Δ Path integrals

- QM review
- QFT (free theory)

△ Path Integral :

QM :

$[Q, P] = i$

$H = \frac{1}{2m} P^2 + V(Q)$

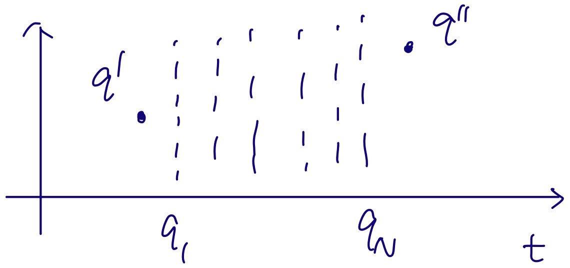
$|q, t\rangle \equiv e^{+iHt} |q\rangle$, $Q|q\rangle = q|q\rangle$

$Q(t)\, |q, t\rangle = e^{iHt} Q e^{-iHt} e^{iHt} |q\rangle$

$= q\, |q, t\rangle$

$\langle q'', t'' | q', t' \rangle$

$$= \int dq_N\, dq_{N-1} \cdots dq_1 \quad \langle q'' | e^{-iH\delta t} | q_N \rangle \langle q_N | e^{-iH\delta t} | q_{N-1} \rangle \langle q_{N-1} \cdots | q' \rangle$$



$$e^{-iH\delta t} \sim e^{-i\left(\frac{P^2}{2m}\delta t + V(Q)\,\delta t\right)}$$

$$\sim e^{-i\left(\frac{P^2}{2m}\delta t\right)} e^{-iV(Q)\delta t}$$

$$\langle q_2 | e^{-iH\delta t} | q_1 \rangle = e^{-iV(q_1)\delta t} \langle q_2 | e^{-i\frac{P^2}{2m}\delta t} | q_1 \rangle$$

$$= \int_{p_1} e^{-i V(q_1)\,\delta t} \qquad e^{-i\, p_1^2/2m\, \delta t}$$

$$\langle q_2 | p \rangle \langle p | q_1 \rangle$$

$$= \int_{p_1} e^{-i H(q_1, p_1)\,\delta t} \qquad e^{+i p \cdot (q_2 - q_1)}$$

$$= \int_{p_1} \exp\Big(\underbrace{i\Big(p \cdot \frac{\delta q}{\delta t} - H(q_1, p_1)\Big) \cdot \delta t}_{\text{number!}} \Big)$$

$\Rightarrow$ $\langle q'', t'' | q', t' \rangle$

$$= \int \prod dq_i \prod \frac{dp_i}{2\pi} \cdot \exp\Big(i \cdot \int_{t'}^{t''} dt \, \big(p\dot{q} - H(q,p)\big)\Big)$$

$$= \int \mathcal{D}q \; \mathcal{D}p \; \exp\Big(i \cdot \int_{t'}^{t''} dt \, \big(p\dot{q} - H(q,p)\big)\Big)$$

If H is quadratic in p : $\dot{q} - \frac{\partial H}{\partial p} = 0$

$$= \int \mathcal{D}q \; \exp\Big(i \int_{t'}^{t''} dt \cdot L(q(t), \dot{q}(t)) \Big)$$

Correlation functions

$$\langle q'', t' | Q(t_1) | q', t' \rangle$$

$$= \int \mathcal{D}q \, \mathcal{D}p \;\; q(t_1) \; e^{iS}$$

What about

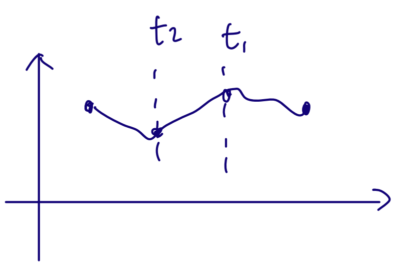
$$\int \mathcal{D}q\, \mathcal{D}p \quad q(t_1)\, q(t_2)\, e^{iS}$$

$$Q(t) = e^{iHt} Q e^{-iHt}$$

$$\rightarrow \quad \langle \psi_{out} | Q(t_1)\, Q(t_2) | \psi_{in} \rangle$$

$$= \langle q'' | e^{-i\hat{H}t''} \left(e^{+i\hat{H}t_1} Q e^{-i\hat{H}t_1} \right) \left(e^{+i\hat{H}t_2} Q e^{-i\hat{H}t_2} \right) e^{+i\hat{H}t'} | q' \rangle$$

$$= \langle q'' | e^{-i\hat{H}(t''-t_1)} Q e^{-i\hat{H}(t_1-t_2)} Q e^{-i\hat{H}(t_2-t')} | q' \rangle$$



