National Center for Theoretical Sciences
Physics Division 國家理論科學研究中心 物理組

NTU-NCTS Holography and Quantum Information Workshop Sep.29-Oct.3

Time-like Entanglement Entropy and Traversable Wormholes

Tadashi Takayanagi

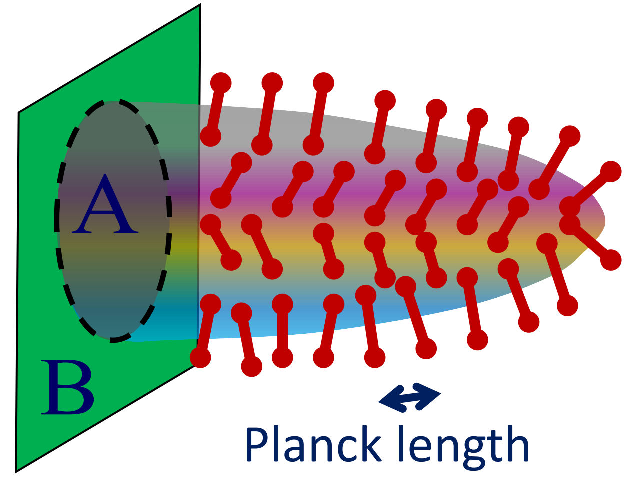
Yukawa Institute for Theoretical Physics
Kyoto University

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

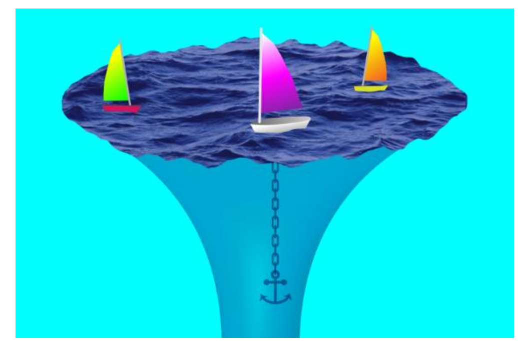
CGPQI Center for Gravitational Physics and Quantum Information
Yukawa Institute for Theoretical Physics, Kyoto University

Grant-in-Aid for Transformative Research Areas A
Extreme Universe

① Introduction



The relations between holography and quantum information implies that the space coordinate in gravity may emerge from quantum entanglement.

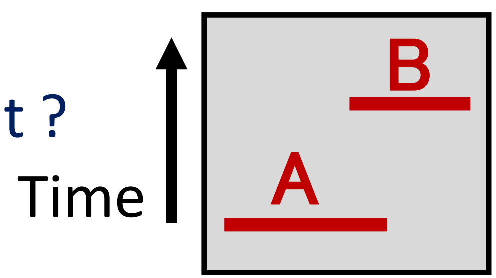


What about the time coordinate ?

Relevant questions

[Q1] How the time in de Sitter spaces emerge from CFTs ?

[Q2] What is a “time-like vesion” of entanglement ?
➔ causal connection vs entanglement ?



[Q3] Is traversability of wormholes related to quantum information ?

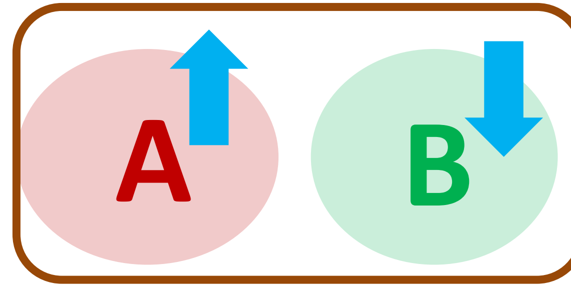
In this talk, we will argue that these are directly related to a generalization of quantum entanglement to the case where the density matrices are not hermitian.

The generalization of entanglement entropy to the above cases is called **pseudo entropy**.

[Ref: arXiv:2005.13801
Yoshifumi Nakata (YITP, Kyoto), Yusuke Taki (YITP, Kyoto)
Kotaro Tamaoka (Nihon U.), Zixia Wei (Harvard U.) and TT]

Quantum Entanglement (QE)

Two subsystems A and B in a total system are quantum mechanically correlated.



e.g. Bell state: $|\Psi_{Bell}\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_{\mathrm{B}}\right]$ ➡ Minimal Unit of Entanglement

Pure States: Non-zero QE $\Leftrightarrow |\Psi\rangle_{AB} \neq |\Psi_1\rangle_A \otimes |\Psi_2\rangle_B$.

Direct Product

The best (or only) measure of quantum entanglement for pure states is known to be **entanglement entropy (EE)**.

EE ＝ # of Bell Pairs between A and B

Entanglement entropy (EE) in HEP/CMP

Divide a quantum system into two subsystems A and B:

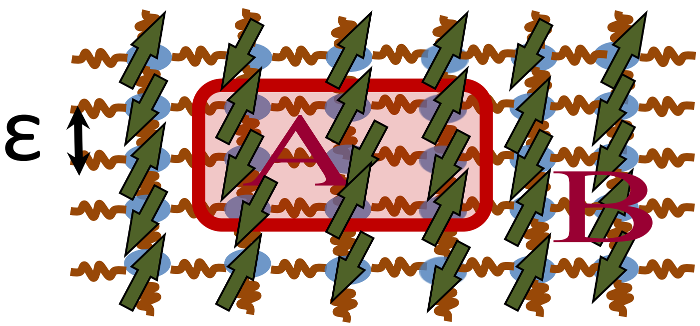
$$H_{tot} = H_A \otimes H_B \quad .$$

Define the reduced density matrix by $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$.

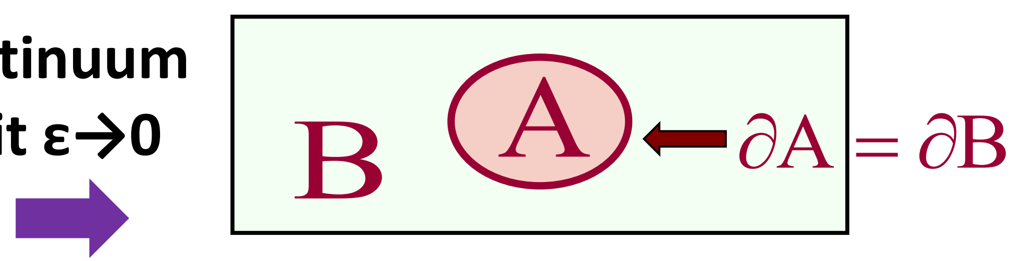
The entanglement entropy S_A is defined by the von-Neumann entropy

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \, .$$

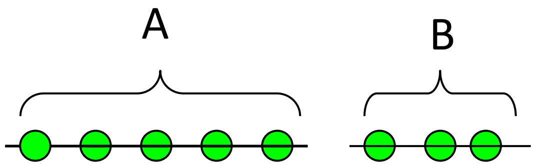
Quantum Many-body Systems | **Quantum Field Theories (QFTs)**



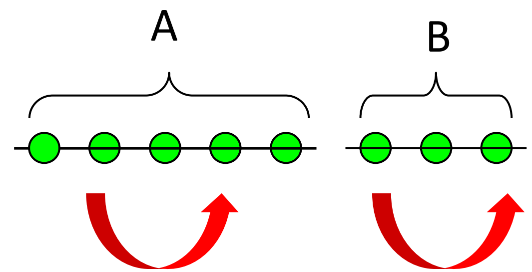
Continuum Limit ε→0



Entanglement Entropy (EE) in Quantum Information

Setup 

$\Rightarrow \; H_{tot} = H_A \otimes H_B$



LO (=Local Operations)

Projection measurements and unitary trfs.

which act either A or B only.

CC (=Classical Communications between A and B)

⇒These operations are combined and called LOCC.

A basic example of LOCC: quantum teleportation

$$(|\Psi\rangle_{AB}\langle\Psi|)^{\otimes M}$$

Entangled in a very complicated way

N Bell pairs

$$(|\Psi\rangle_{AB}\langle\Psi|)^{\otimes M} \Rightarrow (|\mathrm{Bell}\rangle\langle\mathrm{Bell}|)^{\otimes N}$$

Well-known fact in QI:

$$S(\rho_A) = \lim_{M\to\infty} \frac{N}{M}$$

$$\rho_A \equiv \mathrm{Tr}_B[|\Psi\rangle_{AB}\langle\Psi|]$$

[Bennett-Bernstein-Popescu-Schumacher 95, Nielsen 98]

Holographic Entanglement Entropy

[Ryu-TT 2006, Hubeny-Rangamani-TT 2007]

A generic Lorentzian asymptotic AdS spacetime is dual to a time dependent state $|\Psi(t)\rangle$ in the dual CFT.

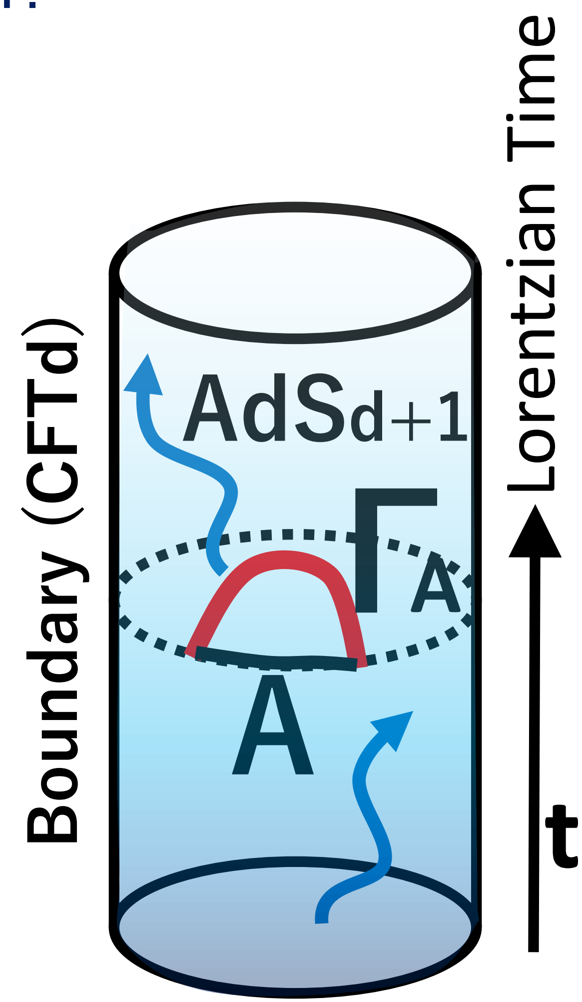
The time-dependent entanglement entropy

$$\rho_A(t) = \mathrm{Tr}_B[|\Psi(t)\rangle\langle\Psi(t)|] \Rightarrow S_A(t).$$

is computed from an extremal surface area:

$$S_A(t) = \mathrm{Min}_{\Gamma_A}\mathrm{Ext}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

$\partial A = \partial\gamma_A$ and $A \sim \gamma_A$.



Question: More general formula ?

Minimal areas in *Euclidean time dependent* asymptotically AdS spaces

= What kind of QI quantity in CFT ?

⇒ The answer is Pseudo Entropy !

[Nakata-Taki-Tamaoka-Wei-TT, 2020]

Contents

Main References

③,⑤➔ arXiv:2210.09457, arXiv:2302.11695

with Kazuki Doi (YITP), Jonathan Harper (YITP),

Ali Mollabashi (IPM), Yusuke Taki (YITP).

▶Time-like EE and PE in dS/CFT

④➔ arXiv: 2502.03531 + in preparation

with Jonathan Harper (YITP), Taishi Kawamoto (YITP),

Ryota Maeda (YITP), Nanami Nakamura (YITP)

▶PE in traversable AdS wormhole

② Pseudo Entropy and Holography

(2-1) Definition of Pseudo (Renyi) Entropy

Consider two quantum states $|\psi\rangle$ and $|\varphi\rangle$, and define the *transition matrix*:

$$\tau^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}.$$

We decompose the Hilbert space as $H_{tot} = H_A \otimes H_B$.

and introduce the reduced transition matrix:

$$\tau_A^{\psi|\varphi} = \mathrm{Tr}_B\left[\tau^{\psi|\varphi}\right]$$

Pseudo Entropy

$$S\left(\tau_A^{\psi|\varphi}\right) = -\mathrm{Tr}\left[\tau_A^{\psi|\varphi}\log\tau_A^{\psi|\varphi}\right].$$

Renyi Pseudo Entropy

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \frac{1}{1-n}\log\mathrm{Tr}\left[\left(\tau_A^{\psi|\varphi}\right)^n\right].$$

(2-2) Basic Properties of Pseudo Entropy (PE)

- In general, $\tau_A^{\psi|\varphi}$ is not Hermitian. Thus PE is complex valued.

