Wigner negativity, Random matrices and Gravity

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Based on: arXiv:2402.13694 [hep-th], arXiv:2506.02110 [hep-th] and work in progress,

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NTU-NCTS Holography and Quantum Information workshop

Motivation 1: Why AdS/CFT?

- The AdS/CFT correspondence gives us a map between a gravitational theory in AdS and an "ordinary" quantum system without gravity living on the boundary.
- Remarkably, AdS/CFT allows us to compute some aspects of the strong-coupling, large-N dynamics of the boundary theory.
- One might say that gravity provides the most efficient set of variables to describe the strongly-coupled boundary dynamics.
- Given the sucess of AdS/CFT, it is natural to ask whether we can derive it from some underlying fundamental principles.
- Equivalently, given a suitable quantum system, how do we go about finding gravity within its Hilbert space?

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- This construction uses a discrete version of the Wigner function.
- Here, we will argue that this formalism may be well-suited in looking for efficient classical variables for quantum dynamics.

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- Semi-classical notion of spacetime protected by complexity.
- This conjecture was geometrized in the form of the python's lunch conjecture [Brown et al '19].

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- In the second half of this talk, we will argue that the discrete Wigner function gives a useful language to make progress towards the python's lunch conjecture.

Review of discrete Wigner function

Review: Wigner function

- In standard quantum mechanics, the Wigner function for a state ψ is a *quasi-probability* distribution in phase space:

$$W_\psi(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\,\langle q - \frac{y}{2}|\psi\rangle\langle\psi|q + \frac{y}{2}\rangle\, e^{-ipy}.$$

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- Equivalently, one can write:

$$W(q,p) = \frac{1}{2\pi}\langle\psi|\widehat{A}(q,p)|\psi\rangle,$$

where $\widehat{A}(q,p)$ are Fourier transforms of displacement operators:

$$\widehat{A}(q,p) = \int \frac{dp'\,dq'}{2\pi} e^{i(qp'-pq')} e^{i(p'\widehat{q}-q'\widehat{p})},$$

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- The Wigner function is the inverse of the Weyl map from classical functions to Weyl-ordered operators.

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- States for which the Wigner function is everywhere positive may be regarded as *classical* states (more on this below).
- For instance, a pure state has a positive Wigner function if and only if it is Gaussian (i.e., a generalized coherent state).

  [Hudson '74, Soto & Claverie '83]

Review: Discrete Wigner function

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- For a Hilbert space of dimension D, the phase space is taken to be the lattice $\mathcal{P} = \mathbb{Z}_D \times \mathbb{Z}_D$.
- The formalism works best when D is prime, but can be generalized to arbitrary D.

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$$\widehat{A}(q,p) = \sum_{q',q''=0}^{D-1} \tilde{\delta}_{2q,q'+q''} e^{-\frac{2\pi i(q'-q'')p}{D}} |q'\rangle\langle q''|,$$

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- But as before, the Wigner function is not a probability distribution, in that it can take negative values.
- States with positive Wigner functions can be thought of as being classical in the following sense:

Gottesman-Knill theorem

Any quantum circuit which starts with a Wigner positive state and only involves stabilizer (i.e., positivity preserving) operations can be simulated efficiently on a classical computer. [Aaronson and Gottesman '04, Mari & Eisert '12, Veitch et al '12]

Wigner negativity

- For a general state ψ, the negativity of the Wigner function (sometimes called Mana), defined as

$$\mathcal{N}_\psi = \sum_{q,p=0}^{D-1} |W_\psi(q,p)|$$

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- It is a monotone under stabilizer operations [Veitch et al '14].
- Intuitively, one can regard it as a measure of the complexity of simulating the quantum circuit on a classical computer [Stahlke '14, Pashayan et al '15].

Wigner negativity and Uncertainty

- Wigner negativity is also related to quantum uncertainty:

$$S_{1/2}(q) \geq \log \mathcal{N}_\psi,$$
$$S_{1/2} = 2\log \sum_q P_q^{1/2}, \quad P_q = |\langle q|\psi\rangle|^2.$$

where $S_{1/2}(q)$ is the $1/2$-Renyi entropy of the probability distribution in the q-basis.

- More generally:

$$\min\left(S_{1/2}(q), S_{1/2}(p)\right) \geq \log \mathcal{N}_\psi.$$

- Thus, Wigner negativity necessarily implies some amount of quantum spreading in phase space.

