△ QFT introduction

* Why QFT ?

* Scales

* QM → QFT

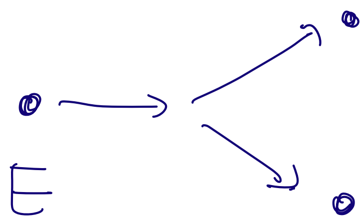
△ Motivation :

QM : single particle states

e.g. hydrogen atom

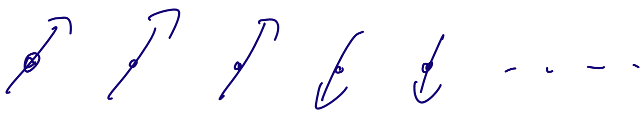
→ absorption , decay ?

But when $E > m$



(e.g. μ^- , nucleon decay, etc.)

- Condensed matter ⇒ many-body systems.



- High energy physics ⇒ Lorentz inv.

- Astrophysics $\begin{cases} h_{\mu\nu}(x) & \text{(classical)} \\ \delta\phi(x) & \text{(inflation)} \end{cases}$

Δ Scales :

Units :

$$\boxed{\hbar = c = 1}$$

c = 1 :

$$E = mc^2 \rightarrow m$$

$$[E] = [m] = 1$$

"length" = "time"

$\hbar = 1$:

$$E = \hbar \omega$$

$$[E] = [\omega] = -[t] = -[L]$$

$$E \propto 1/L$$

$\Rightarrow$ high energy $\longleftrightarrow$ short distance ,

e.g. 1 GeV (proton) $\sim$ 1 fm $\sim 10^{-15}$ m .

E.g.

$$[d^d x] = -d$$

$$S = \int d^d x \cdot \mathcal{L}$$

$$e^{iS} \Rightarrow [S] = 0$$

$$\Rightarrow \qquad = [d^d x] + [\mathcal{L}]$$

$$\Rightarrow [\mathcal{L}] = d$$

△ From QM to QFT :

QM :

$$H = \frac{p^2}{2m} + V$$

1) single particle state

2) state → probability interpretation.

3) $\left\{ \begin{array}{l} \hat{H}, \hat{x}, \hat{p} \quad \text{promoted into operators} \\ t \quad : \quad \text{parameter labeling the state} \end{array} \right.$

Problems :

1) multi-particle states ?

2) relativity

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu$$

no unique notion of t .

- Free theory :

QM :

$$H = \frac{\vec{p}^2}{2m} \Rightarrow - \frac{\nabla_x^2}{2m}$$

We can write : Same as SHO in QM.

$$[a(\vec{x}), a(\vec{x}')] = 0$$

$$[a^\dagger(\vec{x}), a^\dagger(\vec{x}')] = 0$$

$$[a(\vec{x}), a^\dagger(\vec{x}')] = \delta(\vec{x} - \vec{x}')$$

$$H = \int_x a^\dagger(\vec{x}) \left(- \frac{\nabla^2}{2m} \right) a(\vec{x})$$

State :

$|0\rangle$ vacuum $\qquad a(\vec{x})|0\rangle = 0$

1- particle state :

$$|\psi, t\rangle = \int_{x_1} \psi(\vec{x}_1, t) \, a^\dagger(\vec{x}_1) |0\rangle$$

$$\hat{H}|\psi, t\rangle = i\frac{\partial}{\partial t}|\psi\rangle$$

$$\Rightarrow \int_{x_1} \hat{H}\left(\psi(\vec{x}_1, t)\, a^\dagger(\vec{x}_1)\right)|0\rangle$$

$$= \int_{x_1, x} \left(a^\dagger(\vec{x})\left(-\frac{1}{2m}\nabla^2\right)a(\vec{x})\right)\psi(\vec{x}_1, t)\, a^\dagger(\vec{x}_1)|0\rangle$$

$$= \int_{x_1, x} a^\dagger(\vec{x})\left(-\frac{1}{2m}\nabla^2\right)\left(\psi(\vec{x}_1, t)\,\delta(\vec{x}-\vec{x}_1)\right)|0\rangle$$

$$= \int_x a^\dagger(\vec{x})\left(-\frac{1}{2m}\nabla^2\right)\psi(\vec{x}, t)\,|0\rangle$$

$$\text{RHS} = \int_{\vec{x}_1} i\frac{\partial}{\partial t}\psi(\vec{x}_1, t)\, a^\dagger(\vec{x}_1)|0\rangle$$

$$\Rightarrow -\frac{1}{2m}\nabla^2\psi(\vec{x}, t) = i\frac{\partial}{\partial t}\psi(\vec{x}, t)$$

HW: add potential to H.