More generally, we call $S(\tau_A)$ pseudo entropy when τ_A is not hermitan.

- If either $|\psi\rangle$ or $|\varphi\rangle$ has no entanglement (i.e. direct product state) , then

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = 0.$$

- We can show $S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \left[S^{(n)}\left(\tau_A^{\varphi|\psi}\right)\right]^{\dagger}.$

- We can show $S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = S^{(n)}\left(\tau_B^{\psi|\varphi}\right).$

→"SA=SB"

This implies a local holographic formula !

(2-3) Pseudo Entropy and Quantum Phases

[Mollabashi-Shiba-Tamaoka-Wei-TT 20, 21]

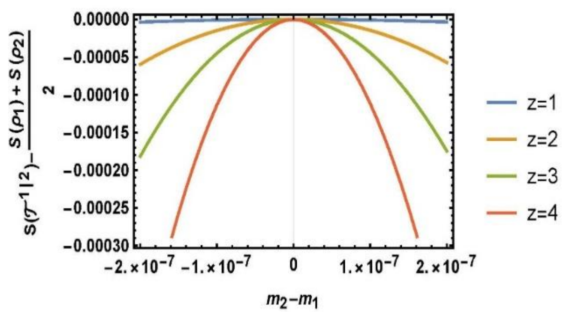
Properties of Pseudo entropy in QFTs

[1] Area law

$$S_A \sim \frac{\text{Area}(\partial A)}{\varepsilon^{d-1}} + (\text{subleading terms}),$$

[2] The difference

$$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{2|1}\right) - S(\rho_A^1) - S(\rho_A^2)$$



is negative if $|\psi_1\rangle$ and $|\psi_2\rangle$ are in a same phase.

PE in a 2 dim. free scalar when we change its mass.

What happen if they belong to different phases ?
Can ΔS be positive ?

Quantum Ising Chain with a transverse magnetic field

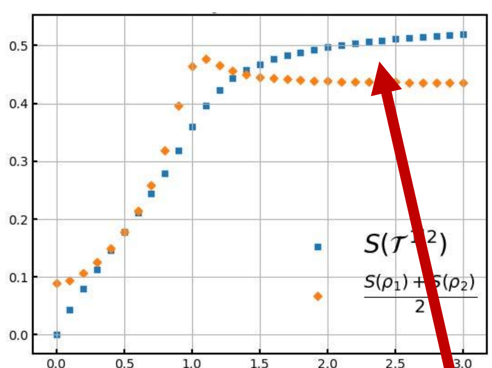
$$H = -J\sum_{i=0}^{N-1}\sigma_i^z\sigma_{i+1}^z - h\sum_{i=0}^{N-1}\sigma_i^x,$$

Ψ 1→ vacuum of H(J1)
Ψ 2→ vacuum of H(J2)
(We always set h=1)

J<1 Paramagnetic Phase
J>1 Ferromagnetic Phase

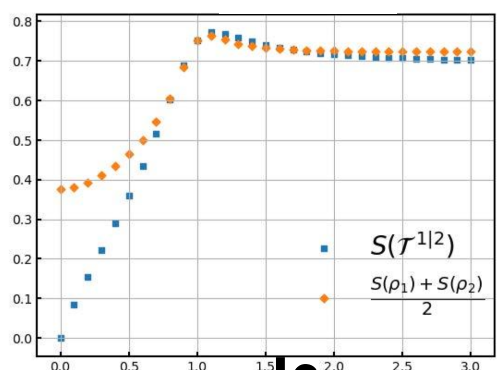
N=16, NA=8

J1=1/2



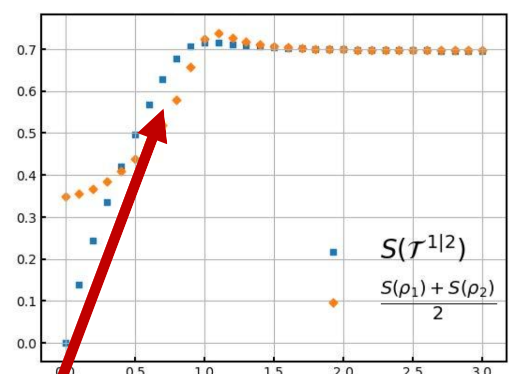
J2

J1=1



J2

J1=2

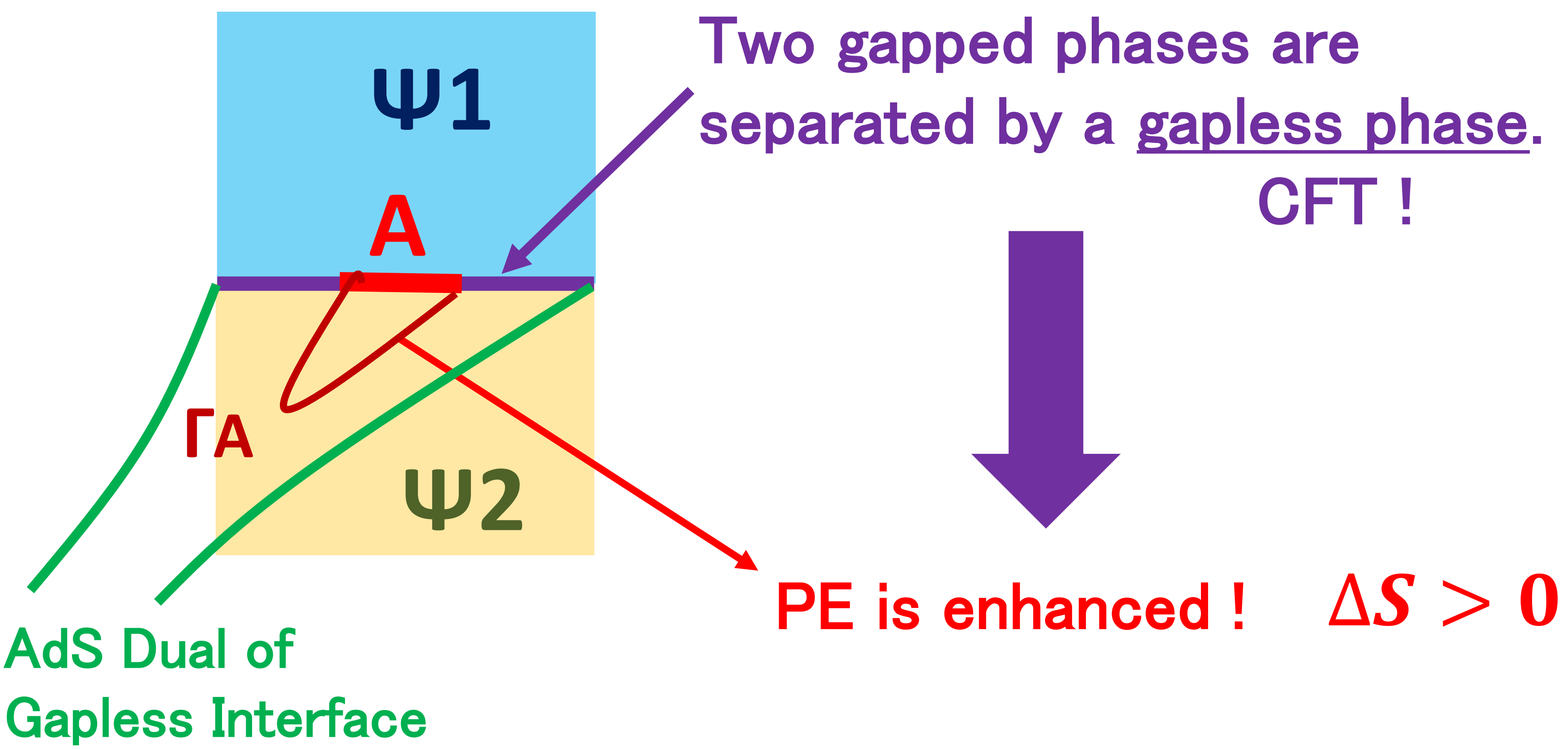


J2

$$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{2|1}\right) - S(\rho_A^1) - S(\rho_A^2) > 0$$

We find
when Ψ 1 and Ψ 2 are in different phases !

Heuristic Interpretation



The gapless interface (edge state) also occurs in topological orders.

➔Topological pseudo entropy

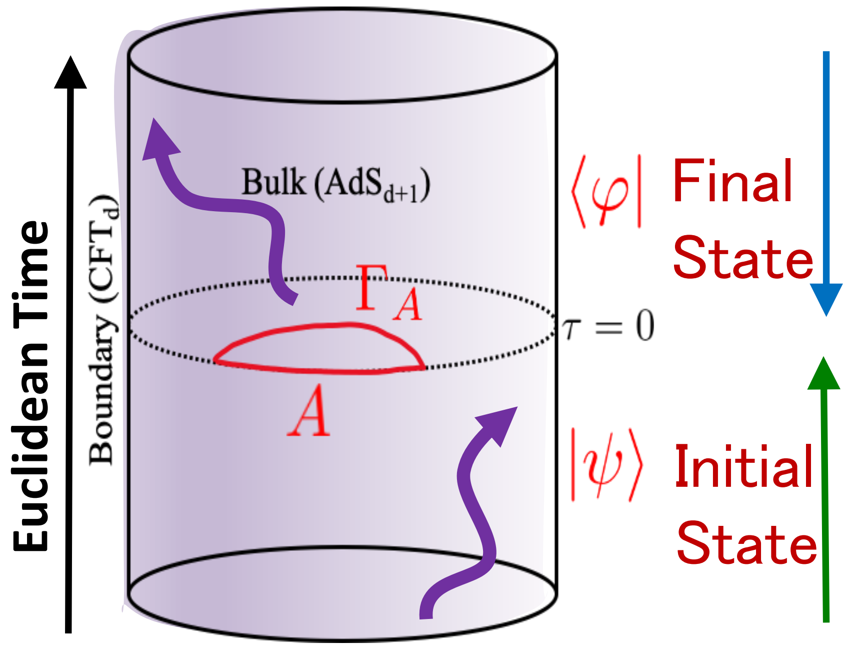
[Nishioka-Taki-TT 2021, Caputa-Purkayastha-Saha-Sułkowski 2024]

(2-4) Holographic Pseudo Entropy (HPE) Formula

[Nakata–Taki–Tamaoka–Wei–TT, 2020]

In **Euclidean time dependent setups**, the minimal surface area coincides with the pseudo entropy.

$$S\left(\tau_A^{\psi|\varphi}\right) = \mathrm{Min}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$



Below we will apply HPE to Lorentzian spacetimes, where **non-Hermitian density matrices** show up.
Key question: "Is the time coordinate encoded in QI quantity ?"