Negativity growth under time evolution

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- However, recall that the Wigner function is defined with respect to an ordered basis.

So, given an initial state ψ_0 and the time evolution operator e^{-itH}, our task is to find an ordered basis for the Hilbert space such that the Wigner negativity growth of the state under time-evolution is "minimized".

Minimizing negativity growth

Claim

The early time Wigner negativity growth is minimized by the Krylov basis $\mathcal{K}$ (up to individual phases) [Basu, Ganguly, Nath & OP '24].

Krylov basis

- The Krylov basis is obtained by orthonormalizing the set of states $\psi_0, H\psi_0, H^2\psi_0 \cdots$:

$$|0\rangle_{\mathcal{K}} = |\psi_0\rangle,$$

$$|1\rangle_{\mathcal{K}} = \frac{1}{\sqrt{N_1}} \left(H|\psi_0\rangle - \langle 0_{\mathcal{K}}|H|\psi_0\rangle |0\rangle_{\mathcal{K}} \right),$$

$$|2\rangle_{\mathcal{K}} = \frac{1}{\sqrt{N_2}} \left(H^2|\psi_0\rangle - \langle 0_{\mathcal{K}}|H^2|\psi_0\rangle |0\rangle_{\mathcal{K}} - \langle 1_{\mathcal{K}}|H^2|\psi_0\rangle |1\rangle_{\mathcal{K}} \right),$$

- The Krylov basis is known to minimize the "spread of the wavefunction" [Balasubramanian et al '22].
- The same idea has also appeared previously in the context of operator spreading [Parker et al '18, Swingle et al '20, Rabinovici et al '21].

Minimizing negativity growth: perturbative argument

- By minimizing the early time negativity growth, we mean the following: if we wish to minimize the negativity at $t = 0$, we can simply take ψ_0 to be a basis vector, and without loss of generality, we take it to be the 0th basis vector.

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- Now, for any choice of the first basis vector, the coefficient of the linear in time growth of the negativity is always larger than the coefficient of the linear in time growth in the Krylov basis.
- Similarly, for any basis which agrees with the Krylov basis up to the mth vector, but differs at $m + 1$, the coefficients in the Taylor approximation of the Wigner function agree between the two bases up to t^m, but at $O(t^{m+1})$, the negativity in the Krylov basis is smaller than any other such basis.

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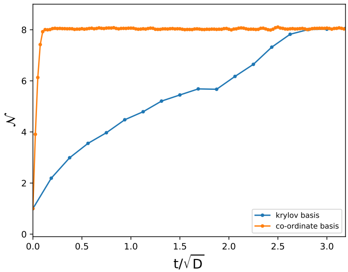
- We will choose a Hamiltonian (i.e., a $D \times D$ Hermitian matrix) from the Gaussian unitary ensemble. Chaotic systems are expected to show random matrix theory behavior, so we expect our analysis to apply in such systems.
- The initial state must also be sufficiently generic w.r.t the Hamiltonian. In the basis where the Hamiltonian is a random matrix, we can simply take the state $(1, 0, \cdots, 0)$ as the initial state.

Negativity growth in random matrix theory

Claim

The Wigner negativity w.r.t a generic basis grows rapidly and saturates to an exponentially large value within an O(1) amount of time evolution. On the other hand, in the Krylov basis, the Wigner negativity grows gradually and takes an exponential amount of time to saturate.



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- Recall that the initial state ψ_0 will be the first basis vector $|0\rangle$ in this basis.
- We would like to compute the average Wigner negativity as a function of time:

$$\overline{\mathcal{N}(t)} = \frac{1}{Z}\int dH\, e^{-D\,\mathrm{Tr}\,H^2}\mathcal{N}_{\psi_t}, \quad \psi_t = e^{-itH}\psi_0.$$

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- Consider a change of basis which leaves ψ_0 invariant but rotates the rest of the basis vectors by a Haar random unitary U_{ij} $(i, j \neq 0)$.

Negativity growth in generic basis

- From the definition of the Wigner function:

$$W(q,p) = \frac{1}{D} \sum_{k,\ell=0}^{D-1} \tilde{\delta}_{2q,k+\ell} e^{-\frac{2\pi i(k-\ell)p}{D}} \langle \psi_0 | e^{itH} | k \rangle \langle \ell | e^{-itH} | \psi_0 \rangle$$

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- This follows from the unitary invariance of the GUE.
- Thus, we can write the averaged negativity as

$$\overline{\mathcal{N}(t)} = \int DU_{ij} \frac{1}{Z} \int dH \, e^{-D \mathrm{Tr}\, H^2} \sum_{q,p} |W_U(q,p)|,$$

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- It is convenient to do the U integral first using standard techniques for Haar integration.

