for TMG-AdS_3 given by [Kraus, Larsen '05]

$$c_L = \frac{3\ell}{2G_N^{(3)}} \left(1 - \frac{1}{\mu\ell} \right) , \quad c_R = \frac{3\ell}{2G_N^{(3)}} \left(1 + \frac{1}{\mu\ell} \right) .$$

Holographic entanglement entropy in TMG-AdS$_3$

- For locally AdS$_3$ solutions to TMG, the holographic principle dictates that the primary operators in the CFT correspond to massive spinning particles propagating along extremal worldlines in the bulk geometry.
- The holographic entanglement entropy is given by the on-shell action of a massive spinning particle in the topologically massive gravity as [Castro, Detournay, Iqbal and Perlmutter '14]

$$S_{\text{on-shell}} = \frac{1}{4G_N^{(3)}} \int_C ds \left(\sqrt{g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} + \frac{i}{\mu} n_2 \cdot \nabla n_1 \right)_E .$$



- The first part corresponds to the Ryu-Takayanagi term originating from the Einstein-Hilbert action.
- The second part arises from the Chern-Simons term due to the spin of the particle and quantifies the extrinsic properties of the worldline.
- Such extrinsic properties can be studied through the change of the normal frame $(\dot{X}, n_1, n_2)$ as the worldline is traversed.

- The second term corresponds to the Chern-Simons contributions and can be written in terms of parallel transported vectors as [Castro, Detournay, Iqbal and Perlmutter '14]

$$S_{\text{anom}}^{\text{spacelike}} = \frac{1}{4G_N^{(3)}\mu} \log\left(\frac{q(s_f)\cdot n_f - \tilde{q}(s_f)\cdot n_f}{q(s_i)\cdot n_i - \tilde{q}(s_i)\cdot n_i}\right).$$

- n_i and n_f are the boundary values of n which are determined from the boundary CFT data.
- For a timelike path (or geodesic), the tangent vector is timelike and the other two normal vectors are spacelike $(n, \tilde{n})$ which is different from the spacelike geodesic case.
- The anomaly contribution from the timelike geodesic can be obtained in terms of q and $\tilde{q}$ as

$$S_{\text{anom}}^{\text{timelike}} = \frac{1}{4G_N^{(3)}\mu} \log\left[\frac{q(\tau_f)\cdot n_f - i\,\tilde{q}(\tau_f)\cdot n_f}{q(\tau_i)\cdot n_i - i\,\tilde{q}(\tau_i)\cdot n_i}\right].$$

- The parallel transported vectors satisfy the following condition

$$v^2 = -1, \quad q^2 = \tilde{q}^2 = 1, \quad q \cdot \tilde{q} = v \cdot q = v \cdot \tilde{q} = 0$$
$$\nabla v = 0 \ , \quad \nabla q = 0 \ , \quad \nabla \tilde{q} = 0.$$

- As holographic TEE involves both spacelike and timelike geodesics, it is determined by the total of the on-shell actions as follows

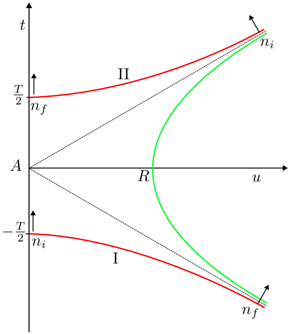
$$S_{\text{TEE}} = \text{ext}_C \ \left(S^{\text{spacelike}}_{\text{on}-\text{shell}} + S^{\text{timelike}}_{\text{on}-\text{shell}} \right).$$

- This expression incorporates the geodesic length (RT formula) along with the anomalous contributions arising from Chern-Simons term considering both spacelike and timelike geodesic contributions.

Holographic TEE at zero temperature

- Consider a pure timelike interval A of length T described by $A \equiv [-\frac{T}{2}, \frac{T}{2}]$ in a dual CFT_2 with a gravitational anomaly at zero temperature.
- The corresponding holographic dual is given by the AdS_3 spacetime in Poincaré coordinates.
- The equation for the spacelike and timelike geodesics in the Poincaré metric is given by [Basak, Chakraborty, Chu, Giataganas and HP '23]

$$t^2 - u^2 = \frac{T^2}{4}, \quad u^2 - t^2 = R^2.$$



Spacelike geodesic contribution

- The total length of these geodesics can be obtained as

$$L_A^{\text{spacelike}} = 2T\ell \int_\epsilon^\infty \frac{du}{2u} \frac{1}{\sqrt{u^2 + \frac{T^2}{4}}} = 2\ell \log \frac{T}{\epsilon}.$$

- Now for anomalous contribution, the tangent vector and parallel transported normal frame vectors along this spacelike geodesic is given by

$$v^\mu = -\frac{2u}{T\ell}(u, 0, t), \quad q^\mu = -\frac{2u}{T\ell}(t, 0, u), \quad \tilde{q}^\mu = (0, \frac{u}{\ell}, 0).$$