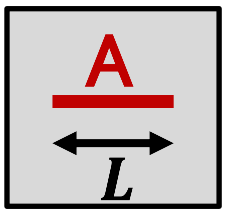
③ Time-like Entanglement Entropy

[Doi–Harper–Mollabashi–Taki–TT 2022]

Consider a time-like version of entanglement entropy

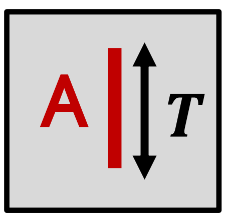
by rotating the subsystem A into a time-like one:

CFT on an infinite line



$$S_A = \frac{C}{3}\log\left[\frac{L}{\varepsilon}\right]$$

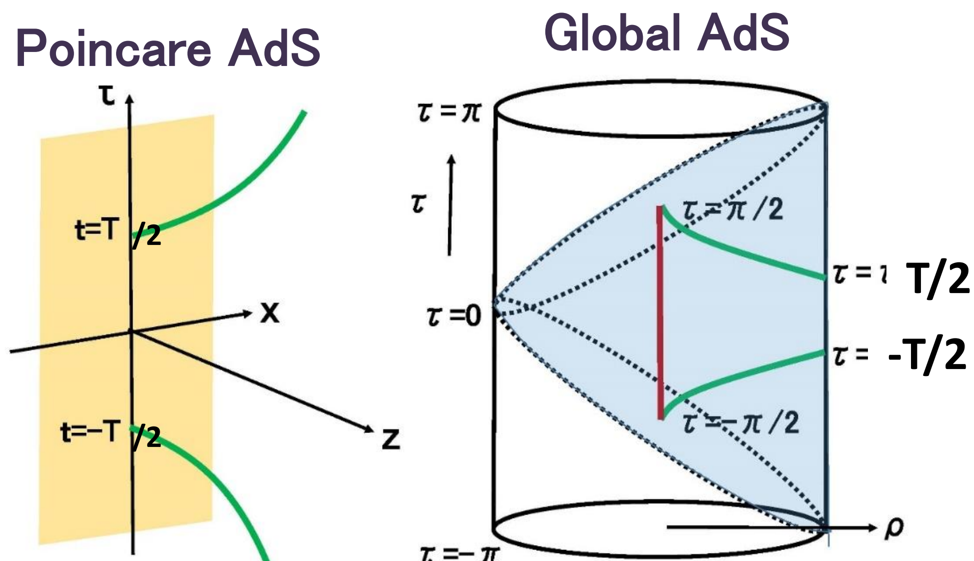
$L \to iT$



$$S_A = \frac{C}{3}\log\left[\frac{T}{\varepsilon}\right] + \frac{\pi}{6} iC$$

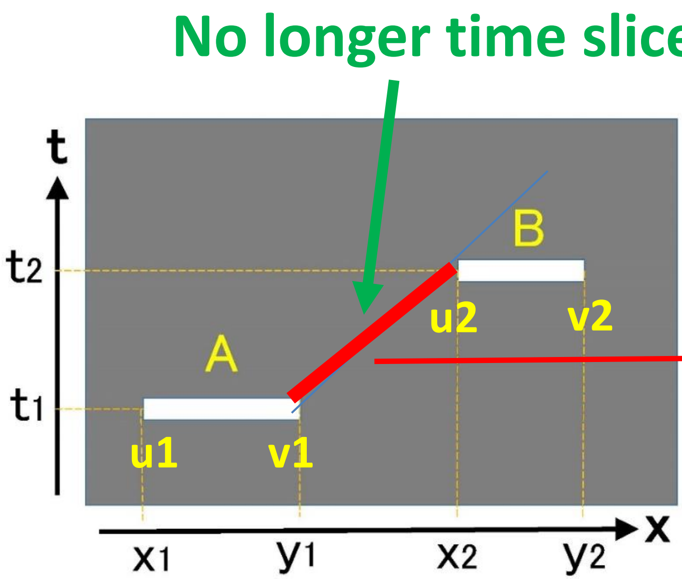
Imaginary part !

Holographic calculation



[More generally we need to consider extremal surfaces in complexified AdS as shown in Heller–Ori–Sereantes 23]

We can find an essentially same phenomenon in a more standard setup of entanglement entropy for double intervals:



e.g. Free Dirac fermion CFT c=1

$$S_{AB} = \frac{c}{6}\log\frac{|v_1-u_1|^2}{\epsilon^2} + \frac{c}{6}\log\frac{|v_2-u_2|^2}{\epsilon^2} + \frac{c}{6}\log\frac{|v_1-u_2|^2}{\epsilon^2} + \frac{c}{6}\log\frac{|v_2-u_1|^2}{\epsilon^2} - \frac{c}{6}\log\frac{|u_1-u_2|^2}{\epsilon^2} - \frac{c}{6}\log\frac{|v_1-v_2|^2}{\epsilon^2}.$$

If this interval is time-like, entropy gets complex valued !

The imaginary part of TEE is explained by the time-like geodesic in AdS.

[Kawamoto−Maeda−Nakamura−TT 25 refer also to Parzygnat−Fullwood 22]

ρ_{AB} is not Hermitian ⟺ A and B are causally connected

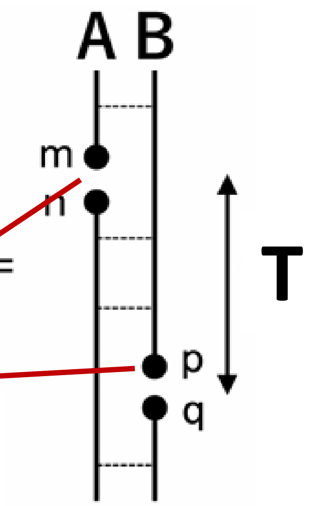
TEE is a special example of pseudo entropy.

A Toy Example: Coupled Harmonic Oscillators

$$H = \frac{1}{\sqrt{1-\lambda^2}}\left[a^\dagger a + b^\dagger b + \lambda(a^\dagger b^\dagger + ab) + 1 - \sqrt{1-\lambda^2}\right].$$

$$\lambda = \tanh 2\theta$$

$$(\rho_{AB})^{m\,p}_{n\,q} =$$

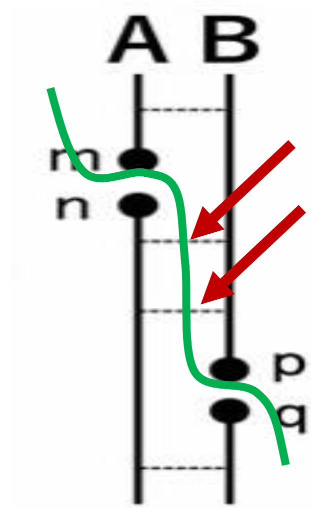


$$[\rho_{AB}]^{a_2,b_2}_{a_1,b_1} = \langle\Psi_0|_{12}\cdot(|b_2\rangle\langle b_1|)_2\cdot\mathcal{P}e^{-i\int_{t_1}^{t_2}dtH_{12}(t)}\cdot(|a_2\rangle\langle a_1|)_1\cdot|\Psi_0\rangle_{12},$$

$$\rho^\dagger_{AB} \neq \rho_{AB}$$

$$S^{(2)}_{AB} = \log\left[\frac{1+e^{-2iT}+(1-e^{-2iT})\cosh 4\theta}{2}\right].$$

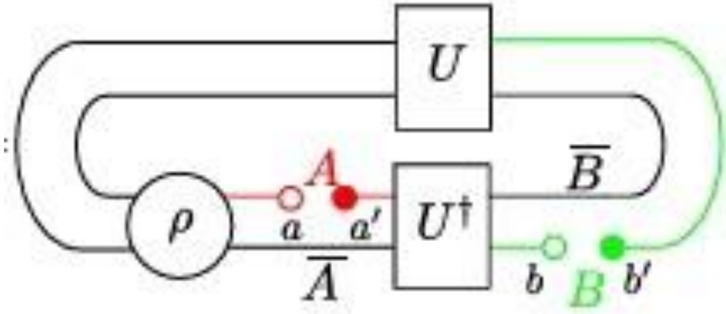
$$S(\rho_{AB}) \neq 0 \qquad \rho_{AB} = \text{mixed}$$



Purification needs extra Hilbert space

Recently, a clear theorem was provided by [Milekhin–Adamska–Preskill 2025]

$$\left(\rho_{\mathrm{AB}}\right)_{ab}^{a'b'} =$$



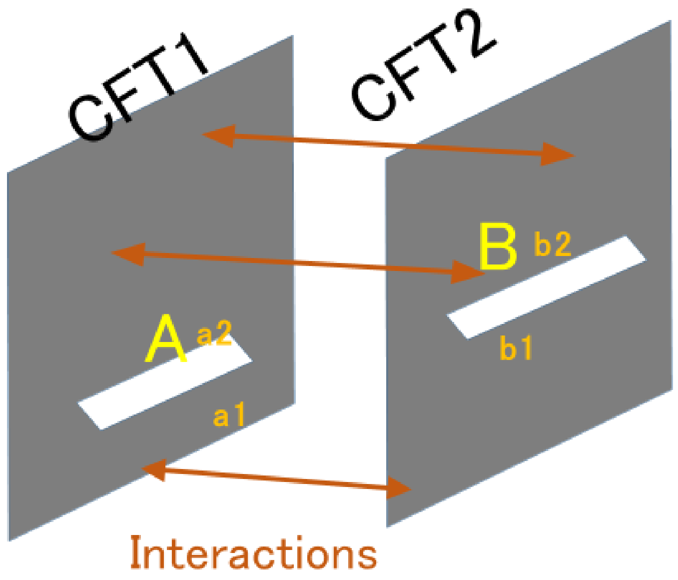
$$\left\langle \left[O_A(0), O_B(t) \right] \right\rangle = \mathrm{Tr}[(\rho_{\mathrm{AB}} - \rho_{AB}^{\dagger}) O_A O_{\mathrm{B}}]$$

Interactions between A and B

$$\frac{1}{\dim H_A} \left\| \rho_{\mathrm{AB}} - \rho_{AB}^{\dagger} \right\|_2 \le \frac{\left| \left\langle \left[O_A(0), O_B(t) \right] \right\rangle \right|}{\left\| O_A \right\|_2 \cdot \left\| O_B \right\|_2} \le \left\| \rho_{\mathrm{AB}} - \rho_{AB}^{\dagger} \right\|_2$$

A similar situation occurs when two CFTs are interacting.

$$\rho_{AB}^{\dagger} \neq \rho_{AB}$$



Indeed, we can easily find

$$\boldsymbol{H_{tot} \neq H_{CFT1} \otimes H_{CFT2}}$$

because A and B are causally connected.

This motivates us to consider traversable AdS wormholes.

