Eikonal quasinormal modes and photon orbits of deformed Schwarzschild black holes

Che-Yu Chen

Institute of Physics, Academia Sinica, Taiwan

- **CYC, HWC, JST,** arXiv:2205.02433

The Future is Illuminating, NCTS, Taiwan, 28 June, 2022

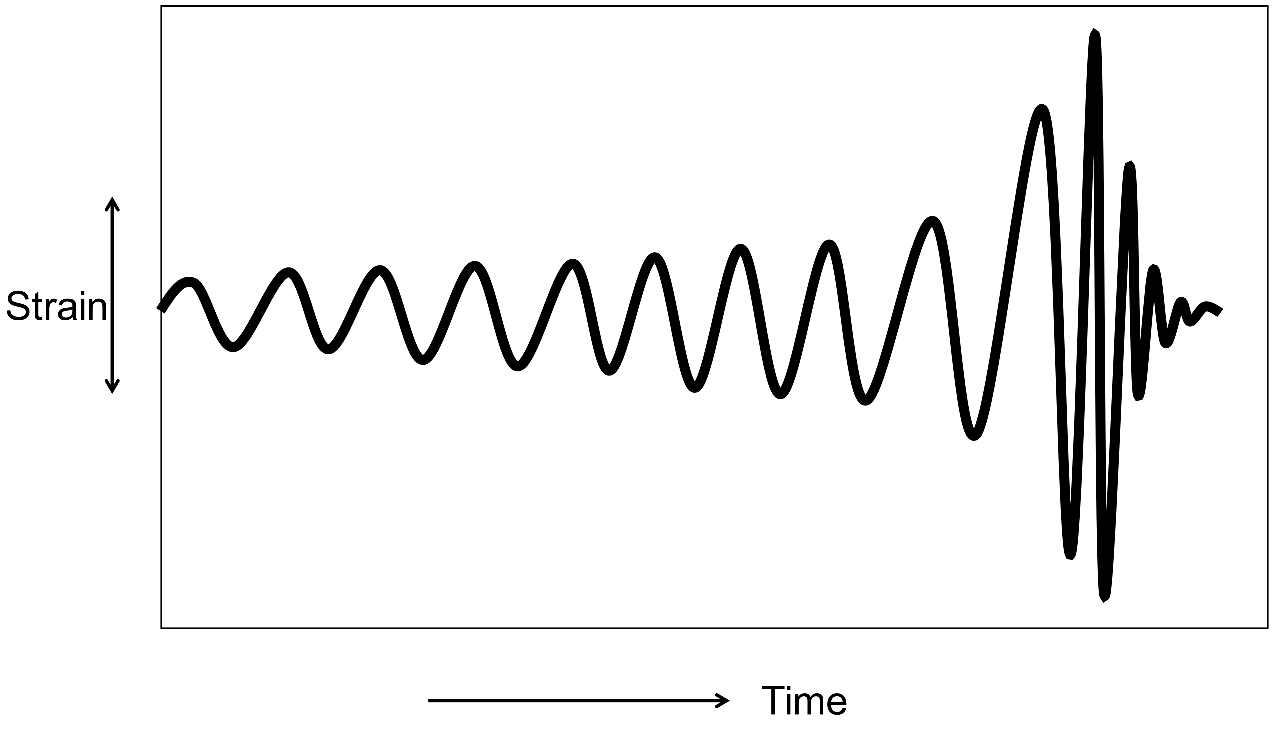
- Geometric-optics approximations for BHs
- Black hole spacetimes with less symmetry
  - Deformed Schwarzschild spacetime
- Conclusions

