Feynman rules from

1) Path integral

1) Feynman Rules : [Srednicki Chap 1-8 to 1-10.]

Consider a scalar field theory

$$Z(J) = \int \mathcal{D}\phi \, \exp\left(i \left(S + \int J \cdot \phi \right) \right)$$

$$= \langle 0 | 0 \rangle_J$$

$$\mathcal{L}_{int} = -\frac{\lambda}{4!} \phi^4 + c\, (\partial\phi)^4 + \cdots$$

$$S = S_0 + \int d^4x \, \mathcal{L}_{int} \Rightarrow$$

$$Z_0(J) = \int \mathcal{D}\phi \left(\exp\left(i S_0 + i \int J \cdot \phi \right) \right)$$

$$Z(J) = \int \mathcal{D}\phi \left(\exp\left(i S_0 + i \cdot \int J \phi \right) \right) \exp\left(i \cdot S_{int} \right)$$

$$= \int \mathcal{D}\phi \, \exp\left(i \int \mathcal{L}_{int}\left(\frac{\delta}{i\delta J}, \partial_\mu \left(\frac{\delta}{i\delta J} \right) \right) \right) \times \exp\left(i S_0 + i \cdot \int J \phi \right)$$

$$= \exp\left(i \int \mathcal{L}_{int}\left(\frac{\delta}{i\delta J}, \partial_\mu \left(\frac{\delta}{i\delta J} \right) \right) \right) \times Z_0(J)$$

Correlation fuction :

$$\langle 0 | T(\phi(x_1) \cdots \phi(x_n)) | 0 \rangle$$

$$= \int \mathcal{D}\phi \;\; \phi(x_1) \cdots \phi(x_n) \; e^{iS}$$

$$= \left(\frac{\delta}{i\,\delta J(x_1)} \cdots \frac{\delta}{i\,\delta J(x_n)} \right) \cdot e^{i(S + \int J\phi)} \Big|_{J=0}$$

$$= \left(\frac{\delta}{i\,\delta J(x_1)} \cdots \frac{\delta}{i\,\delta J(x_n)} \right) Z(J)$$

$$= \left(\frac{\delta}{i\,\delta J(x_1)} \cdots \frac{\delta}{i\,\delta J(x_n)} \right)$$

$$\underbrace{\exp\left(i \int \mathcal{L}_{int}\left(\frac{\delta}{i\delta J} , \partial_\mu\left(\frac{\delta}{i\delta J}\right)\right) \right)}_{\text{expanded by perturbation.}} \times \underbrace{Z_0(J)}_{\text{free theory}}$$

$$\text{free theory} \;\; = \;\; \exp\left(-\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{J(k)\, J(-k)}{k^2 - m^2 + i0} \right)$$

$$= \exp\left(\frac{1}{2} \;\; \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \right)$$

$$\text{———} \; : \; \frac{1}{i} G(x, y)$$

$$\bullet \qquad i \int d^4x \, J(x)$$

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-1}{k^2-m^2+i0} e^{ik(x-y)}$$

$$(\Box + m^2)\, G(x,y) = + \delta(x-y)$$

e.g. Free theory :

$$\langle 0 | T(\phi(x_1)\, \phi(x_2)) | 0 \rangle$$

$$= \frac{1}{i} G(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i0} e^{ik(x-y)}$$

$$\langle 0 | T(\phi(x_1) \cdots \phi(x_{2n})) | 0 \rangle$$

$$= \frac{1}{i^n} \Big\{ G(x_1-x_2)\, G(x_3-x_4) \cdots G(x_{2n-1}-x_{2n})$$

$$+ \text{distinguishable pairings} \Big\}$$

$\Rightarrow$ Wick's theorem.