$\Rightarrow$

$t_1 > t_2 \qquad \int \mathcal{D}p\, \mathcal{D}q \quad q(t_1)\, q(t_2)\, e^{iS}$

$t_1 < t_2 \qquad$ no simple expression.

$$\langle q'', t'' | T(Q(t_1)\, Q(t_2)) | q', t' \rangle$$

$$= \int \mathcal{D}p\, \mathcal{D}q \quad q(t_1)\, q(t_2)\, e^{iS}$$

Δ Functional derivative :

$$\frac{\partial y_i}{\partial y_j} = \delta_{ij}$$

$$\frac{\delta f(x)}{\delta f(x')} = \delta(x-x')$$

s.t.

$$\frac{\delta}{\delta f(x)} \left(\int d^4x' \, f(x') \, g(x') \right)$$

$$= \int d^4x' \, \delta(x-x') \, g(x') = g(x)$$

Generating function for corelation function

Consider

$$S = \int dt \, \left(p\dot{q} - H + f(t) \cdot q(t) + h(t) \cdot p(t) \right)$$

$$\langle q'', t'' | q', t' \rangle_{f,h}$$

$$= \int \mathcal{D}p \, \mathcal{D}q \, \exp\left(i \cdot \int dt \, (p\dot{q} - H + f \cdot q + h \cdot p) \right)$$

$$\Rightarrow \quad \frac{\delta}{i\,\delta f(t_1)} \langle q'', t'' | q', t' \rangle_{f,h}$$

$$= \int \mathcal{D}p\, \mathcal{D}q \cdot \frac{\delta}{i\,\delta f(t_1)} \exp\left(i \int dt\, (p\dot{q} - H + fq + h\cdot p) \right)$$

$$= \int \mathcal{D}p\, \mathcal{D}q \cdot q(t_1) \cdot \exp\left(i \cdot S(q,p) \right)$$

$$= \langle q'', t'' | Q(t_1) | q', t' \rangle_{f,h}$$

$$\frac{\delta}{i\,\delta h(t_1)} \langle q'', t'' | q', t' \rangle_{f,h} = \langle q'', t'' | P(t_1) | q', t' \rangle_{f,h}$$

$$\left(\frac{\delta}{i\,\delta f(t_1)} \right) \left(\frac{\delta}{i\,\delta f(t_2)} \right) \cdots \left(\frac{\delta}{i\,\delta h(t'_1)} \right) \cdots \langle q'', t'' | q', t' \rangle_{f,h}$$

$$\xrightarrow[f=h=0]{} \langle q'' t'' | T\left(Q(t_1)\, Q(t_2) \cdots P(t'_1) \cdots P(t'_n) \right) | q', t' \rangle$$

Δ Projection to ground state :

$$|q',t'\rangle = e^{iHt'}|q'\rangle$$

$$= \sum_n e^{iHt'}|n\rangle\langle n|q'\rangle$$

$$= \sum_n e^{iE_n t'}\psi_n(q')|n\rangle$$

$$\xrightarrow[\substack{H\to H(1-i\epsilon)\\ t'\to -\infty}]{} \sum_n e^{iE_n(1-i\epsilon)t'}\psi_n(q')|n\rangle$$

$$\approx e^{iE_0(1-i\epsilon)t'}\psi_0(q')|0\rangle$$

send $\begin{cases} H\to H(1-i\epsilon) \\ t'\to -\infty \end{cases}$ project the state into ground state.

$$\langle q'',t''| = \langle q''|e^{-iHt''}$$

$$\xrightarrow[\substack{H\to H(1-i\epsilon)\\ t''\to \infty}]{} \langle q''|e^{-iH(1-i\epsilon)t''} = \langle q''|0\rangle\langle 0|e^{-iE_0(1-i\epsilon)t''}$$