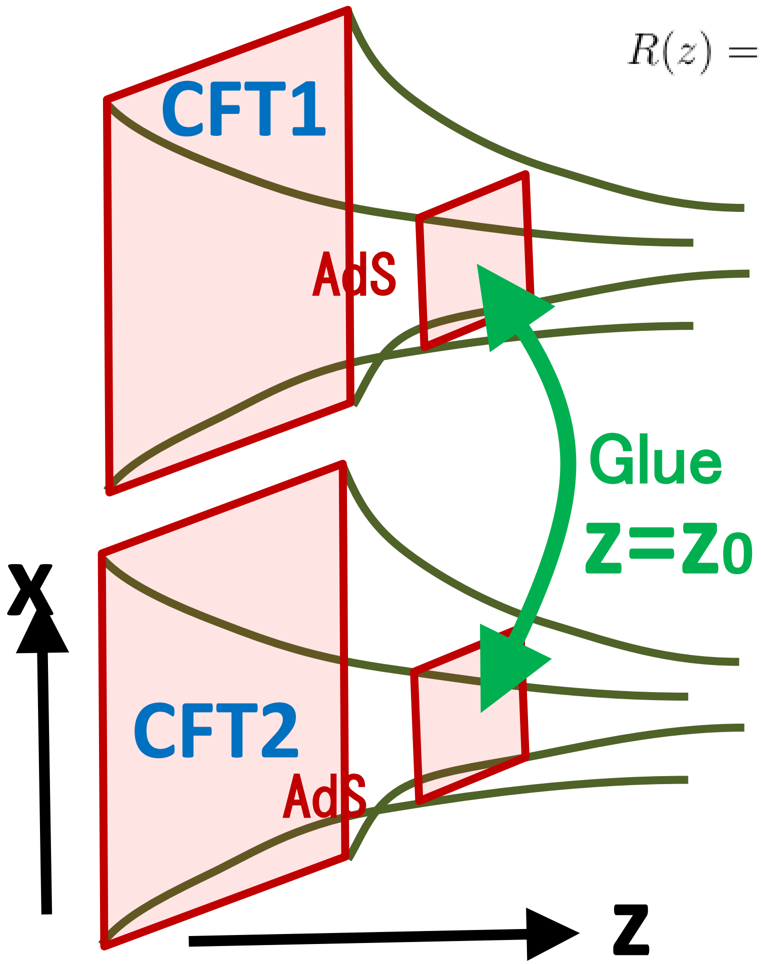
④ Traversable AdS Wormhole

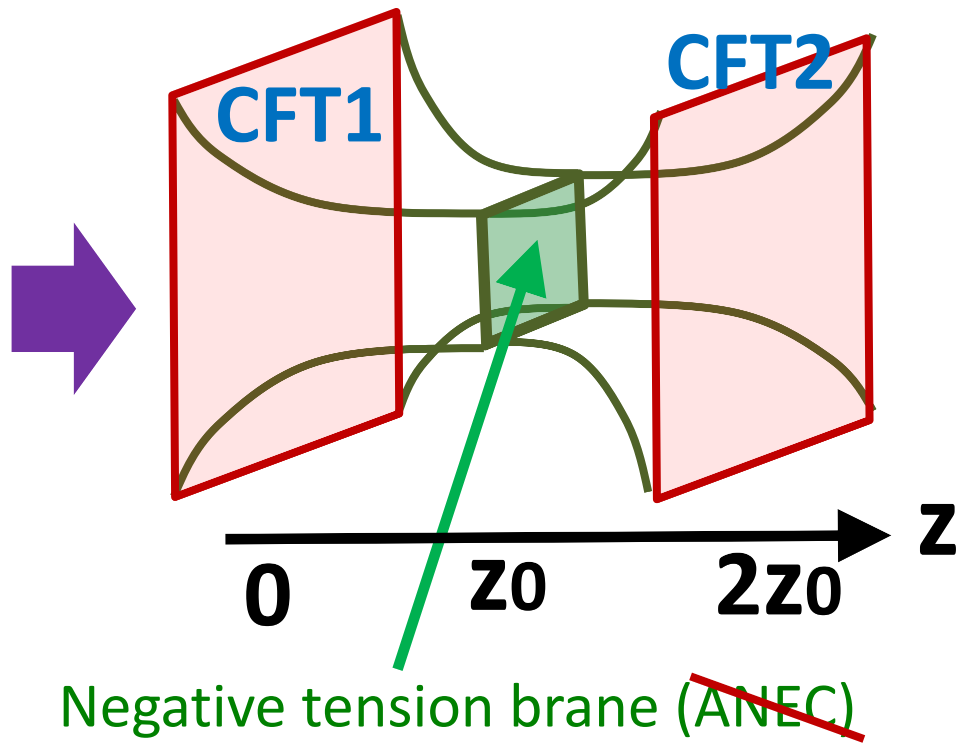
(4-1) General setup

Consider a simple model of traversable AdS wormhole:

$$ds^2 = R(z)\left(dz^2 + \sum_{i=0}^{d-1} dx_i^2\right),$$

$$R(z) = \frac{1}{z^2} \quad (0 < z < z_0), \qquad R(z) = \frac{1}{(2z_0 - z)^2} \quad (z_0 < z < 2z_0).$$



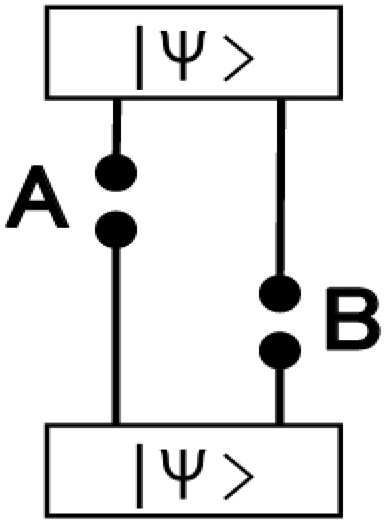


Two constructions of AdS Traversable wormhole

Non-traversable

[Maldacena 01]

Thermofield double



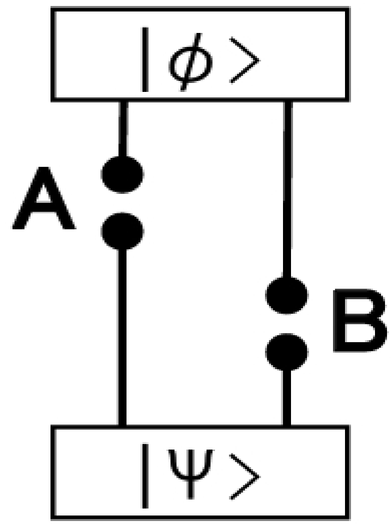
$\rho_{AB}^{\dagger} = \rho_{AB}$

$S(\rho_{AB}) = 0$

Traversable

[Kawamoto-Maeda -Nakamura-TT 2025]

Model A (Janus)



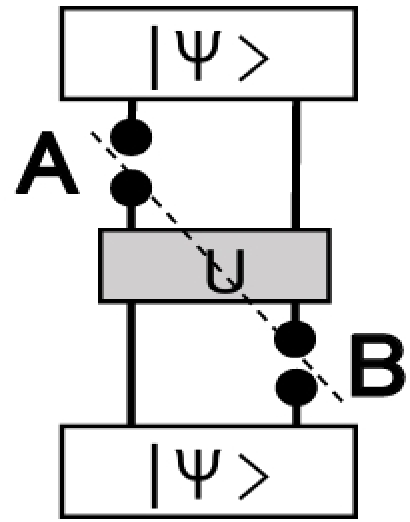
$\rho_{AB}^{\dagger} \neq \rho_{AB}$

$S(\rho_{AB}) = 0$

- ♦ No interactions between A and B
- ♦ H is non-hermitian

[Gao-Jafferis-Wall 2016, Maldacena-Qi 2018, Harvey-Jensen 2023, Lin's talk]

Model B (Double trace)



$\rho_{AB}^{\dagger} \neq \rho_{AB}$

$S(\rho_{AB}) \neq 0$

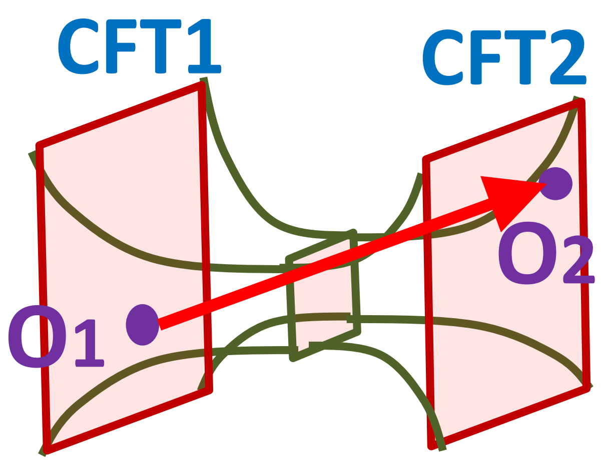
- ♦ ∃Interactions between A and B
- ♦ H is hermitian

Lorentzian 2pt functions of scalar operators

In Lorentzian signature x_0=it, the scalar two point function <O1O2> gets divergent at $-t^2+x^2+4z_0^2=0$ as two points are null separated:

$$\langle \mathcal{O}_1(t,x)\mathcal{O}_2(0,0)\rangle \sim \frac{1}{(-t^2+x^2+4z_0^2)^{d+2\nu-\frac{1}{2}}}.$$

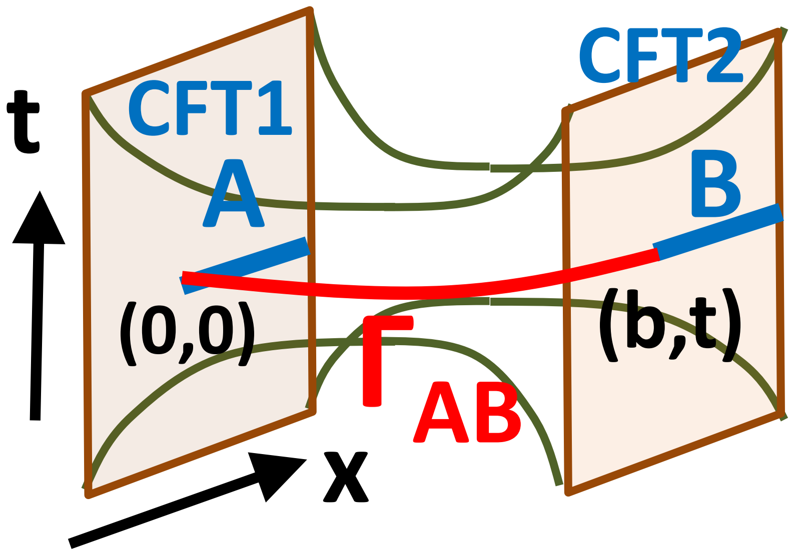
$$\nu = \sqrt{m^2+\frac{d^2}{4}}$$



➡ A characteristic feature of traversable AdS black hole

Pseudo entropy (Time-like entanglement entropy)

How does S_{AB} look like ?



When $t^2 < b^2 + 4z_0^2$:

$$S_{AB} = \frac{c}{3}\log\frac{\frac{b^2-t^2}{4}+z_0^2}{\epsilon z_0}.$$

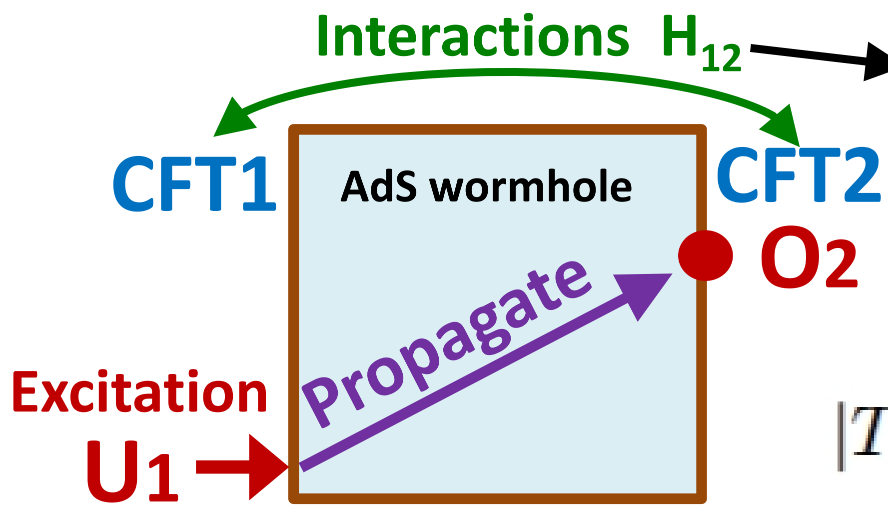
When $t^2 > b^2 + 4z_0^2$

$$S_{AB} = \frac{c}{3}\log\frac{\frac{t^2-b^2}{4}-z_0^2}{\epsilon z_0}+\frac{c}{3}\pi i.$$

Γ_{AB} can be time-like in a traversable wormhole.

S_{AB} becomes complex valued because $\rho_{AB}^{\dagger} \neq \rho_{AB}$.
Thus, S_{AB} should be regarded as pseudo entropy.

(4-2) Double trace deformation of External BH (Model B)



$$\int dxdy\lambda(x,y)O_1(x)O_2(y)$$

$$H_{tot}=H_1+H_2+H_{12}$$

$$|TFD\rangle = \mathcal{N}\sum_n e^{-\frac{\beta}{4}H_{tot}}|n\rangle_1|n\rangle_2$$

$$U_1 = e^{i\lambda O_1}$$

$$[\rho_{12}]_{ab}^{a'b'} = \langle TFD|e^{it_2H_{tot}}|b'\rangle\langle b|e^{-i(t_2-t_1)H_{tot}}|a'\rangle\langle a|e^{-it_1H_{tot}}|TFD\rangle,$$

$$\rho_2 = \mathrm{Tr}_1\left[e^{-it_2H_{tot}}\left(e^{it_1H_{tot}}U_1e^{-it_1H_{tot}}\right)|TFD\rangle\langle TFD|\left(e^{it_1H_{tot}}U_1^{\dagger}e^{-it_1H_{tot}}\right)e^{it_2H_{tot}}\right]$$

$$\langle O_2\rangle = \mathrm{Tr}[O_2\rho_2]$$
$$\simeq \langle TFD|O_2|TFD\rangle + i\lambda\langle TFD|[O_1(t),O_2]|TFD\rangle + O(\lambda^2).$$

Non-vanishing due to the interactions ↔ $\rho_{AB}^{\dagger} \neq \rho_{AB}$

(4-3) Wormhole via Janus deformation (Model A)

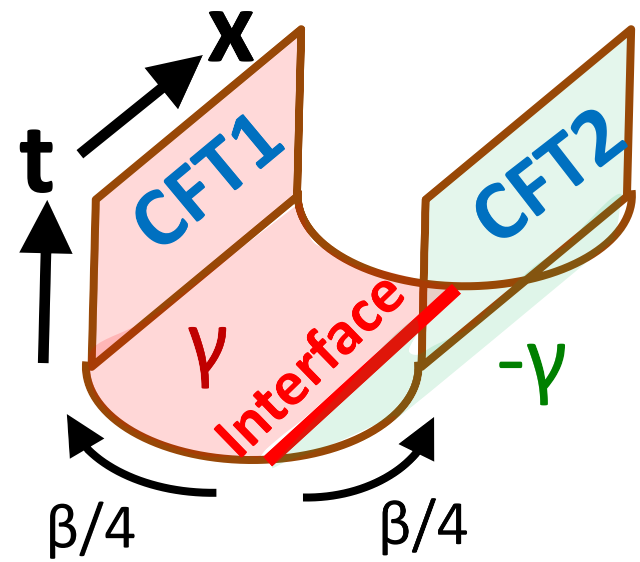
[Harper-Kawamoto-Maeda-Nakamura-TT, in preparation]

Janus deformation = asymmetric exactly marginal perturbations in a pair of CFTs

[Bak-Gutperle-Hirano 03]

$$S_{\mathrm{CFT1}} = S^{(0)}_{\mathrm{CFT}} + \gamma \int dx^d O_1(x)$$

$$S_{\mathrm{CFT2}} = S^{(0)}_{\mathrm{CFT}} - \gamma \int dx^d O_2(x)$$



◆We consider the TFD state of the doubled CFT for d=2.