Multiparticle state :

$$|\Psi, t\rangle = \int_{x_1,\cdots x_n} \Psi(\vec{x}_1, \cdots \vec{x}_n ; t) \; a^\dagger(\vec{x}_1)\, a^\dagger(\vec{x}_2) \cdots a^\dagger(\vec{x}_n)|0\rangle$$

Property :

$$\Psi(\cdots x_i, \cdots x_j \cdots ; t) = \Psi(\cdots x_j, \cdots, x_i, \cdots ; t) \Rightarrow \text{boson.}$$

Canonical Quantization :

$$a(\vec{p}) = \int d^3\vec{x} \; e^{-i\vec{p}\cdot\vec{x}} a(\vec{x}) \; , \quad a(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} a(\vec{p})$$

$$[a(\vec{p}), a^\dagger(\vec{p}')] = \int_{\vec{x},\vec{x}'} e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{p}'\cdot\vec{x}'} \delta(\vec{x}-\vec{x}')$$

$$= \int_{\vec{x}} e^{-i(\vec{p}-\vec{p}')\cdot\vec{x}}$$

$$= (2\pi)^3 \cdot \delta(\vec{p}-\vec{p}')$$

$$[a(\vec{p}), a(\vec{p}')] = 0$$

$$[a^\dagger(\vec{p}), a^\dagger(\vec{p}')] = 0$$

- Lorentz invariance ?

$$H = \int_{\vec{x}} a^\dagger(\vec{x}) \left(-\frac{\nabla^2}{2m}\right) a(\vec{x})$$

$$= \int_{\vec{x}} \int_{\vec{p},\vec{p}'} a^\dagger_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} \left(-\frac{\nabla^2}{2m}\right) \left(a_{\vec{p}'}\, e^{-i\vec{p}'\cdot\vec{x}}\right)$$

$$= \underline{\underline{\int_{\vec{x}}}} \int_{\vec{p},\vec{p}'} \underline{e^{i(\vec{p}-\vec{p}')\cdot\vec{x}}}\, a^\dagger_{\vec{p}}\, a_{\vec{p}'} \cdot \frac{\vec{p}'^2}{2m}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\frac{\vec{p}^2}{2m}\right) \cdot a^\dagger_{\vec{p}}\, a_{\vec{p}}$$

Relativistic ?

$$\hat{H} = \int_{\vec{p}} \sqrt{\vec{p}^2 + m^2}\; a^\dagger_{\vec{p}}\, a_{\vec{p}}$$

Whether it's relativistic or not is not obvious

because time evolution

$e^{i\hat{H}t}$ breaks covariance

△ Lagrangian :

$$\hat{H}\,\phi = i\frac{\partial}{\partial t}\phi$$

$$\hat{H}^2\phi = -\partial_t^2\,\phi$$

$$= (\vec{P}^2 + m^2)\,\phi$$

$$= (-\nabla^2 + m^2)\,\phi$$

$$\Rightarrow (\partial_t^2 - \nabla^2 + m^2)\,\phi = 0$$

$\Rightarrow$ Klein - Gordon equation !