This is equivalent to

$$S\to S(1+i\epsilon) \qquad \int_{-\infty}^{\infty}dt \to \int_{-\infty(1-i\epsilon)}^{\infty(1+i\epsilon)}dt$$

$$\langle 0|0\rangle_{f,h}$$

$$= \int \mathcal{D}p\,\mathcal{D}q \quad \exp\left(i \cdot \int_{-\infty}^{\infty} dt \left(p\dot{q} - H(1-i\epsilon) + f\cdot q + h\cdot p \right)\right)$$

✡ interacting theory

$$H = H_0 + H_1$$

$$\langle 0|0\rangle_{f,h}$$

$$= \int \mathcal{D}p\,\mathcal{D}q \quad \exp\left(i \int dt \left(p\dot{q} - H_0 - H_1 + f\cdot q + h\cdot p \right)\right)$$

$$= \int \mathcal{D}p\,\mathcal{D}q \cdot \exp\left(-i\int dt \; H_1(p,q)\right) \exp\left(i\, S_0(f,h)\right)$$

$$\Rightarrow \int \mathcal{D}p\,\mathcal{D}q \quad \exp\left(-i\int dt \; H_1\left(\frac{\delta}{i\,\delta f(t)}, \frac{\delta}{i\,\delta h(t)}\right)\right) \cdot \exp\left(i \cdot S_0(f,h)\right)$$

$$= \exp\left(-i\int dt \; H_1\left(\frac{\delta}{i\,\delta f(t)}, \frac{\delta}{i\,\delta h(t)}\right)\right) \langle 0|0\rangle_{f,h} \,(\text{free})$$

$$\langle 0|0\rangle = \int \mathcal{D}p\,\mathcal{D}q \ \exp\left(i \cdot S\,(f=h=0) \right)$$

is not necessary normalized

Master formula :

$$\langle 0|\, T(\,Q(t_1)\cdots \ P(t'_1)\cdots)\,|0\rangle \Big|_{\langle 0|0\rangle = 1}.$$

$$= \left(\frac{\delta}{i\,\delta f(t_1)} \cdots \ \frac{\delta}{i\,\delta h(t_1)} \cdots \right)$$

$$\frac{\exp\left(-i\int dt \ H_1\left(\frac{\delta}{i\,\delta f(t)}, \frac{\delta}{i\,\delta h(t)}\right)\right) \langle 0|0\rangle_{f,h}\,(\text{free})}{\exp\left(-i\int dt \ H_1\left(\frac{\delta}{i\,\delta f(t)}, \frac{\delta}{i\,\delta h(t)}\right)\right) \langle 0|0\rangle_{f,h}\,(\text{free})\Big|_{f=h=0}}$$

$\triangle$ QFT :

$Q(t) \longrightarrow \phi(x)$ (quantum field (operator))

$q(t) \longrightarrow \phi(x)$ classical value

$f(t) \longrightarrow J(x)$ classical source.

$$Z(J) = \langle 0 | 0 \rangle_J$$

$$= \int \mathcal{D}\phi \; \exp\left(i \cdot \int d^4x \cdot (\mathcal{L} + J\phi) \right)$$

$$\mathcal{D}\phi \Rightarrow \prod_x d\phi(x)$$

infinite in $\vec{x}, t$

$$\langle 0 | T(\phi(x_1) \cdots \phi(x_n)) | 0 \rangle$$

$$= \frac{\left(\frac{\delta}{i\,\delta J(x_1)} \cdots \frac{\delta}{i\,\delta J(x_n)} \right) Z(J)}{Z(0)}$$

△ Free theory :

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$Z_0(J) = \int \mathcal{D}\phi \quad \exp\left(i\int d^4x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi \right)\right)$$

Since ϕ is classical,

$$\phi(x) = \int d^4x \; e^{-ikx} \; \phi(k)$$

$$\phi(k) = \int \frac{d^4k}{(2\pi)^4} \; e^{ikx} \; \phi(x)$$

$$S = \int d^4x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi \right)$$

$$= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} d^4x \left(-\frac{1}{2}(k\cdot p)\,\phi(k)\phi(p) - \frac{1}{2}m^2\phi(k)\phi(p) + J(k)\,\phi(p) \right) \cdot e^{-i(k+p)\cdot x}$$

$$= \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{2}\cdot(k^2 - m^2 + i\epsilon)\,\phi(-k)\phi(k) + J(-k)\,\phi(k) \right]$$