◆In the standard Janus deformation, γ is real valued.
We will extend γ to imaginary values.

Explicit construction from Janus deformation

We start with 3D Janus BH solutions in [Bak-Gutperle-Hirano 2007].

The model is given by the 3d gravity action

$$I = \frac{1}{16\pi G_N}\int d^3x\left[R - g^{ab}\partial_a\phi\partial_b\phi + 2\right].$$

The solution ansatz looks like

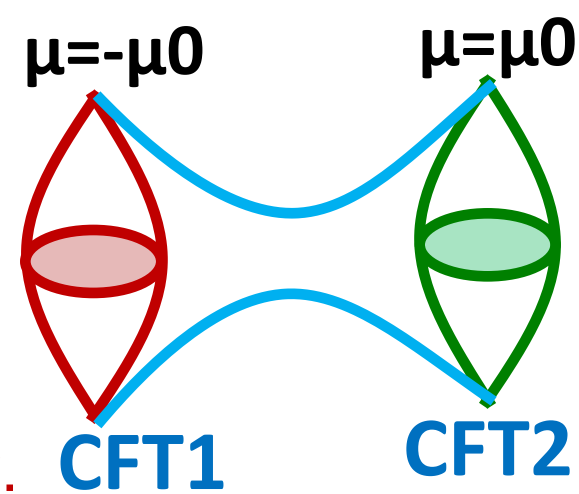
γ is Janus deformation Parameter.

$$ds^2 = f(\mu)(d\mu^2 + ds^2_{AdS2}), \qquad \phi = \phi(\mu).$$

$$ds^2_{AdS2} = -d\tau^2 + r_0^2\cos^2\tau d\theta^2$$

$$\frac{d\phi(\mu)}{d\mu} = \frac{\gamma}{\sqrt{f(\mu)}},$$

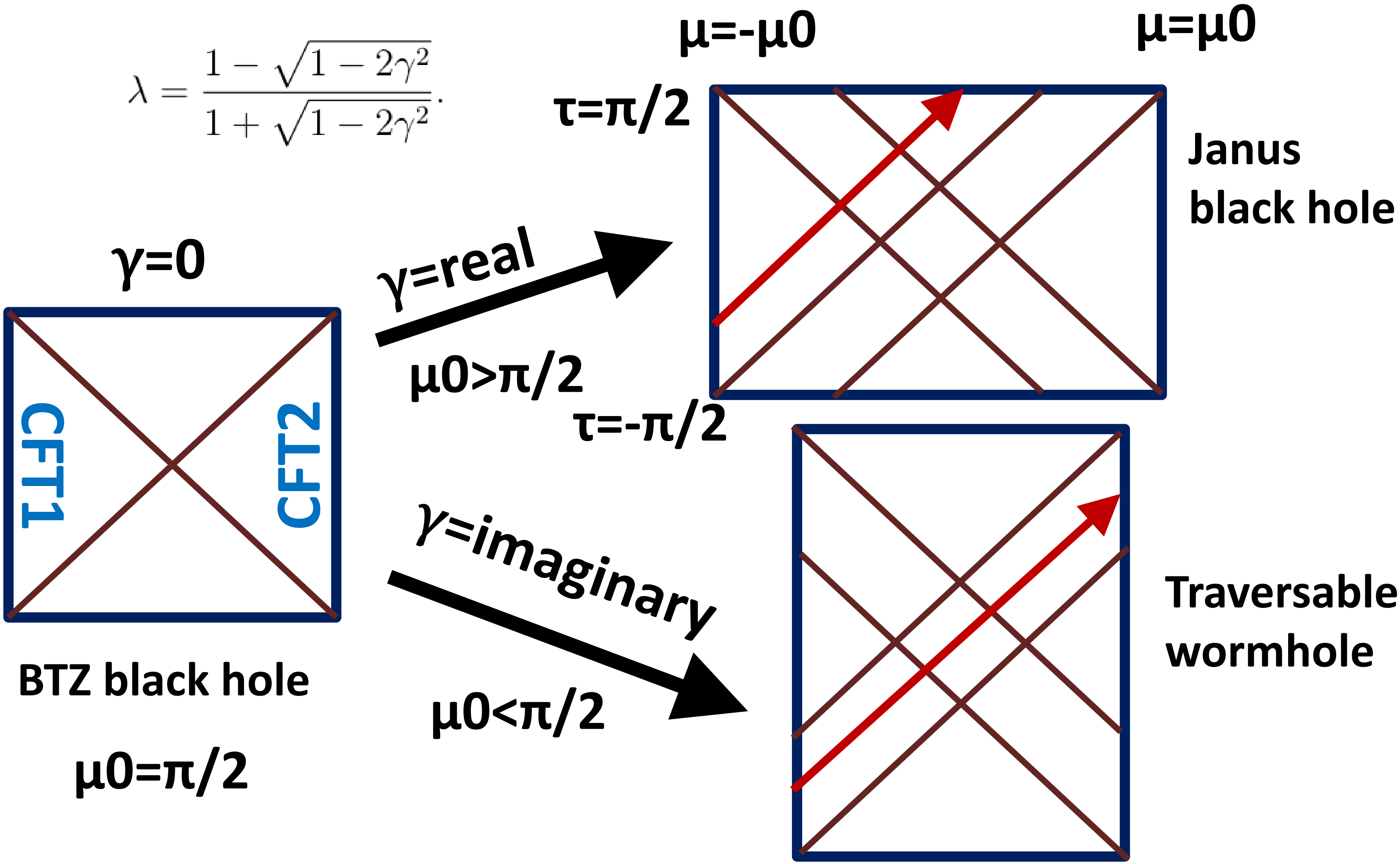
$$\frac{df(\mu)}{d\mu} = \sqrt{f(4f^2 - 4f + 2\gamma^2)}.$$



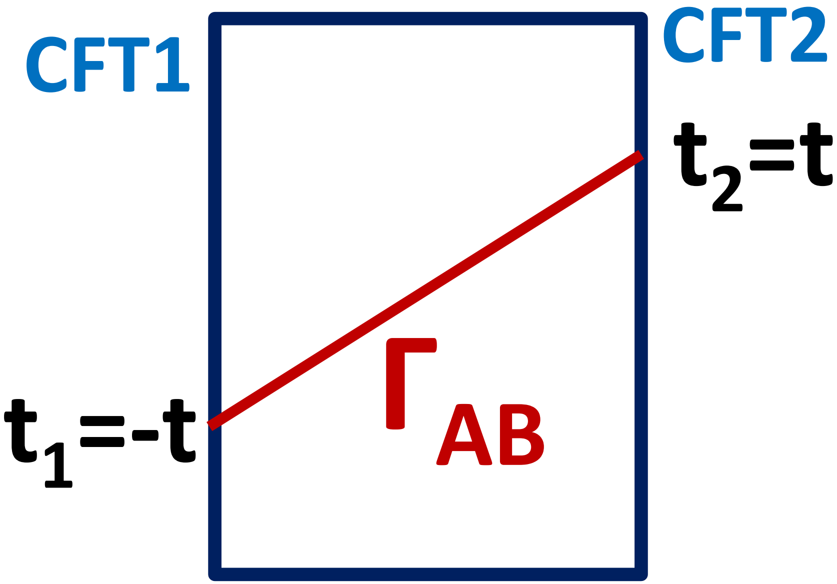
We now extend this solution to **imaginary γ**.

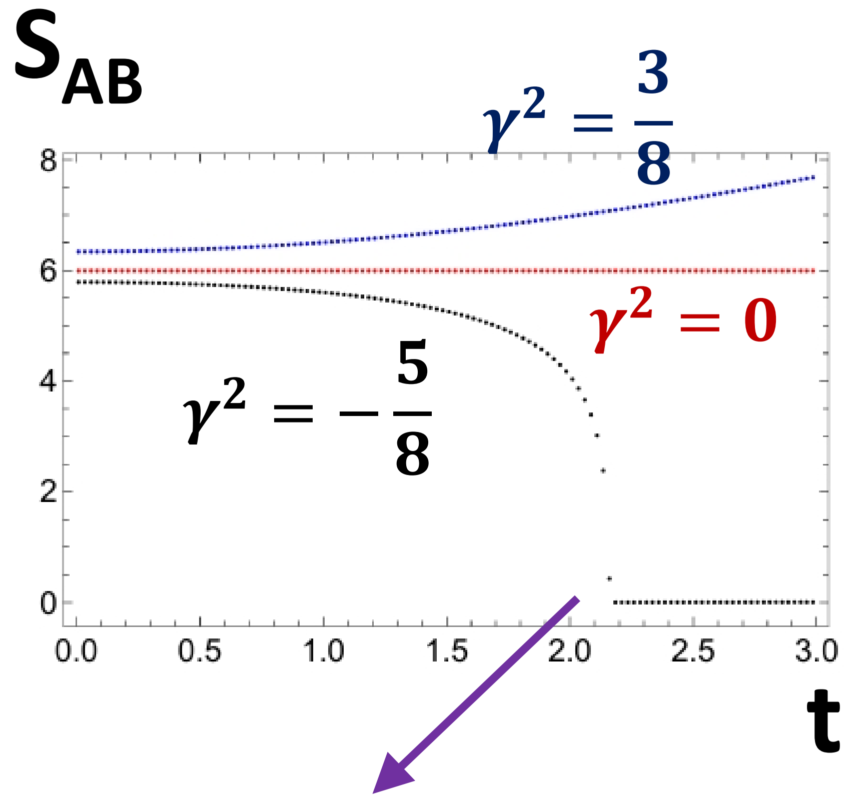
$$\mu_0 = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}},$$

$$\lambda = \frac{1-\sqrt{1-2\gamma^2}}{1+\sqrt{1-2\gamma^2}}.$$



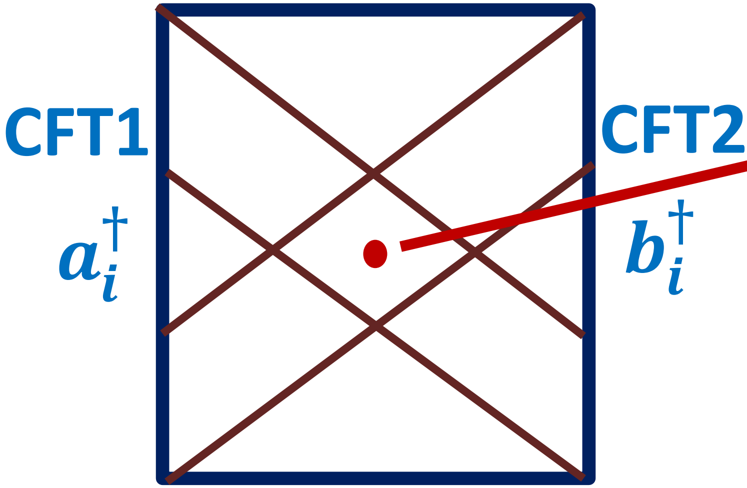
Holographic pseudo entropy for half lines





Γ_{AB} becomes light-like !
The characteristic feature of traversable wormhole.