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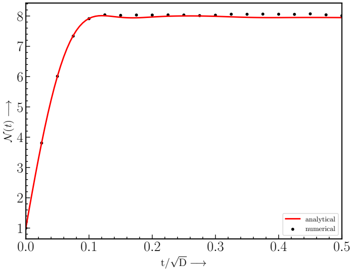
for integer n, and then analytically continue the answer to $n \to \frac{1}{2}$.
- At large D, this calculation can be done by summing up a set of leading diagrams, and we get

$$\overline{N(t)} = S + \sqrt{\frac{2D}{\pi}}\sqrt{1-S^2} + O(1/\sqrt{D}),$$

$$S(t) = \overline{|\langle\psi_0|e^{-itH}|\psi_0\rangle|^2},$$

where $S(t)$ is called the survival probability or the spectral form factor.

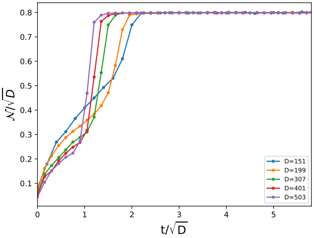
Negativity growth in generic basis

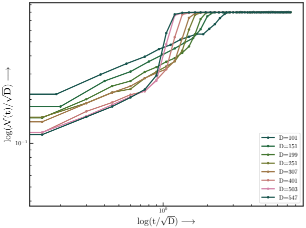


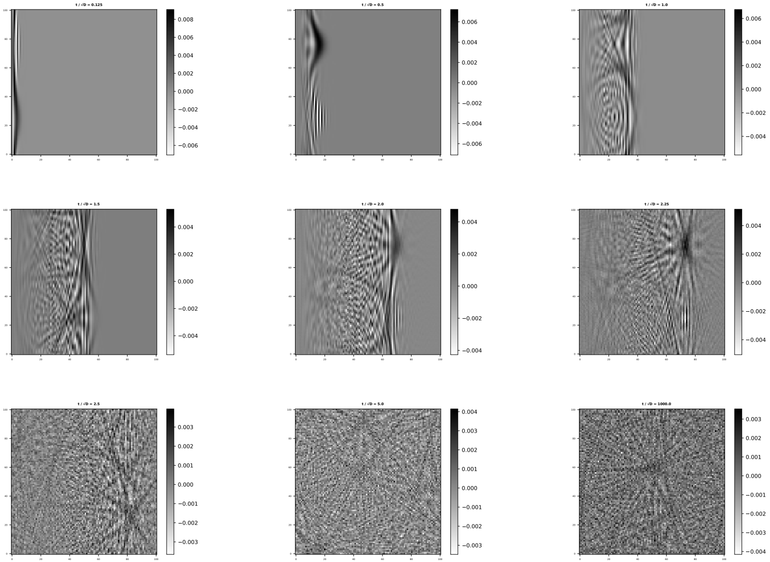
- In particular, $S(t)$ decays away from 1 in an $O(1)$ amount of time evolution.
- So, we see that the negativity grows rapidly and saturates to its maximum value of $\sqrt{\frac{2D}{\pi}}$ in $O(1)$ time.

Negativity growth in the Krylov basis

- On the other hand, the negativity in the Krylov basis grows gradually (power law) for a time of $O(\sqrt{D})$, then hits a sharp ramp and saturates to a final value close to $\sqrt{\frac{2D}{\pi}}$.







t / √D = 0.125
0.008
0.006
0.004
0.002
0.000
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t / √D = 0.5
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0.000
−0.001
−0.002
−0.003
t / √D = 5.0
0.004
0.003
0.002
0.001
0.000
−0.001
−0.002
−0.003
t / √D = 1000.0
0.003
0.002
0.001
0.000
−0.001
−0.002
−0.003
−0.004

Negativity growth in the Krylov basis

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- The average values of the Lanczos coefficients in GUE are known. In the large D limit, one finds [Balasubramanian et al, '22]

$$\overline{a_n} = 0, \ \ \overline{b_n} = 1, \cdots (D \to \infty, n \text{ fixed})$$

and the variances are $O(1/D)$.