- Geometric-optics approximations for BHs

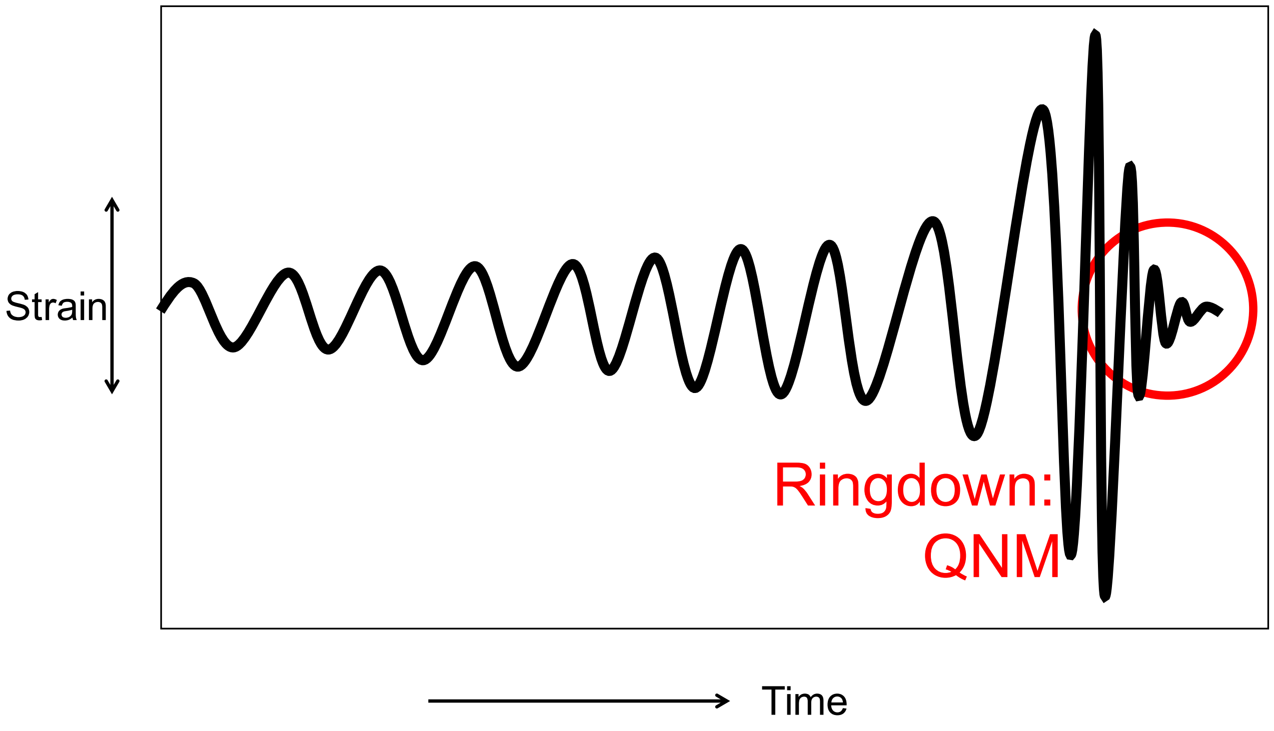
- Black hole spacetimes with less symmetry
  - Deformed Schwarzschild spacetime