(Pseudo) Entanglement entropy between CFT1 and CFT2



$$S_1 = \frac{\pi}{6} c r_0 \sqrt{1+\sqrt{1-2\gamma^2}}$$

Area of throat

Monotonically decreasing function of γ^2

In the dual CFT, this is dual to the PE/EE in the deformed TFD state:

$$|\mathrm{TFD}(\beta,\gamma)\rangle = \tilde{\mathcal{N}} \exp\left[\sum_{i=1}^{\infty} e^{-\frac{\beta}{2}E_i}\left(\sin 2\theta\ a_i^\dagger b_i^\dagger + \cos 2\theta\left((a_i^\dagger)^2 - (b_i^\dagger)^2\right)\right)\right]|0\rangle$$

$$\langle\mathrm{TFD}(\beta,\gamma)| = \tilde{\mathcal{N}}\langle 0|\exp\left[\sum_{i=1}^{\infty} e^{-\frac{\beta}{2}E_i}\left(\sin 2\theta\ a_i b_i + \cos 2\theta\left((a_i)^2 - (b_i)^2\right)\right)\right] \quad \theta \equiv \frac{\pi}{4} + \gamma$$

S_1 becomes its maximum at $\theta=\pi/4$ (i.e. no deformation) and decreases as γ^2 gets larger. For imaginary γ, it increases. This is consistent with the gravity dual.

Why traversable ?

The Hamiltonians H_1 and H_2 of CFT1 and CFT2 for γ=imaginary becomes non-Hermitian:

$$H_1 = H_0 + \gamma V, \quad H_2 = H_0 + \gamma^* V, \quad \text{such that } H_1^\dagger = H_2$$

They have different eigen-vectors with complex eigen-values:

$$H_1 |n_+\rangle = E_n |n_+\rangle, \qquad H_2 |n_-\rangle = E_n^* |n_-\rangle,$$

$$\langle n_+| H_2 = \langle n_+| E_n^*, \qquad \langle n_-| H_1 = \langle n_-| E_n \ ,$$

where we introduce their (Hermitian) conjugations by $|n_\pm\rangle^\dagger = \langle n_\pm|$.

They satisfy $\langle n_\mp | m_\pm \rangle = \delta_{n,m}, \qquad \sum_n |n_\pm\rangle\langle n_\mp| = 1.$

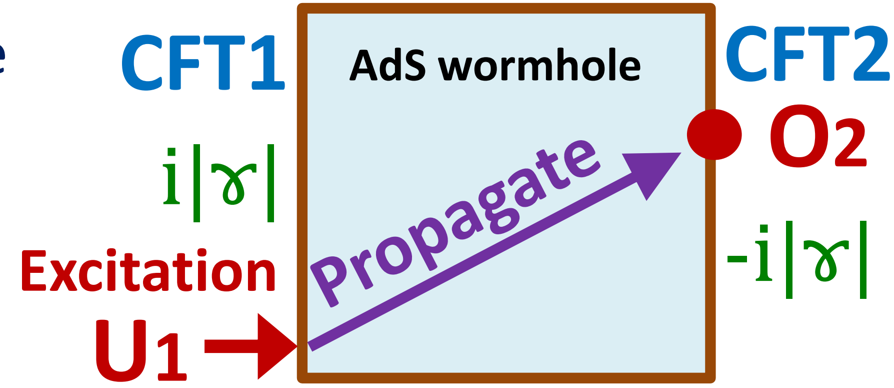
This motivates us to define the modified conjugation ‡ by

$$\langle n_+ | O | m_- \rangle^* = \langle m_+ | O^\ddagger | n_- \rangle .$$

The initial and final TFD state look like

$$\left|\mathrm{TFD}\right\rangle = \sum_n e^{-\frac{\beta}{4}(H_1+H_2)} \left|n_+\right\rangle_1 \left|n_+\right\rangle_2 \quad ,$$

$$\left\langle\overline{\mathrm{TFD}}\right| = \sum_n \left\langle n_-\right|_1 \left\langle n_-\right|_2 e^{-\frac{\beta}{4}(H_1+H_2)} \quad .$$



The density matrix $\rho = \left|\mathrm{TFD}\right\rangle\left\langle\overline{\mathrm{TFD}}\right|$ is not Hermitian $\rho^\dagger \neq \rho$.

However, it satisfies $\rho^\ddagger = \rho$, implying ‡ is good for the conjugation.

An observer in CFT2 probes the state:

$$\rho_2 = e^{-it_2 H_-^{(2)}} \mathrm{Tr}_1 \left[(e^{-it_1 H_+^{(1)}} e^{i\alpha O^{(1)}} e^{it_1 H_+^{(1)}}) |TFD\rangle\langle\overline{TFD}| (e^{-it_1 H_+^{(1)}} e^{-i\alpha O^{(1)\ddagger}} e^{it_1 H_+^{(1)}}) \right] e^{it_2 H_-^{(2)}} .$$

When the CFT1 is excited by $U_1 = e^{i\alpha O^{(1)}}$, the CFT2 observer sees

$$\begin{aligned} \langle O^{(2)}(t_2)\rangle &= \mathrm{Tr}[\rho_2 O^{(2)}] \\ &\simeq \langle\overline{TFD}|O^{(2)}|TFD\rangle + i\alpha\langle\overline{TFD}| \underline{\left(O^{(2)}(t_2)O^{(1)}(t_1) - O^{(1)\ddagger}(t_1)O^{(2)}(t_2)\right)} |TFD\rangle, \end{aligned}$$

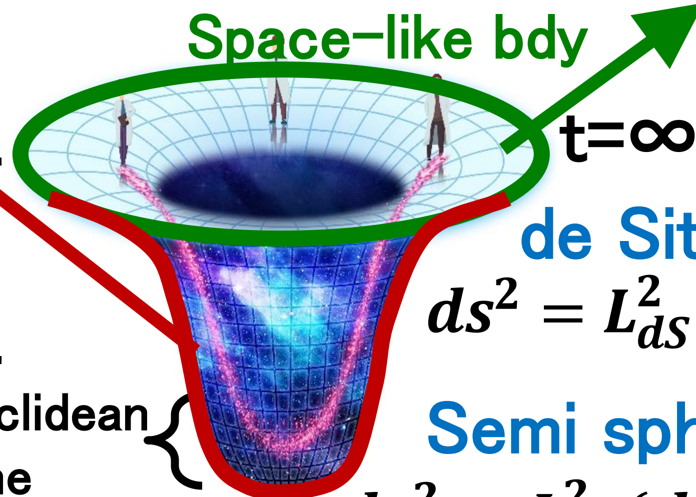
We have $[O_1,O_2]=0$, but this is non-vanishing !

⑤ dS/CFT correspondence

A Sketch of dS/CFT [Strominger 2001, Witten 2001, Maldacena 2002,⋯.]

d+1 dim. Lorentzian de−Sitter spacetime **Dual** ⟺ Euclidean d. dim CFT on S^d

Space−like bdy



$t=\infty$

Time

Lorentzian time

Euclidean time

Time emerges from Euclidean CFT !

de Sitter

$$ds^2 = L_{dS}^2(-dt^2 + \mathrm{Cosh}^2 t\, d\Omega^2)$$

Semi sphere

$$ds^2 = L_{dS}^2(d\theta^2 + \mathrm{Sin}^2\theta\, d\Omega^2)$$

Ψ [dS gravity] = Z [CFT]

Central charge (even d → imaginary)

$$c \sim \frac{L_{AdS}^{d-1}}{G_N} = i^{d-1} \cdot \frac{L_{dS}^{d-1}}{G_N}$$

Why dS/CFT is much more difficult than AdS/CFT ?

[1] Dual Euclidean CFTs should be exotic and non-unitary !

A "standard" Euclidean CFTs is dual to gravity on hyperbolic space.

e.g. dS3/CFT2 ➔ ***Imaginary valued*** central charge $c \approx i\frac{3L_{dS}}{2G_N}$!

Unsual conjugation: $(L_n)^\dagger = (-1)^{n+1}\widetilde{L_n}$ [Doi-Ogawa-Shimyo-Suzuki-TT 2024]

[2] Time should emerge from Euclidean CFT !

From a usual Euclidean CFT, a space-like direction will emerge as RG scale.

How does a ***time-like direction emerge*** from a Euclidean CFT ?

[3] "Entanglement entropy" looks complex valued !

Extremal surfaces in dS which end on its boundary are ***time-like*** !

Non-unitary CFT dual of 3 dim. dS

[Hikida–Nishioka–Taki–TT, 2021–22, Chen–Hikida–Taki–Uetoko 2022–24,..]

Large c limit of SU(2)$_k$ × SU(2)$_{-k}$ WZW model (a 2dim. CFT)
= Einstein Gravity on 3 dim. de Sitter (radius L_{ds})

Level $k \approx -2 + \frac{4iG_N}{L_{dS}}$ ⟶ **Central charge** $c = \frac{3k}{k+2} \approx i\frac{3L_{dS}}{2G_N}$

$$Z[S^3, R_j] = |S_j^0|^2 \approx e^{\frac{\pi L_{ds}}{2G_N}\sqrt{1-8G_N E}}$$

CFT partition function — De Sitter Entropy

This non-unitary CFT is equivalent to the Liouville CFT

at $b^{-2} \approx \pm \frac{i}{4G_N}$. $\quad I_{CFT}[\phi] = \int d^2x \left[\frac{1}{4\pi}(\partial_a\varphi\partial_a\varphi) + \mu e^{2b\varphi}\right]$.

[Hikida–Nishioka–Taki–TT, 2022] — complex !

The same Liouville CFT appears in [Verlinde–Zhang 2024] via DSSYK.
→Why two different holographic constructions lead to the same CFT ?

Holographic Entanglement Entropy in dS3/CFT2 ?

[Hikida–Nishioka–Taki–TT 2022, Doi–Harper–Mollabashi–Taki–TT 2022]

In dS3/CFT2, the geodesic Γ_A becomes time–like and we find:

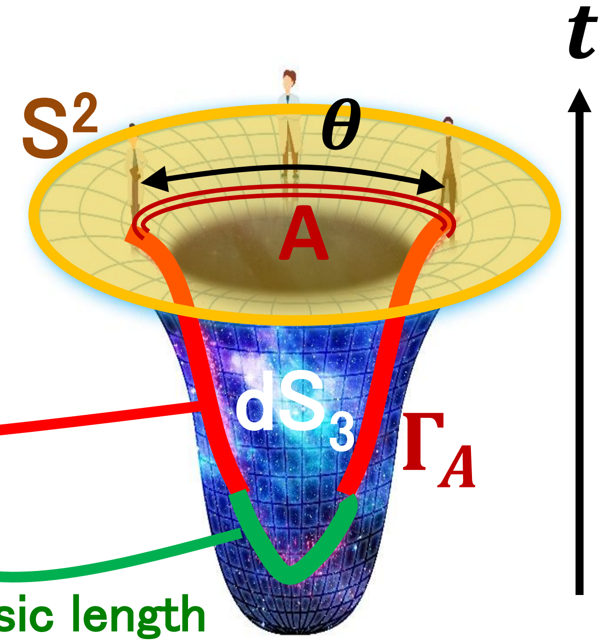
$$S_A = \frac{L(\Gamma_A)}{4G_N} = i\frac{c_{ds}}{3}\log\left(\frac{2}{\epsilon}\mathrm{Sin}\frac{\theta}{2}\right) + \frac{c_{dS}}{6}\pi.$$

$S_{dS}/2$

Time–like geodesics length
➔imaginary part !

Agree !