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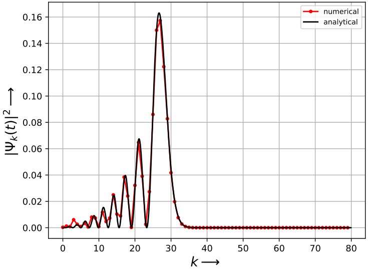
- Using this, we can compute the time evolution of the initial state:

$$\langle n|e^{-itH_{\text{eff}}}|0\rangle = i^n\frac{(n+1)}{t}J_{n+1}(2t).$$

Negativity growth in the Krylov basis



- The wavefunction is *localized* in the region $n \leq 2t$, and decays exponentially beyond. We can use this to bound the growth of Wigner negativity.
- Note that we have neglected statistical fluctuations of the Lanczos coefficients, which is a good approximation for $t \ll \sqrt{D}$.

Bound on Negativity growth

- Recall that the Wigner negativity is upper bounded by the spread in the state:

$$\log \mathcal{N} \leq S_{1/2}(q), \quad S_{1/2}(q) = 2 \log \sum_n |\langle n|\psi_t\rangle|.$$

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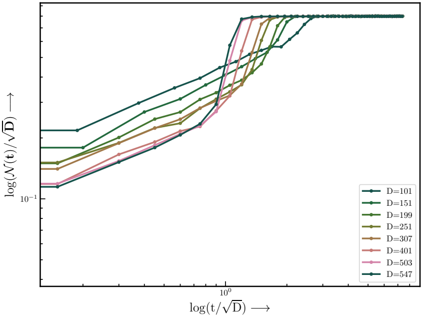
- One can actually do better – by using the Jensen's inequality, it is possible to show that

$$\mathcal{N} \leq \text{const.}\sqrt{t}.$$

Bound on Negativity growth



- Thus, the negativity in the Krylov basis cannot become exponentially large in any finite $O(1)$ amount of time.
- Of course, at times of $O(\sqrt{D})$, these arguments break down. Precisely at this time, numerics show a sharp ramp followed by saturation close to the maximum value of $\sqrt{\frac{2D}{\pi}}$.

Stabilizer complexity of Hawking radiation

Stabilizer complexity of Hawking radiation

- We now turn to our second application – to compute the Wigner negativity of Hawking radiation, thought of as stabilizer complexity from the resource theory of stabilizer computation.
- In order to model an evaporating scenario, we entangle a black hole B with a quantum mechanical bath/reservoir R [PSSY '19]:

$$|\Psi\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |\psi_i\rangle_B \otimes |i\rangle_R,$$

where the states $\{|i\rangle\}$ in R are orthonormal, $|\psi_i\rangle$ are end-of-the-world brane states in JT gravity.

Stabilizer complexity of Hawking radiation

- The reduced density matrix for the bath is given by:

$$\rho_R = \frac{1}{D\, e^{S_0} Z_1} \sum_{i,j=1}^{D} \langle \psi_i | \psi_j \rangle_B \, |j\rangle\langle i|_R .$$

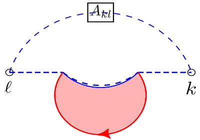
- For small D, one can treat the states $\{|\psi_i\rangle\}$ as being approximately orthogonal. This leads to a maximally mixed state on the bath whose entropy grows as $\log D$.
- However, when D becomes $O(e^{S_0})$, one can no longer treat the states $\{|\psi_i\rangle\}$ as being orthogonal.
- Indeed, it was shown in [PSSY '19] that there is a phase transition in the entanglement entropy at $D \sim e^{S_0}$. This cuts off the naive growth of entropy with D and realizes the expected Page curve.
- This happens in gravity via replica wormholes.

Stabilizer complexity of Hawking radiation

- Our goal is to compute the Wigner negativity of the reduced density matrix ρ_R using the rules of the gravitational path integral.
- In order to compute the Wigner function, we must choose an orthonormal basis for the radiation Hilbert space.
- In general, the Wigner negativity very much depends on the choice of basis. However, in our gravity calculation, the negativity turns out to have a universal basis independent form, which can then only depend on information theoretic properties of the state ρ_R.