$$= \int_k \left[\frac{1}{2}(k^2 - m^2 + i\epsilon)\left(\phi(k) + \frac{J(k)}{k^2 - m^2 + i\epsilon}\right)\left(\phi(-k) + \frac{J(-k)}{k^2 - m^2 + i\epsilon}\right) - \frac{1}{2}\frac{J(k)J(-k)}{k^2 - m^2 + i\epsilon} \right]$$

$$\chi(k) \equiv \phi(k) + \frac{J(k)}{k^2-m^2} \qquad \mathcal{D}\phi = \mathcal{D}\chi$$

$$Z_0(J) = \int \mathcal{D}\chi \; \exp\left(i \int_k \frac{1}{2}(k^2-m^2)\, \chi(k)\, \chi(-k) \right)$$

$$\exp\left(-\frac{i}{2} \cdot \int \frac{J(k)\, J(-k)}{k^2-m^2+i\epsilon} \right)$$

$$= \exp\left(-\frac{i}{2} \cdot \int \frac{J(k)\, J(-k)}{k^2-m^2} \right) \cdot \langle 0 | 0 \rangle$$

normalize to 1.

$$= \exp\left(-\frac{i}{2} \cdot \int_{x,y,k} J(x)\, \frac{e^{ik(x-y)}}{k^2-m^2}\, J(y) \right)$$

$$= \exp\left(+\frac{i}{2} \cdot \int_{x,y} J(x)\, G(x,y)\, J(y) \right)$$

$$G(x,y) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2-m^2+i\epsilon}$$

$$\langle 0 | T(\phi(x_1)\, \phi(x_2)) | 0 \rangle$$

$$= \frac{\delta}{i\,\delta J(x_1)} \frac{\delta}{i\,\delta J(x_2)} Z_0(J) \Big|_{J=0}$$

$$= \frac{\delta}{i\,\delta J(x_1)} \left(\underbrace{\left[+\frac{1}{2} \int_{x,y} \delta(x-x_2)\, G(x,y)\, J(y) + (x \leftrightarrow y) \right]}_{= -\int_y G(x_2,y)\, J(y)} Z_0(J) \right) \Bigg|_{J=0}$$

$$= -i\, G(x_1, x_2)$$

Feynman rule :

$$Z_0(J) = \exp\left(\frac{i}{2} \cdot \int_{x,y} J(x)\, G(x,y)\, J(y) \right)$$

$$= \exp\left(\frac{1}{2} \cdot \int_{x,y} \left(i J(x) \right) \left(\frac{1}{i} G(x,y) \right) \left(i J(y) \right) \right)$$

$$= \exp\left(\frac{1}{2} \cdot \right.$$  $$\left. \right)$$

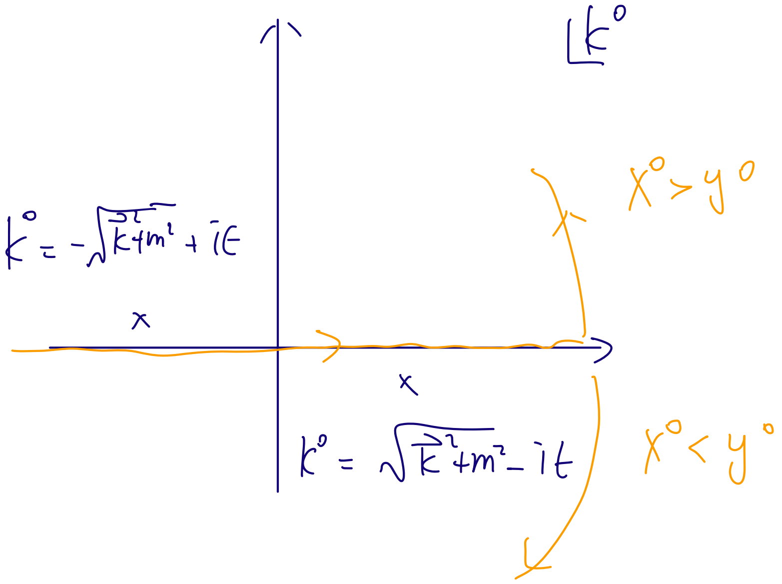
$i \int d^4x\, J(x)$ $\qquad$ $\frac{1}{i} G(x,y)$ $\qquad$ $i \int d^4y\, J(y)$

$$\frac{1}{i} G(x,y) = \int \frac{d^4k}{(2\pi)^4} \left(\frac{i}{k^2 - m^2 + i\epsilon} \right) \cdot e^{ik(x-y)}$$

k $\quad : \quad \dfrac{i}{k^2 - m^2 + i\epsilon}$

There is a lot of information in the 2pt function

$$\langle 0 | T(\phi(x)\,\phi(y)) | 0 \rangle = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon}$$



$$\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon} = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{dk^0}{2\pi} \cdot \frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\, e^{ik^0(x^0-y^0)}}{k^2 - m^2 + i\epsilon}$$

