- Correlation functions

  → Observables

Lehmann - Symanzik - Zimmermann formula

(LSZ)

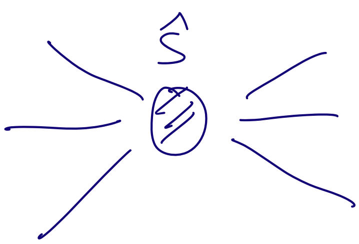
△ Lehmann - Symanzik - Zimmermann formula

(LSZ) [Peskin Sec. 7.2]

Q: How to convert correlation functions to observables ?

Observables → S-matrix

$\langle f | S | i \rangle$

$|i\rangle$  $|f\rangle$

initial & final states : well-separated wave packets.

Key : the singularity of the correlation functions encode long-distance behavior.

Consider

$$\int d^4x \, e^{iP\cdot x} \langle 0 | T(\phi(x_1) \cdots \phi(x) \cdots \phi(x_n)) | 0 \rangle$$

$$\Rightarrow \int dt = \underbrace{\int_{T^+}^{\infty} dt}_{\text{singular region}} + \underbrace{\int_{T^-}^{T^+} dt}_{\text{other } x_i \text{ are in here}} + \underbrace{\int_{-\infty}^{T^-} dt}_{\text{singular region}}$$

$$\mathbb{1} = \sum_{\lambda} \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{2E_{\vec{q}}(\lambda)} |\lambda_{\vec{q}}\rangle\langle\lambda_{\vec{q}}|$$

λ: other quantum numbers.

$$\int_{T^+}^{\infty} dt \, d^3x \, e^{iP^0 t} e^{-i\vec{P}\cdot\vec{x}} \langle 0 | \phi(x) T(\phi(x_1) \cdots) | 0 \rangle$$

$$= \int_{T^+}^{\infty} dt \, d^3x \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{2E_{\vec{q}}} e^{iP^0 t} e^{-i\vec{P}\cdot\vec{x}}$$

$$\langle 0 | \phi(x) | \lambda_{\vec{q}} \rangle \langle \lambda_{\vec{q}} | T(\phi(x_1) \cdots) | 0 \rangle$$

By Lorentz invariance

$$\langle 0|\phi(x)|\lambda_{\vec q}\rangle$$

$$= \langle 0| e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}\cdot x}|\lambda_{\vec q}\rangle$$

$$= \langle 0|\phi(0)|\lambda_{\vec q}\rangle e^{-iq\cdot x}$$

$$= \langle 0|\phi(0)\underbrace{U^{-1}U}_{\text{boost}}|\lambda_{\vec q}\rangle e^{-iq\cdot x}$$

$$= \langle 0|\phi(0)|\lambda_{\vec 0}\rangle e^{-iq\cdot x}$$

e.g. free theory

$$\langle 0|\phi(x)|\lambda_{\vec q}\rangle$$

$$= \int d\tilde{k}\, \langle 0| a_{\vec k}\, e^{-ik\cdot x}|\vec q\rangle$$

$$= e^{-iq\cdot x}\Big|_{q^0 = E_{\vec q}}$$

$$\int_{T^+}^{\infty} dt\, d^3x \int \frac{d^3\vec q}{(2\pi)^3}\frac{1}{2E_{\vec q}}\, e^{iP^0 t}\, e^{-i\vec P\cdot\vec x}\, e^{-i\overset{E_{\vec q}}{\overset{\parallel}{q^0}}t}\, e^{+i\vec q\cdot\vec x}$$

$$\langle 0|\phi(0)|\lambda_{\vec q}\rangle\langle\lambda_{\vec q}|T(\phi_1\cdots)|0\rangle$$

$$= \int dt \int \frac{d^3\vec q}{(2\pi)^3}\frac{1}{2E_{\vec q}}\, e^{i(P^0 - E_{\vec p})t}\cdot(2\pi)^3\delta(\vec p - \vec q)$$

$$\langle 0|\phi(0)|\lambda_{\vec q}\rangle\langle\lambda_{\vec q}|T(\phi_1\cdots)|0\rangle$$

$$= \int_{T^+}^{\infty} dt\, e^{i(P^0 - E_{\vec p})t}\frac{1}{2E_{\vec p}}\langle 0|\phi(0)|\lambda_{\vec p}\rangle\langle\lambda_{\vec p}|T(\phi_1\cdots)|0\rangle$$

$$= \frac{+i \quad e^{i(p^0 - E_{\vec p})T^+}}{2E_{\vec p}\left[(p^0 - E_{\vec p}) + \underbrace{i\epsilon}_{\text{needed to suppress } t=+\infty}\right]} \quad \langle 0|\phi(0)|\lambda_{\vec p}\rangle\langle\lambda_{\vec p}|T(\phi_1 \cdots)|0\rangle$$

$$p^0 \to E_{\vec p}$$

$$\int d^4x \; e^{ip\cdot x} \langle 0|\, T(\phi(x_1) \cdots \phi(x) \cdots)\,|0\rangle$$

$$\to \frac{i}{p^2 - m^2 + i\epsilon} \cdot \sqrt{R} \qquad \langle \vec p\,|\, T(\phi(x_1)\cdots)\,|0\rangle$$

where $\quad \langle 0|\phi(0)|\lambda_p\rangle = \sqrt{R}$

Similar analysis for $\displaystyle\int_{-\infty}^{T^-} dt$

$$\Rightarrow \quad p^0 \to -E_{\vec p}$$

$$\int d^4x \; e^{ip\cdot x} \langle 0|\, T(\phi(x_1) \cdots \phi(x) \cdots)\,|0\rangle$$

$$\to \frac{i}{p^2 - m^2 + i\epsilon} \cdot \sqrt{R} \qquad \langle 0\,|\, T(\phi(x_1)\cdots)\,|-\vec p\,\rangle$$