Lagrangian :

$$S = \int dt\; L \qquad \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \text{ is the flat metric.}$$

$$= \int dt\, d^3x\; \mathcal{L}$$

$$= \int d^4x \left(\tfrac{1}{2}\,\eta^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}\,m^2\phi^2 \right)$$

Euler - Lagrange eq :

$$\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\delta \partial_\mu \phi} \right) = 0$$

$$\Rightarrow \quad -m^2 \phi - \partial_\mu \left(\eta^{\mu\nu} \partial_\nu \phi \right)$$

$$\Rightarrow \quad \left(-m^2 - \eta^{\mu\nu} \partial_\mu \partial_\nu \right) \phi = 0$$

$$\Rightarrow \quad \left(-m^2 - \partial_t^2 + \nabla^2 \right) \phi = 0$$

Classical field:

$$\Box \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial_\mu \partial^\mu$$

$$(-\Box - m^2)\phi = 0$$

incoming / outgoing

positive / negative energy

Ansatz:

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{1}{f(|\vec{p}|)} \left(\underbrace{a_{\vec{p}}}_{} \, e^{i\vec{p}\cdot\vec{x} - i\omega t} + \underbrace{b_{\vec{p}}}_{} \, e^{i\vec{p}\cdot\vec{x} + i\omega t} \right)$$

functions

$$(-\Box - m^2)\phi = 0$$

$$\Rightarrow \int_{\vec{p}} \frac{1}{f(\vec{p})} \Big(a_{\vec{p}} \cdot (\omega^2 - \vec{p}^2 - m^2) \, e^{i\vec{p}\cdot\vec{x} - i\omega t}$$

$$+ \, b_{\vec{p}} \, (\omega^2 - \vec{p}^2 - m^2) \, e^{i\vec{p}\cdot\vec{x} + i\omega t} \Big)$$

$$\Rightarrow \quad \boxed{\omega^2 - \vec{p}^2 - m^2 = 0}$$

$$\omega = \sqrt{\vec{p}^2 + m^2} > 0$$

Real field : $\phi(x) = \phi^*(x)$

$$\Rightarrow \quad \phi^*(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{f^*(\vec{p})} \left(a^* e^{-i\vec{p}\cdot\vec{x} + i\omega t} + b^* e^{-i\vec{p}\cdot\vec{x} - i\omega t} \right)$$

$$a^*(\vec{p}) = b(-\vec{p})$$

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{f(|\vec{p}|)} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^*_{\vec{p}}\, e^{ip\cdot x} \right)$$

Need to find f to be consistent w/ Lorentz

Trick :

$$\int \frac{d^4 p}{(2\pi)^4} \cdot \delta(p^2 - m^2) \cdot 2\pi$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^4} \cdot \frac{dp^0}{2\pi} \cdot 2\pi \cdot \delta\left((p^0 - \omega)(p^0 + \omega) \right)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \int dp^0 \cdot \delta(p^0 - \omega) \cdot \frac{1}{2\omega} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega}$$

$$\equiv \int \widetilde{dp}$$

c.f. $\int dx\ \delta(f(x)) = \int dx\ \delta(f(x_0) + (x-x_0) f'(x=x_0))$

$$= \int dx\ \delta((x-x_0) f'(x_0))$$

$u \equiv (x-x_0)\cdot f'(x_0)$

$$= \int du\ \delta(u) \frac{1}{f'(x_0)}$$

Classical field :

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^*_{\vec{p}}\, e^{ip\cdot x} \right)$$

$$S = \int d^4x\ \mathcal{L}$$

$$= \int d^4x \left(\tfrac{1}{2}\cdot(\partial\phi)^2 - \tfrac{1}{2} m^2\phi^2 \right)$$

$$H = \int d^3\vec{x}\ \mathcal{H} \qquad \pi = \frac{\delta\mathcal{L}}{\delta\dot\phi}$$

$$= \int d^3\vec{x} \left(\pi\dot\phi - \mathcal{L} \right) \qquad = \dot\phi$$

free $\Rightarrow$ $$= \int d^3\vec{x} \left(\dot\phi^2 - \mathcal{L} \right) = \int d^3\vec{x} \left[\tfrac{1}{2}\left(\dot\phi^2 + (\nabla\phi)^2\right) + \tfrac{1}{2} m^2\phi^2 \right]$$

HW: Plug in

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^*_{\vec{p}}\, e^{ip\cdot x} \right)$$