$\omega_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m^2}$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \cdot \Big\{ \theta(x^0 - y^0) \frac{i}{-2\omega_{\vec{k}}} e^{-i\omega\Delta t} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} + \theta(y^0 - x^0) \frac{-i}{2\omega_{\vec{k}}} e^{i\omega\Delta t} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \Big\}$$

$$= (-i) \int \widetilde{dk}\, \Big\{ \theta(x^0 - y^0)\, e^{-ik\cdot(x-y)} + \theta(y^0 - x^0)\, e^{ik\cdot(x-y)} \Big\}$$

$\Rightarrow$ Same as canonical quantization.

- The 2pt function is also called "propagator" or "Green's function"

Consider

$$\partial_t \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle$$

$$= \partial_t \Big\{ \theta(x^0 - y^0) \langle 0 | \phi(x)\phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y)\phi(x) | 0 \rangle \Big\}$$

$$= \delta(x^0 - y^0) \Big(\langle 0 | \underbrace{[\phi(x), \phi(y)]}_{=0 \text{ when } x^0 = y^0} | 0 \rangle \Big) + \theta(x^0 - y^0) \langle 0 | \dot\phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \dot\phi(x) | 0 \rangle$$

$$\partial_t^2 \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle$$

$$= \delta(x^0 - y^0) \langle 0 | \underbrace{(\dot\phi(x)\phi(y) - \phi(y)\dot\phi(x))}_{= [\pi(x), \phi(y)] \text{ at } x^0 = y^0 \; = -i\delta(\vec{x} - \vec{y})} | 0 \rangle + \langle 0 | T(\ddot\phi(x)\phi(y)) | 0 \rangle$$

$$(\partial_t^2 - \nabla^2 + m^2) \langle 0 | T(\phi(x)\,\phi(y)) | 0 \rangle$$

$$= -i\,\delta(x-y) + \langle 0 | T((\partial_t^2 - \nabla^2 + m^2)\phi(x)\,\phi(y)) | 0 \rangle$$

Assuming that the quantum field also satisfy the same EOM

$$(\Box + m^2) \langle 0 | T(\phi(x)\,\phi(y)) | 0 \rangle = -i \cdot \delta(x-y)$$

$$\leftrightarrow \quad (\Box + m^2)\, G(x,y) = \delta(x-y)$$

in classical field theory.

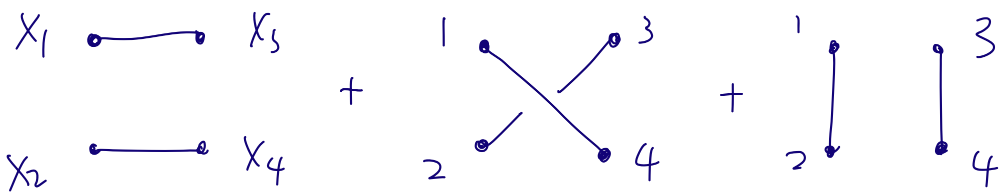
Exercise :

$$\langle 0 | T(\phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4)) | 0 \rangle$$

$$= \frac{\delta}{i\,\delta J(x_1)} \frac{\delta}{i\,\delta J(x_2)} \frac{\delta}{i\,\delta J(x_3)} \frac{\delta}{i\,\delta J(x_4)} Z_0(J)$$

$$\frac{\delta}{i\,\delta J(x)} Z_0(J) = \frac{\delta}{i\,\delta J(x)} \exp\left(\frac{1}{2}\right.$$  $$\left.\right)$$

$=$  $Z_0(J)$

$=$ 

$$= \left(\frac{1}{i} G(x_1, x_3)\right)\left(\frac{1}{i} G(x_2, x_4)\right)$$

$+ \ \ldots$

☆ Try this w/ canonical quantization !