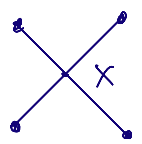
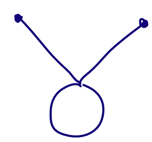
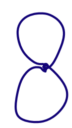
Example ① : $\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$

$$\Rightarrow \quad -\frac{\lambda}{4!}\left(\frac{\delta}{i\,\delta J(x)}\right)^4$$

$Z(J)$

$$= \exp\left(i \int_x \left(\frac{-\lambda}{4!}\right)\left(\frac{\delta}{i\,\delta J(x)}\right)^4 \right) \cdot Z_0(J)$$

$$= Z_0(J)$$

$$+\ i \cdot \left(\frac{-\lambda}{4!}\right) \cdot \int_x \Bigg[$$  $$+ \frac{1}{2!\,2!}$$  $$+ \frac{1}{2!\,2!}$$  $$\Bigg]$$

$\langle 0 | T(\phi(x_1) \cdots \phi(x_4)) | 0 \rangle$

$$= i\, \underbrace{4!}_{\text{4! permutations.}} \left(\frac{-\lambda}{4!}\right) \int d^4x \cdot \left(i\,\underbrace{G(x_1 - x)}_{\int \frac{d^4k_1}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i0} e^{ik_1(x_1 - x)}}\right)\left(i\,G(x_2 - x)\right)\left(i\,G(x_3 - x)\right)\left(i\,G(x_4 - x)\right)$$

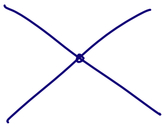
$\Rightarrow$

$$= \prod_{a=1}^{4} \int \frac{d^4k_a}{(2\pi)^4} e^{ik_a x_a} \cdot \Big\{ (2\pi)^4 \delta(k_1 + k_2 + k_3 + k_4) \cdot$$

$$i \cdot (-\lambda) \cdot \left(\frac{i}{k_1^2 - m^2 + i0}\right)\left(\frac{i}{k_2^2 - m^2 + i0}\right)\left(\frac{i}{k_3^2 - m^2 + i0}\right)\left(\frac{i}{k_4^2 - m^2 + i0}\right)$$

Feynman rules:

 : $\frac{i}{k^2-m^2+i0}$

 : $i(-\lambda)$

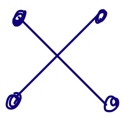
Consider higher orders:

$$Z(J) = \exp\left(-\frac{i\lambda}{4!}\int d^4x \left(\frac{\delta}{i\,\delta J(x)}\right)^4\right) \exp\left(-\frac{i}{2}\int_{x,y}\right.$$

$$\left.\right)$$

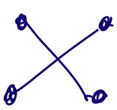
$$= Z_0(J)$$

$$-\frac{i\lambda}{4!}$$



$$\cdot\, Z_0(J)$$

$$+\frac{1}{2!}\left(-\frac{i\lambda}{4!}\right)^2 \Big($$



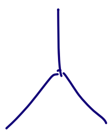
$$\Big)^2 Z_0(J)$$

$$= \exp\left(\sum D\right) \times Z_0(J)$$

sum over connected interacting diagram

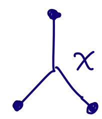
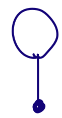
Similarly :

$$V = -\frac{g}{3!}\phi^3$$



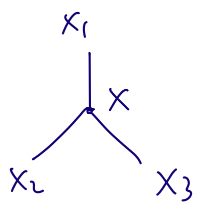
$-ig$

e.g.

$$Z(J) = Z_0(J) \times \Big\{ 1 + \frac{(-ig)}{3!}\int_x \Big($$  $$+ \frac{1}{2}$$  $$\Big)$$

$$+ \quad \cdots \quad \Big\}$$

$$\langle 0 | T(\phi(x_1)\,\phi(x_2)\,\phi(x_3)) | 0 \rangle$$

$$= (-ig) \cdot \int d^dx$$ 

$$= (-ig)\int d^4x \; e^{i(P_1+P_2+P_3)\cdot x}\, e^{-iP_1x_1}\, e^{-iP_2x_2}\, e^{-iP_3x_3}$$

$$\times \left(\frac{i}{P_1^2-m^2+i\epsilon}\right)\left(\frac{i}{P_2^2-m^2+i\epsilon}\right)\left(\frac{i}{P_3^2-m^2+i\epsilon}\right)$$

$$= (-ig)\,(2\pi)^4\,\delta(P_1+P_2+P_3)\left(\frac{i}{P_1^2-m^2+i\epsilon}\right)\left(\frac{i}{P_2^2-m^2+i\epsilon}\right)\left(\frac{i}{P_3^2-m^2+i\epsilon}\right)$$