Find H in terms of a

$\Rightarrow$

$$H = \int \widetilde{dp} \, \sqrt{\vec{p}^{\,2}+m^2} \left(a^*_{\vec{p}}\, a_{\vec{p}} \right)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{1}{2} \, a^*_{\vec{p}}\, a_{\vec{p}} \quad \longleftrightarrow \quad \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \sqrt{\vec{p}^{\,2}+m^2}\; a^\dagger_{\vec{p}}\, a_{\vec{p}}$$

$\Rightarrow$ Almost same as the previous H,

Except :

$$a(\vec{p})\Big|_{\text{classical}} \longrightarrow a_{\vec{p}} \quad (\text{operator})$$

Now

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{ip\cdot x} \right)$$

becomes a quantum operator too!

$$[a_{\vec{p}}, a^\dagger_{\vec{k}}] = 2\sqrt{\vec{p}^{\,2}+m^2}\;(2\pi)^3\,\delta(\vec{p}-\vec{k})$$

$$[\phi(\vec{x},t), \phi(\vec{y},t)]$$

$$= [\int \widetilde{dp} \, (a_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x}),$$

$$\int \widetilde{dk} \, (a_{\vec{k}} e^{-ik\cdot y} + a^\dagger_{\vec{k}} e^{ik\cdot y})]$$

$$= \int \widetilde{dp} \, \widetilde{dk} \, (+(2\pi)^3 \delta(\vec{p}-\vec{k}))(-e^{i(p\cdot x - k\cdot y)} + e^{-i(p\cdot x - k\cdot y)})$$

$$= \int \widetilde{dp} \cdot \frac{1}{2E} (-e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{i\vec{p}(\vec{x}-\vec{y})}) = \emptyset$$

$$[\phi(\vec{x},t), \pi(\vec{y},t)]$$

$$= [\phi(\vec{x},t), \dot{\phi}(\vec{y},t)]$$

$$= \int \widetilde{dp} \, \widetilde{dk} \, [(a_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x}),$$

$$(-i\sqrt{\vec{k}^2+m^2} \, a_{\vec{k}} e^{-ik\cdot x} + i\sqrt{\vec{k}^2+m^2} \, a^\dagger_{\vec{k}} e^{ik\cdot x})]$$

$$= \int \widetilde{dp} \, \widetilde{dk} \, 2\sqrt{\vec{p}^2+m^2} (2\pi)^3 \delta(\vec{p}-\vec{k}) \, (i\sqrt{\vec{k}^2+m^2})(+e^{i(p\cdot x - k\cdot y)} + e^{-i(p\cdot x - k\cdot y)}$$

$$= i \cdot \int \widetilde{dp} \, (e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}) \sqrt{\vec{p}^2+m^2}$$

$$= i \cdot \delta(\vec{x}-\vec{y})$$

- Correlation function :

$$\langle 0 | \phi(x)\, \phi(y) | 0 \rangle$$

$$= \langle 0 | \int \widetilde{dp}\, \widetilde{dk} \left(a_p e^{-ipx} + \cancel{a_p^{+}} e^{ipx} \right) \left(\cancel{a_k} e^{-ik\cdot y} + a_k^{+} e^{ik\cdot y} \right) | 0 \rangle$$

$$= \int \widetilde{dp}\, \widetilde{dk}\, (2\pi)^3 \cdot 2\omega\, \delta(\vec{p} - \vec{k})\, e^{i(k\cdot y - p\cdot x)}$$

$$= \int \widetilde{dp}\; e^{-i\vec{p}(\vec{y}-\vec{x})}\; e^{i\omega(y^0 - x^0)}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{1}{2\omega}\; e^{-i\vec{p}(\vec{y}-\vec{x})}\; e^{i\omega(y^0 - x^0)}$$

$$\Delta x = |\vec{x} - \vec{y}|$$

$$\Delta t = x^0 - y^0$$

$$= \int \frac{dp \cdot p^2\, d\cos\theta}{(2\pi)^2 \cdot 2\omega}\; e^{-i\, p \cdot \Delta x \cos\theta}\; e^{i\omega(y^0 - x^0)}$$

$$= \int \frac{dp \cdot p^2}{(2\pi)^2 \cdot 2\omega} \left(\frac{e^{-ip\cdot\Delta x} - e^{ip\cdot\Delta x}}{-i\, p\cdot \Delta x} \right) e^{i\omega(y^0 - x^0)}$$

$$= i \cdot \int \frac{dp \cdot p}{4\pi^2 \cdot \sqrt{p^2 + m^2}} \left(\frac{e^{-ip\cdot\Delta x} - e^{ip\cdot\Delta x}}{\Delta x} \right) e^{-i\omega\cdot\Delta t}$$