Doing this for each field

$$\prod_{i=1}^{n} \lim_{p_i^2 \to m_i^2} \int d^4x_i \, e^{i p_i \cdot x_i} \, \frac{1}{i}(p_i^2 - m_i^2) \frac{1}{\sqrt{R}}$$

$$\times \quad \langle 0 | T(\phi(x_1) \cdots \phi(x_n)) | 0 \rangle$$

$$= \quad S(0 \to p_1, p_2, \cdots p_n)$$

Crucial Point :

The local field $\phi(x)$ is not unique,

e.g. $\phi \to \phi + \phi^2$

for LSZ reduction, we only need

$$\langle 0 | \phi(x) | \vec{p} \rangle = e^{-ip\cdot x} \sqrt{R}$$

overlap the one-particle state and vacuum.

$\therefore \; \phi \to \phi + \phi^2 + \phi^3 + (\partial\phi)^2$

does not change the S-matrix !

△ Interpretation of Z:

$$\langle 0|\phi(x)|P\rangle = e^{-ipx}\sqrt{R}$$

$$\langle 0|T(\phi(x)\phi(y))|0\rangle$$

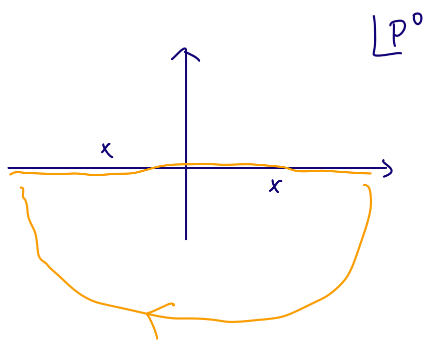
$$= \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} \langle 0|\phi(x)|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|\phi(y)|0\rangle$$

assume $x^0 > y^0$

$$= \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} e^{-ip(x-y)} \cdot \left|\langle 0|\phi(0)|\lambda(\vec{p})\rangle\right|^2$$

$$= \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} e^{-ip(x-y)} \cdot R$$

$$= \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{i\cdot e^{-ip(x-y)}}{p^2-m^2+i0} R$$



→ applies to $x^0 > y^0$ or $x^0 < y^0$

$$= \int \frac{dM^2}{2\pi}\cdot \rho(M^2)\cdot D_F(x-y; M^2)$$

Kallen-Lehmann representation.

$$\rho(M^2) = \sum_\lambda 2\pi\,\delta(M^2-m_\lambda^2)\,|\langle 0|\phi(0)|\lambda_{\vec{p}}\rangle|^2$$

If there's only a single particle state :

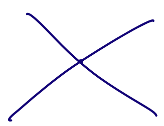
$$\langle 0 | T(\phi(x)\, \phi(y)) | 0 \rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} \cdot \frac{i}{p^2 - m^2 + i0}\, e^{-ip(x-y)} \cdot R$$

R is related to the renormalization factor at $p^2 = m^2$.

Example :

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$$



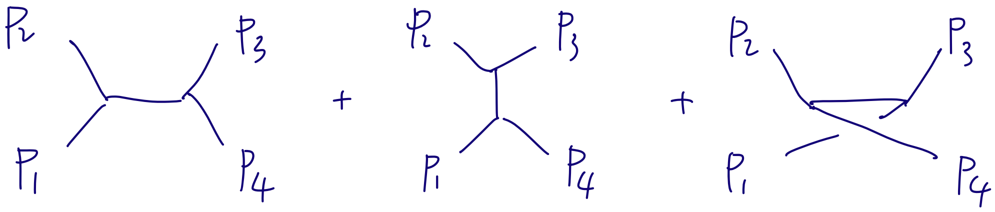
$$\langle T(\phi_1 \cdots \phi_4) \rangle$$

$$= (-i\lambda)\left(\frac{i}{p_1^2 - m^2 + i0}\right)\left(\frac{i}{p_2^2 - m^2 + i0}\right)\left(\frac{i}{p_3^2 - m^2 + i0}\right)\left(\frac{i}{p_4^2 - m^2 + i0}\right)$$

$$\xrightarrow[LSZ]{} (-i\lambda)$$

LSZ simply removes the propagator of external legs.

$$\mathcal{L}_{int} = -\frac{g}{3!}\phi^3$$



$$i\mathcal{A} = (-ig)^2\left[\frac{i}{(P_1+P_2)^2-m^2} + \frac{i}{(P_1+P_4)^2-m^2} + \frac{i}{(P_1+P_3)^2-m^2}\right]$$

When $(P_1+P_2)^2 - m^2 \longrightarrow 0$

$$i\mathcal{A}_4 \longrightarrow (-ig)\frac{i}{(P_1+P_2)^2-m^2}(-ig) + \text{finite}$$

$$(i\mathcal{A}_3)\frac{i}{(P_1+P_2)^2-m^2}(i\mathcal{A}_3)$$

$\Rightarrow$ unitarity !

(factorization)