Space–like geodesic length
➔Real part



CFT calculation

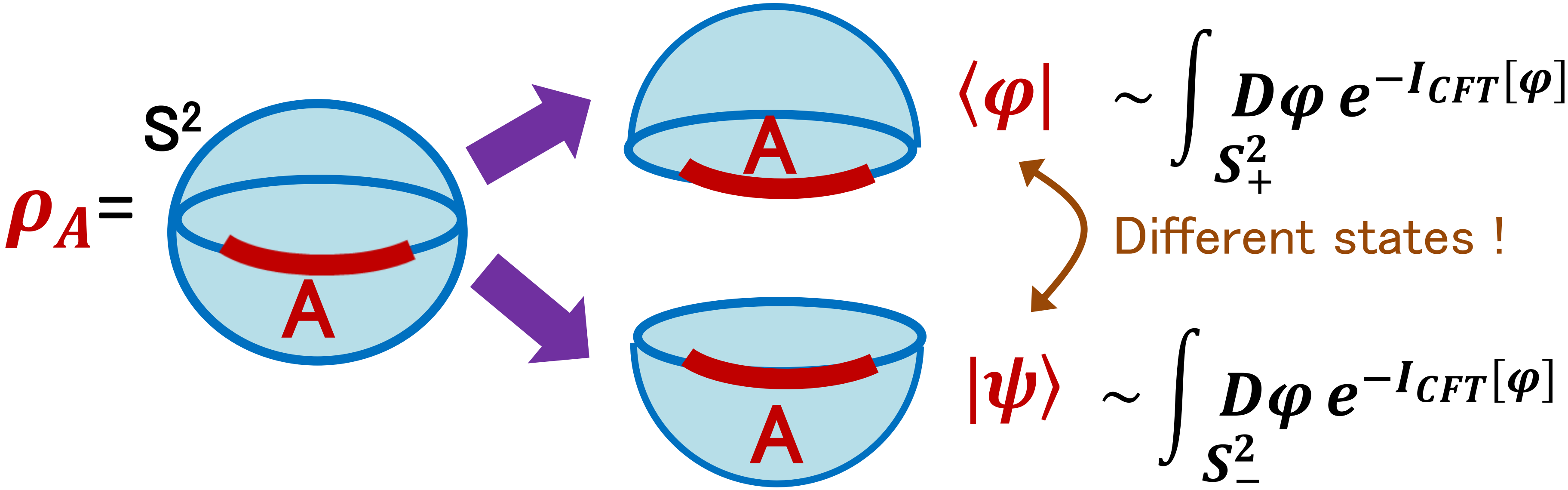
$$S_A = \frac{c_{CFT}}{6}\log\left[\frac{\mathrm{Sin}^2\frac{\theta}{2}}{\tilde{\varepsilon}^2}\right],$$ by setting

$$c_{CFT} = ic_{dS} \text{ and } \tilde{\varepsilon} = i\varepsilon = ie^{-t_\infty}.$$

Complex valued entropy !
(should not be EE !)

We argue it is more properly considered as pseudo entropy (PE).

This is because the reduced density matrix ρ_A **is not Hermitian !**



2D CFT on the space with the metric: $h_{ab} = e^{2\phi}\delta_{ab}$,

$$I_{CFT}[\phi] = i\,\frac{C_{ds}}{24\pi}\int d^2x[(\partial_a\phi)^2 + e^{2\phi}].$$

$\rightarrow \rho_A \neq \rho_A^\dagger$

Note: the emergent time coordinate = imaginary part of PE.

⑥ Pseudo Entropy and Entanglement Distillation

(6-1) Distillation from Post-selection

Let us focus on the following example with real valued PE:

$$|\psi\rangle = \cos\theta_1|00\rangle + \sin\theta_1|11\rangle,$$

$$|\varphi\rangle = \cos\theta_2|00\rangle + \sin\theta_2|11\rangle.$$

$$\Rightarrow \quad \tau_A^{\psi|\varphi} = \frac{\cos\theta_1\cos\theta_2|0\rangle\langle 0| + \sin\theta_1\sin\theta_2|1\rangle\langle 1|}{\cos(\theta_1-\theta_2)}$$

$$S\left(\tau_A^{\psi|\varphi}\right) = -\frac{\cos\theta_1\cos\theta_2}{\cos(\theta_1-\theta_2)}\cdot\log\frac{\cos\theta_1\cos\theta_2}{\cos(\theta_1-\theta_2)} - \frac{\sin\theta_1\sin\theta_2}{\sin(\theta_1-\theta_2)}\cdot\log\frac{\sin\theta_1\sin\theta_2}{\sin(\theta_1-\theta_2)}$$

$$(|\psi\rangle)^{\otimes M} = (\cos\theta_1|00\rangle + \sin\theta_1|11\rangle)^{\otimes M}$$
$$= \sum_{k=0}^{M}(c_1)^{M-k}\,(s_1)^k \sum_{a=1}^{{}_M C_k}|P(k),a\rangle_A|P(k),a\rangle_B$$

$$c_1 \equiv \cos\theta_1,\ s_1 \equiv \sin\theta_1$$

$k = 0$: $\quad |P(0),1\rangle = |00\cdots 0\rangle$

$k = 1$: $\quad |P(1),1\rangle = |10\cdots 0\rangle, |P(1),2\rangle = |01\cdots 0\rangle, \ \cdots$

➡ Projection to maximally entangled states with **Log[MCk]** entropy:

MCk=M!/(M-k)!k!

$$\Pi_k = \sum_{a=1}^{{}_M C_k}|P(k),a\rangle_A\langle P(k),a|$$

probability: $p_k = \langle\varphi|\Pi_k|\psi\rangle/\langle\varphi|\psi\rangle = \dfrac{(c_1c_2)^{M-k}\,(s_1s_2)^k}{(c_1c_2+s_1s_2)^M}\cdot {}_M C_k$

➡ **# of Distillable Bell pairs**: $\mathbf{N} = \sum_{k=0}^{M} p_k \cdot \mathrm{Log}[{}_M C_k]$

$$\approx M\cdot S\left(\tau_A^{\psi|\varphi}\right)\ !$$

(6-2) SVD entropy [Parzygnat–Taki–Wei–TT 2023]

Motivation: Improve PE so that **(i)** it become real and non−negative and **(ii)** it has a better LOCC interpretation.

SVD entropy

$$S_{SVD}\left(\tau_A^{\psi|\varphi}\right) = -\mathrm{Tr}\left[|\tau_A^{\psi|\varphi}|\cdot \log|\tau_A^{\psi|\varphi}|\right].$$

here, $|\tau_A^{\psi|\varphi}| \equiv \sqrt{\tau_A^{\dagger\psi|\varphi}\tau_A^{\psi|\varphi}}$

- This is always non−negative and is bounded by log dim H_A.
- From quantum information theoretic viewpoint, this is the number of Bell pairs distilled from the intermediate state:

$\tau_A^{\psi|\varphi} = U \cdot \Lambda \cdot V$,

$$\frac{\langle\varphi|V^\dagger \sum_k |EPR_k\rangle\langle EPR_k| U^\dagger|\psi\rangle}{\langle\varphi|V^\dagger U^\dagger|\psi\rangle} = \sum_k p_k = 1$$

$$S_{SVD} \approx \sum_k p_k \cdot \#\text{of Bell Pairs in } |EPR_k\rangle$$

⑦ Conclusions

Key ideas

Is gravity the fastest "quantum computer" ?

➔New insights into quantum matter, quantum computation and quantum cryptography

Holography

Universe ＝ Collection of Qubits (=Strings?)

Emergence

Does gravitational spacetime emerge from qubits？

➔New approach to quantum gravity

In this talk we emphasized the use of pseudo entropy (PE).

- PE has a clear gravity dual via holography.
- PE is a useful geometric probe of non-Hermitian dynamics.

  e.g. time-like entanglement, wormholes, and de Sitter spaces…

Imaginary part of Pseudo entropy➔Emergence of Time

(but what is quantum informational meaning of PE ?)

Thank you !

♦ Calculation of two point functions in AdS wormhole

Consider a scalar field Φ in the bulk:

$$I_{\text{scalar}} = \int dz d^d x \left[\frac{1}{z^{d-1}} \left((\partial_z \Phi)^2 + (\partial_x \Phi)^2 \right) + \frac{m^2}{z^{d+1}} \Phi^2 \right].$$

$$\Phi'' - \frac{d-1}{z}\Phi' - \left(k^2 + \frac{m^2}{z^2} \right) \Phi = 0.$$

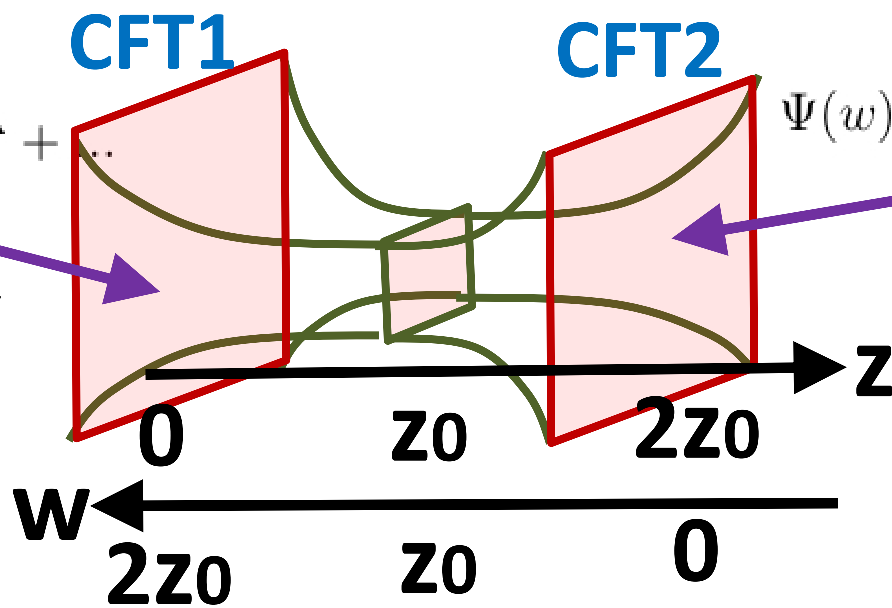
Source J_1 ⟨O1⟩

$$\Phi(z) = \alpha_1 z^{d-\Delta} + \beta_1 z^{\Delta} + \ldots$$

Source J_2 ⟨O2⟩

$$\Psi(w) = \alpha_2 w^{d-\Delta} + \beta_2 w^{\Delta} + \ldots$$

$$\Delta = \frac{d}{2} + \nu, \quad \nu = \sqrt{m^2 + \frac{d^2}{4}}$$



$$ds^2 = R(z) \left(dz^2 + \sum_{i=0}^{d-1} dx_i^2 \right), \quad R(z) = \frac{1}{z^2} \ (0 < z < z_0), \qquad R(z) = \frac{1}{(2z_0 - z)^2} \ (z_0 < z < 2z_0).$$

Two point functions read

$$P(\nu, k, z=z_0, d) := \langle \mathcal{O}_1(k)\mathcal{O}_1(-k)\rangle = -\frac{\beta_1}{\alpha_1}$$

$$= \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu} \frac{kz_0 I_{\nu-1}(kz_0)I_{-\nu}(kz_0) + \left(kz_0 I_{1-\nu}(kz_0) + (d-2\nu)I_{-\nu}(kz_0)\right) I_\nu(kz_0)}{(d-2\nu)I_\nu(kz_0)^2 + 2kz_0 I_{\nu-1}(kz_0)I_\nu(kz_0)}.$$

$$Q(\nu, k, z=z_0, d) := \langle \mathcal{O}_1(k)\mathcal{O}_2(-k)\rangle = \frac{\beta_2}{\alpha_1}$$

$$= \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu} \frac{2\sin\nu\pi}{\pi} \frac{1}{(d-2\nu)I_\nu(kz_0)^2 + 2kz_0 I_{\nu-1}(kz_0)I_\nu(kz_0)}.$$

In the UV limit ($kz_0 \gg 1$), we obtain

⟨O1O1⟩ $$P(\nu, k, z=z_0, d) \simeq \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu}$$

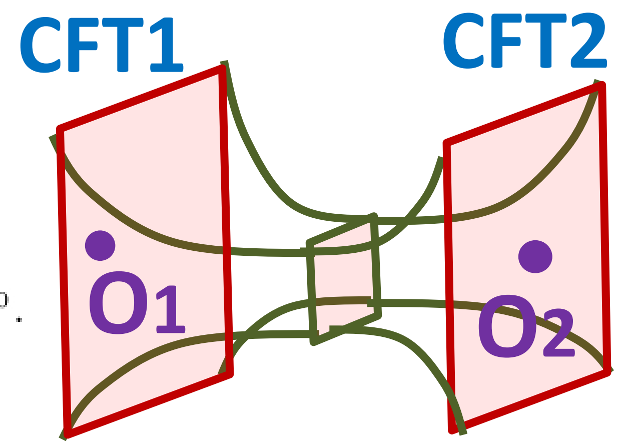
⟨O1O2⟩ $$Q(\nu, k, z=z_0, d) \simeq \frac{2\sin\nu\pi\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu} e^{-2kz_0}.$$

exp decay

In the IR limit ($kz_0 \ll 1$), we obtain

⟨O1O1⟩ $$P(\nu, k, z=z_0, d) \simeq \frac{d}{d+2\nu}\frac{1}{z_0^{2\nu}} + O(kz_0)$$