What about

$$\mathcal{L}_{int} = C \cdot (\partial\phi)^4$$

$$Z(J) = 1 + \int d^4x \, (iC) \cdot \left[\partial_\mu \cdot \left(\int_{x'} \Delta(x - x_1) \, J(x') \right) \right]^4$$

$$\langle T(\phi_1 \, \phi_2 \cdots \phi_4) \rangle$$

$$= \int d^4x \cdot (iC) \cdot 8 \cdot \Big[\left(\partial_\mu \Delta(x - x_1) \cdot \partial^\mu (\Delta(x - x_2)) \right) \left(\partial_\mu \Delta(x - x_3) \cdot \partial^\mu (\Delta(x - x_4)) \right) + (1234 \to 1324) + (1234 \to 1423) \Big]$$

overall 4! permutations.

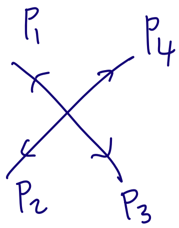
Recall $$\frac{\Delta(x-y)}{i} = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i0} e^{ik(x-y)}$$

$$\frac{\partial}{\partial y^\mu}\left(\frac{\Delta(x-y)}{i}\right) = \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{(k^2-m^2+i0)}\,(-ik_\mu)\,e^{ik(x-y)}$$

$$\therefore \quad \partial_\mu \phi \;\to\; (-ik_\mu)$$

[Sign depends on incoming / outgoing convention]

$$\mathcal{L}_{int} = c\,(\partial\phi)^4$$



Feynman rule :

$$(iV) = (ic)\cdot 8\cdot(-i)^4 \left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right]$$

$$\mathcal{L}_{int} = \tilde{C}\cdot\left(\partial_{\mu\nu}\phi\;\partial^{\mu\nu}\phi\right)^2$$

$$\to \quad iV = i\,\tilde{C}\cdot 8\cdot(-i)^8 \left[(p_1\cdot p_2)^2(p_3\cdot p_4)^2 + (p_1\cdot p_3)^2(p_2\cdot p_4)^2 + (p_1\cdot p_4)^2(p_2\cdot p_3)^2\right]$$

This can be generalized to

multiple scalar

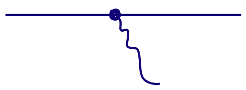
$\Big\{$ spin

Example : QED

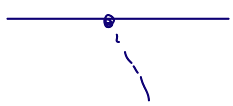
$$\mathcal{L} = i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi + y\,\phi\bar{\psi}\psi$$

$$= i\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi$$

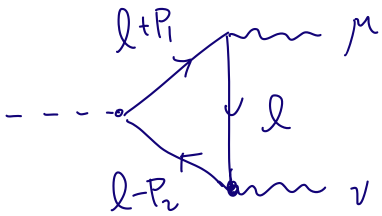
$$- m\bar{\psi}\psi + y\,\phi\bar{\psi}\psi$$



$$(iV) = i \times e \underbrace{\gamma^\mu}_{\text{matrix in spinor space}}$$



$$iV = i \times y \times \underbrace{\mathbb{1}}_{\text{identity matrix}}$$



$$\int \frac{d^4 l}{(2\pi)^4} \cdot i^{6} \cdot \mathrm{Tr}\left[\frac{\gamma^\mu \times (\not{l}-m) \times \gamma^\nu ((\not{l}-\not{P}_2)-m)\cdot((\not{l}+\not{P}_1)-m)}{(l^2-m^2+i0)\,((l+P_1)^2-m^2+i0)\,((l-P_2)^2-m^2+i0)}\right]$$

in the spinor space