- While the normal vector n can be readily fixed at the boundary along the time direction, doing so at the geodesic's other end in the bulk is not straightforward because of the non-covariance of bulk TMG and it requires a particular prescription.
- The boundary values of the (normalised) normal vector n can be written as

$$n^i_\mu = \frac{\ell}{u_\epsilon}(-1, 0, 0), \quad n^f_\mu = \frac{\ell}{u_\infty}(-\cosh\beta, 0, \sinh\beta).$$

- Similarly for curve II, we choose the boundary conditions on n as follows

$$n_\mu^i = \frac{\ell}{u_\infty}(-\cosh\beta, 0, -\sinh\beta), \quad n_\mu^f = \frac{\ell}{u_\epsilon}(-1, 0, 0),$$

- Upon using these boundary normal vectors, we find that the total anomalous contribution from spacelike geodesics I and II vanishes i.e

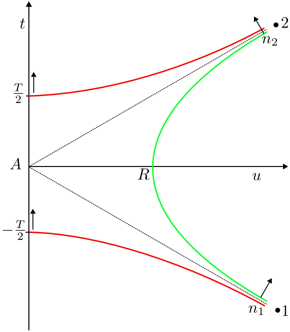
$$S_{\text{anom}}^{\text{spacelike}} = 0.$$

- This is very similar to the case for the entanglement entropy for an unboosted interval (i.e a pure spacelike interval) where anomalous contribution also vanishes. [Castro, Detournay, Iqbal and Perlmutter '14]

Timelike geodesic contribution

- For the timelike geodesic (green color curve), the length of this geodesic can be obtained as

$$L_A^{\text{timelike}} = 2i\ell R \int_R^\infty \frac{du}{u} \frac{1}{\sqrt{u^2 - R^2}} = i\ell\pi.$$



- The timelike geodesic can be parameterized by γ as $u = R\cosh\gamma, t = R\sinh\gamma$ where γ range from $-\infty$ to $+\infty$.

- The tangent vector and parallel transported vector to this timelike geodesic can be written as

$$v^\mu = \frac{u}{\ell R}(u, 0, t), \quad q^\mu = -\frac{u}{\ell R}(t, 0, u), \quad \tilde{q}^\mu = (0, \frac{u}{\ell}, 0).$$

- Given that the timelike geodesic matches up with the spacelike geodesics I, II asymptotically at the null infinities.
- We propose to specify the initial and final normal vectors n_1, n_2 by matching them up with the normal vectors n_i and n_f at the respective null infinities as

$$n^1_\mu = \frac{\ell}{u_{-\infty}}(\sinh\gamma, 0, \cosh\gamma),$$

- Similarly we have $\gamma \to \infty$ for the normal vector near the point 2 at the null infinity

$$n^2_\mu = \frac{\ell}{u_{\infty}}(-\sinh\gamma, 0, -\cosh\gamma),$$

- The anomalous contribution from timelike geodesic is then given by

$$S^{\text{timelike}}_{\text{anom}} = \frac{1}{4G^{(3)}_N \mu} \log\left[\frac{q(\tau_2)\cdot n_2 - i\tilde{q}(\tau_2)\cdot n_2}{q(\tau_1)\cdot n_1 - i\tilde{q}(\tau_1)\cdot n_1}\right] = \frac{1}{4G^{(3)}_N \mu} i\pi.$$

- Hence the total contribution to the on-shell action for timelike geodesic is given by

$$S_A^{\text{timelike}} = \frac{\ell}{4G_N^{(3)}} i\pi + \frac{1}{4G_N^{(3)}\mu} i\pi.$$

- Now upon using on-shell action for holographic TEE, we obtain the holographic TEE in the presence of gravitational anomaly as follows

$$S_A^T = \frac{c_L + c_R}{6} \log \frac{T}{\epsilon} + \frac{c_R}{6} i\pi.$$

- Interestingly, we observe that the above result matches exactly with the dual field theory result.

Summary

- We obtained the timelike entanglement entropy for a pure timelike interval at zero temperature in the context of $\mathrm{AdS}_3/\mathrm{BCFT}_2$.
- The TEE for a timelike interval has three phases in contrast to the usual two phases of the standard entanglement entropy in a BCFT.
- The new Regge phase is unique to the timelike interval when one end point of the interval approaches the light cone of the mirror reflection of other end point.
- We studied the effect of gravitational anomaly on the timelike entanglement entropy (TEE) in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality.
- For a pure timelike interval in all cases, the imaginary part depends on only one central charge in contrast to its real counterpart which depends on both the left and right central charges.
- The holographic TEE involves the sum of length of the geodesics and the twist (rotation) of the normal frame along these geodesics.
- The holographic results agrees with the dual field theory results.

THANK YOU!