Wigner function

- The Wigner function can be represented diagrammatically as follows:

$\overline{W} \approx$ 

- This gravity path integral computes the ensemble averaged Wigner function:

$$\overline{W(q,p)} = \frac{1}{D^2 e^{S_0} Z_1} \sum_{k,\ell=0}^{D-1} A_{k\ell} e^{S_0} Z_1 \delta_{k,\ell} = \frac{1}{D^2} \mathrm{Tr}(A) = \frac{1}{D^2}.$$

- So, on average, the Wigner function looks uniform and positive. However, what we actually want to compute is the ensemble average of the negativity.

Wigner negativity

- So, our goal is to compute the ensemble averaged Wigner negativity of the Hawking radiation using the gravitational path integral:

$$\overline{\mathcal{N}} = \sum_{q,p} \overline{|W(q,p)|},$$

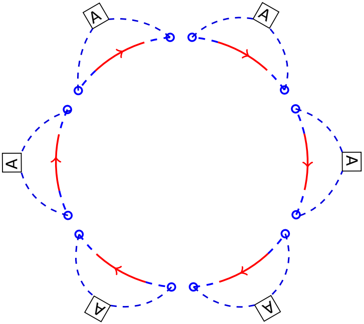
where note that the absolute value is inside the average.

- In order to deal with the absolute value, we employ the *replica trick*: we first evaluate the ensemble average over W^{2n} for integer n, and analytically continue the result to $n = \frac{1}{2}$:

$$\overline{\mathcal{N}} = \lim_{n \to \frac{1}{2}} \sum_{q,\, p} \overline{W^{2n}(q,p)}.$$

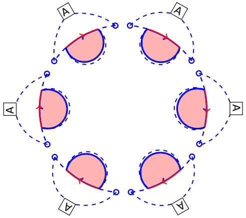
Wigner negativity

From the boundary point of view, this corresponds to the following boundary conditions:



Wigner negativity before Page time

- First, consider the regime $D \ll e^{S_0}$. In this limit, the dominant contribution comes from the completely disconnected diagram:



- This gives:

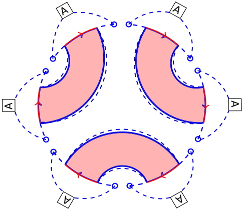
$$\overline{W^{2n}(q,p)} \approx \left(\overline{W(q,p)} \right)^{2n} = \frac{1}{D^{4n}} \quad \cdots \quad (D \ll e^{S_0}).$$

- Upon analytic continuation to $n = \frac{1}{2}$, we find

$$\overline{\mathcal{N}} \approx 1, \quad \cdots \quad (D \ll e^{S_0}).$$

Wigner negativity after Page time

- Next, consider the regime $D \gg e^{S_0}$. In this limit, the dominant contribution comes from the pair-wise connected diagram:



- While such diagrams are subleading in e^{S_0} compared to the fully disconnected diagram, the EOW brane index contractions for such diagrams give an enhancement at large D coming from the fact that $\mathrm{Tr}(A^2) = D$.

Wigner negativity after Page time

- The pairwise connected diagram gives:

$$\overline{W^{2n}(q,p)} \approx \frac{(2n)!}{2^n n!} \frac{\left(e^{S_0} Z_2 \operatorname{Tr}(A^2)\right)^n}{\left(e^{S_0} Z_1 \; D^2\right)^{2n}} = \frac{(2n)!}{2^n n!} \left(\frac{Z_2}{Z_1^2}\right)^n \frac{1}{e^{nS_0} D^{3n}}.$$

- Upon analytic continuation to $n = \frac{1}{2}$, we find

$$\overline{\mathcal{N}} \approx \sqrt{\frac{2}{\pi}} \exp\left[S_{\max} - S_2\right], \quad \cdots \quad (D \gg e^{S_0}),$$

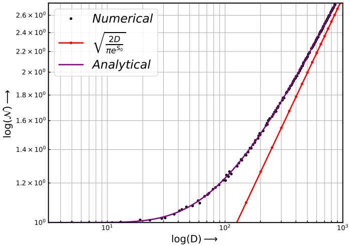
where S_2 is the 2nd Rényi entropy of the radiation post Page time,

$$S_2 = S_0 - \log\left(\frac{Z_2}{Z_1^2}\right),$$

and $S_{\max} = \log D$ is the coarse-grained entropy, or equivalently the entropy of the maximally mixed state on R.

Wigner negativity

- Thus, the Hawking radiation has an $O(1)$ stabilizer complexity before Page time, but an exponentially large complexity after Page time.



- We can attribute this exponentially large complexity past the Page point to the fact that the entanglement wedge of the radiation includes an island region in the black hole interior.