Massless Limit

$$\Rightarrow \frac{i}{4\pi^2} \cdot \frac{1}{\Delta x} \int dp \cdot \left(e^{ip(\Delta t - \Delta x)} - e^{ip(\Delta t + \Delta x)} \right)$$

$$= \frac{i}{2\pi \cdot \Delta x} \left(\delta(\Delta t + \Delta x) - \delta(\Delta t - \Delta x) \right)$$

$$\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle$$

$$= \langle 0 | \phi(x)\phi(y) | 0 \rangle \, \theta(x^0 - y^0) + \langle 0 | \phi(y)\phi(x) | 0 \rangle \, \theta(y^0 - x^0)$$

$$= \int \widetilde{dp} \left[e^{-i\vec{p}(\vec{y}-\vec{x})} e^{i\omega(y^0 - x^0)} \theta(x^0 - y^0) + e^{i\vec{p}\cdot(\vec{y}-\vec{x})} e^{-i\omega(y^0 - x^0)} \theta(y^0 - x^0) \right]$$

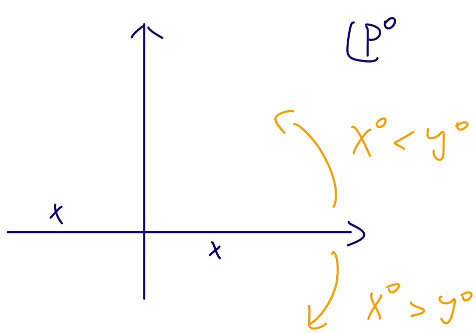
$\Rightarrow$ massless

$$\frac{i}{2\pi \Delta x}\Big[\big(\delta(\Delta t + \Delta x) - \delta(\Delta t - \Delta x)\big)\theta(\Delta t) - \big(\delta(\Delta t + \Delta x) - \delta(\Delta t - \Delta x)\big)\theta(-\Delta t)\Big]$$

$$= \frac{-i}{2\pi \Delta x}\left\{ \delta(\Delta t - \Delta x) + \delta(\Delta t + \Delta x) \right\}$$

$$= \int \frac{d^4p}{(2\pi)^4} \left\{ \delta(p^0 - \sqrt{\vec{p}^2 + m^2}) \, \theta(x^0 - y^0) \, e^{-ip\cdot(x-y)} + \delta(p^0 - \sqrt{\vec{p}^2 + m^2}) \, \theta(y^0 - x^0) \, e^{ip(x-y)} \right\}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$



△ One particle state :

$$|P\rangle = a_p^\dagger |0\rangle$$

$$\langle P | P' \rangle = \langle 0 | a_{p'} a_p^\dagger | 0 \rangle$$

$$= (2\omega)\cdot(2\pi)^3 \delta(\vec{p}-\vec{p}\,')$$

$$\langle P | \phi(x) | 0 \rangle$$

$$= \langle 0 | a_p \int \tilde{dk} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right) | 0 \rangle$$

$$= \int \tilde{dk}\; e^{ik\cdot x} (2\pi)^3 \cdot 2\omega\, \delta(\vec{p}-\vec{k})$$

$$= e^{iP\cdot x}$$

" ϕ interpolate between vacuum & one-particle state.

☒ Lorentz inv.

$$\langle P | e^{-i\hat{P}\cdot x} \phi(0)\, e^{i\hat{P}\cdot x} | 0 \rangle$$

$$= e^{iP\cdot x} \underbrace{\langle P | \phi(0) | 0 \rangle}_{Z(P)}$$

⇒ holds even for interacting theory !

* Exercise :

Try $\langle 0 | T(\phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4)) | 0 \rangle$