⟨O1O2⟩ $$Q(\nu, k, z=z_0, d) \simeq \frac{2\nu}{d+2\nu}\frac{1}{z_0^{2\nu}} + O(kz_0).$$

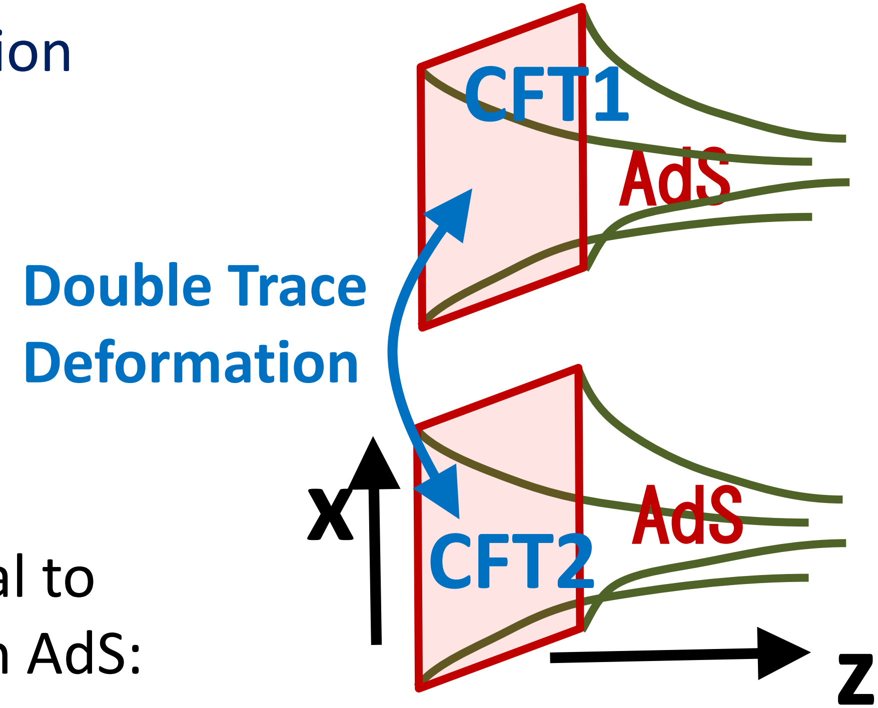


♦ Details of double trace deformation (Model B)

Consider a double trace deformation between CFT1 and CFT2

$$\int dxdy\lambda(x,y)O_1(x)O_2(y)$$

$$\lambda(x,y) = \int d^d k e^{ik(x-y)}\lambda(k)$$



The double trace deformation is dual to the change of boundary condition in AdS:

$$J^{(1)} = \alpha^{(1)} - \lambda\beta^{(2)}, \quad J^{(2)} = \alpha^{(2)} - \lambda\beta^{(1)}$$ [Witten 2001]

Here the scalar field in each AdS is expanded as follows:

$$\Phi^{(i)} \simeq \alpha^{(i)} z_i^{d-\Delta} + \beta^{(i)} z_i^{\Delta} \quad (z_1, z_2 \to 0)$$

$$\frac{\beta^{(i)}}{\alpha^{(i)}} = -G(k), \quad G_p(k) \equiv \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu}$$

In this way we can compute the two point functions:

$$\langle \mathcal{O}_1(k)\mathcal{O}_1(-k)\rangle = \langle \mathcal{O}_2(k)\mathcal{O}_2(-k)\rangle = \frac{G}{1-\lambda^2 G^2},$$
$$\langle \mathcal{O}_1(k)\mathcal{O}_2(-k)\rangle = \frac{\lambda G^2}{1-\lambda^2 G^2}.$$

Two point functions in the simple model of traversable WH is reproduced by setting

$$G(k) = \frac{P(k)^2 - Q(k)^2}{P(k)} = \begin{cases} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu} & (kz_0 \gg 1) \\ \frac{d^2-4\nu^2}{(d+2\nu)d}\cdot\frac{1}{z_0^{2\nu}} & (kz_0 \ll 1). \end{cases}$$

$$\lambda(k) = \frac{Q(k)}{P(k)^2 - Q(k)^2} = \begin{cases} \frac{2\sin\pi\nu\,\Gamma(1+\nu)}{\Gamma(1-\nu)}\left(\frac{k}{2}\right)^{-2\nu} e^{-2kz_0} & ((kz_0 \gg 1) \\ \frac{2(d+2\nu)\nu}{d^2-4\nu^2}\cdot z_0^{2\nu} & (kz_0 \ll 1). \end{cases}$$

UV regularized DT deformation

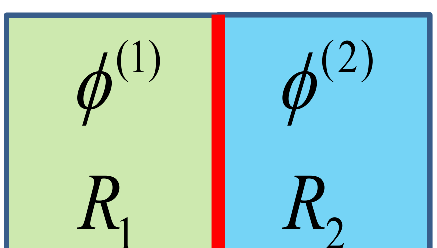
Note: In order to reproduce two point functions for all operators, we need to perform the double trace deformations for all primaries.

◆ A toy model of Janus deformed CFT dual

For a realization of AdS3/CFT2 Janus solution, consider AdS3 × S3 × 4 in IIB string theory, dual to the D1-D5 CFT given by the symmetric product CFT: $\mathrm{Sym}\left[(\mathrm{T}^4)^{Q_1 Q_5}\right]$.

The Janus deformation is performed by shifting the compactification radius R➔R1 in CFT1 and R➔R2 in CFT2.

Below we consider a toy model of Janus CFT based on the c=1 free compactified scalar ϕ (radius R).



$$\tan\theta = \frac{R_2}{R_1}.$$

Janus deformation

$$\theta = \frac{\pi}{4} + \gamma.$$

To probe its dual "geometry", compute the two point function <V1V2>

$$V_1 = e^{i\lambda_+ \phi_L^{(1)}(\tau_1) + i\lambda_- \phi_R^{(1)}(\tau_1)}, \qquad V_2 = e^{i\mu_+ \phi_L^{(2)}(\tau_2) + i\mu_- \phi_R^{(2)}(\tau_2)},$$

In the high temperature limit,

$$\lambda_\pm = \frac{n}{R_1} \pm \frac{wR_1}{2}, \qquad \mu_\pm = \frac{n}{R_2} \mp \frac{wR_2}{2}.$$

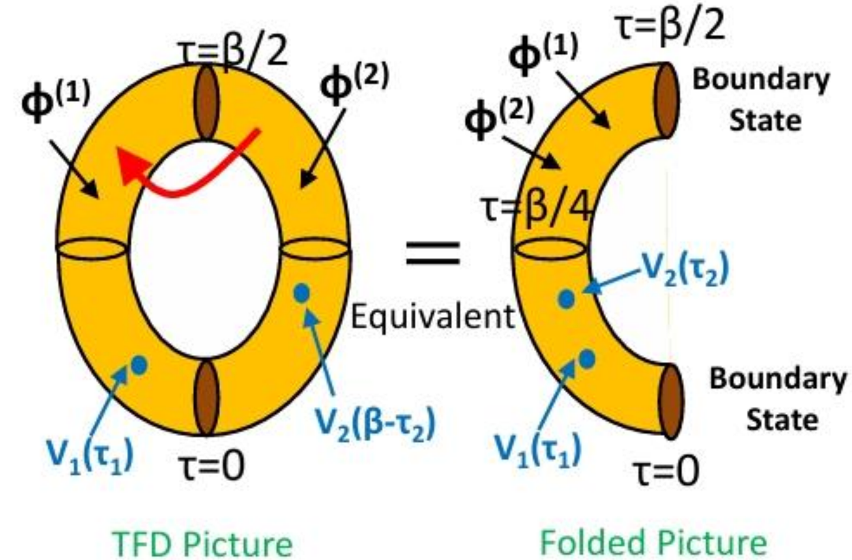
$$\langle V_1(\tau_1)V_2(\tau_2)\rangle$$

$$\simeq \left[\frac{\beta}{\pi}\cdot\sin\left(\frac{2\pi\tau_1}{\beta}\right)\right]^{\left[\left(\frac{n}{R_1}\right)^2-\left(\frac{wR_1}{2}\right)^2\right]\cos 2\theta} \cdot \left[\frac{\beta}{\pi}\cdot\sin\left(\frac{2\pi\tau_2}{\beta}\right)\right]^{\left[-\left(\frac{n}{R_2}\right)^2+\left(\frac{wR_2}{2}\right)^2\right]\cos 2\theta}$$

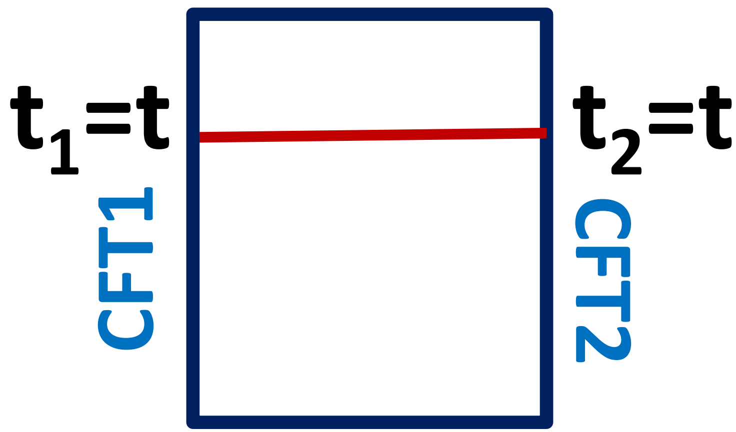
$$\cdot \left[\frac{\beta}{\pi}\cdot\sin\left(\frac{\pi(\tau_1+\tau_2)}{\beta}\right)\right]^{-2\left[\frac{n^2}{R_1R_2}+\frac{w^2R_1R_2}{4}\right]\sin 2\theta}$$

To evaluate the two point function, we employed the doubling trick of interface CFT.

[Bachas-de Boer-Dijkgraaf-Ooguri 2001, Sakai-Saoth 2008]



Case 1 $\tau_1 = \frac{\beta}{4} + it, \qquad \tau_2 = \frac{\beta}{4} + it$

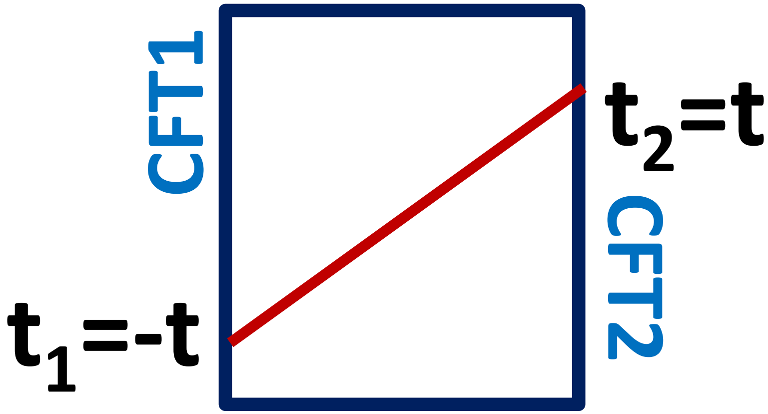


$$\langle V_1(\tau_1) V_2(\tau_2) \rangle \propto \left[\frac{\beta}{\pi} \cdot \cosh \frac{2\pi}{\beta} t \right]^{-\Delta_1 - \Delta_2}$$

Case 2 $\tau_1 = \frac{\beta}{4} + it, \qquad \tau_2 = \frac{\beta}{4} - it$

$$\langle V_1(t_1) V_2(t_2) \rangle \propto \left[\frac{\beta}{\pi} \cdot \cosh \frac{2\pi}{\beta} t \right]^{\eta}$$

$$\eta = -\frac{(R_1^2 - R_2^2)^2}{(R_1^2 + R_2^2) R_1 R_2} \cdot \left(\frac{n^2}{R_1 R_2} + \frac{w^2 R_1 R_2}{4} \right)$$



η<0 for real γ
η>0 for imaginary γ

Qualitatively agree with the gravity dual