- Conclusions

Black Hole Merger



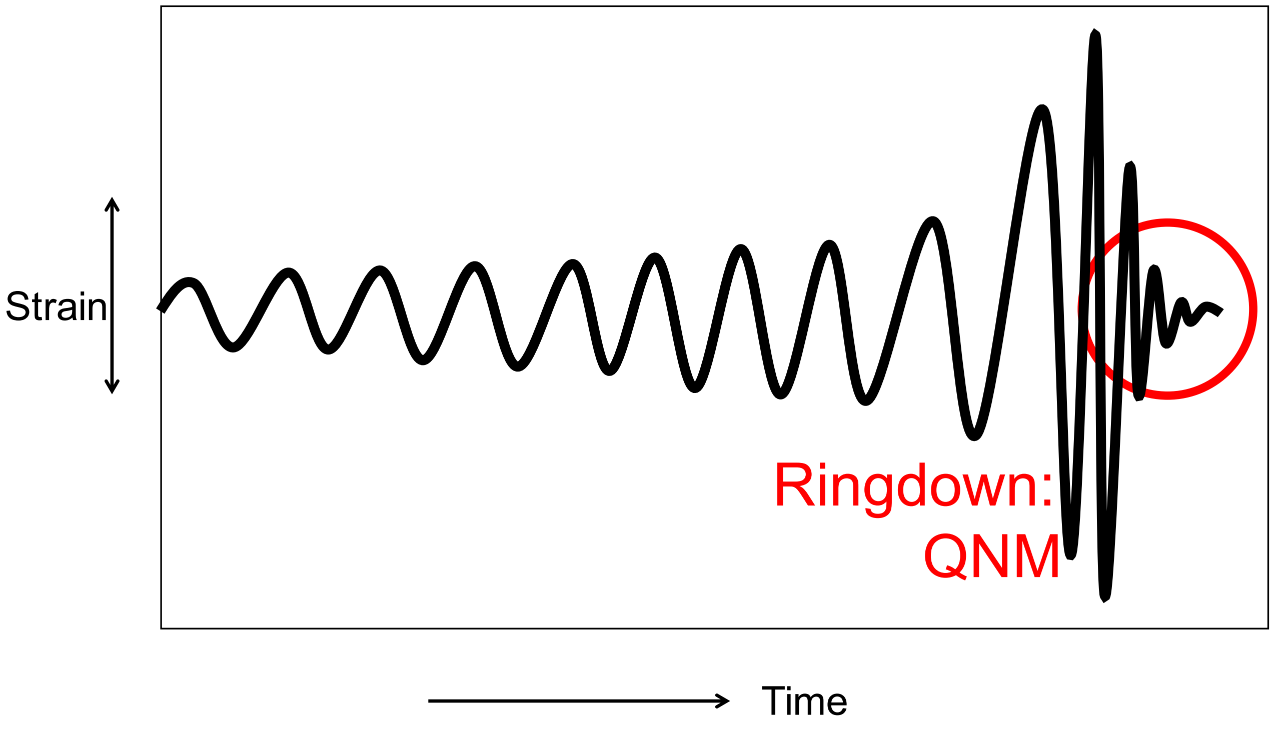
Black Hole Merger: Ringdown and QNMs

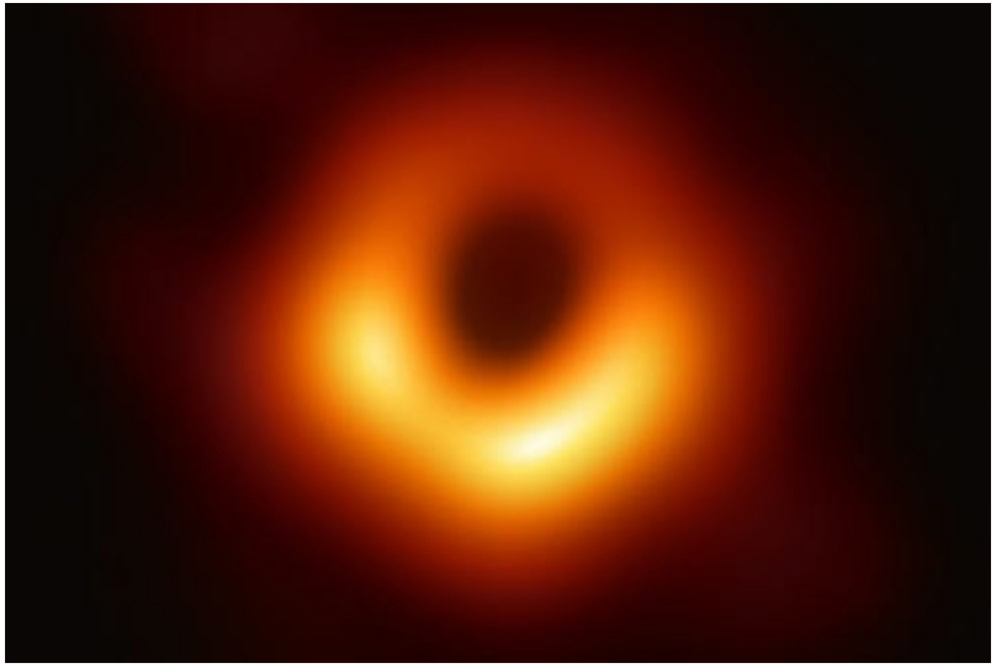


Field propagation in BH spacetimes

$$\nabla^{\alpha}\nabla_{\alpha}A = \cdots$$

Black Hole Images

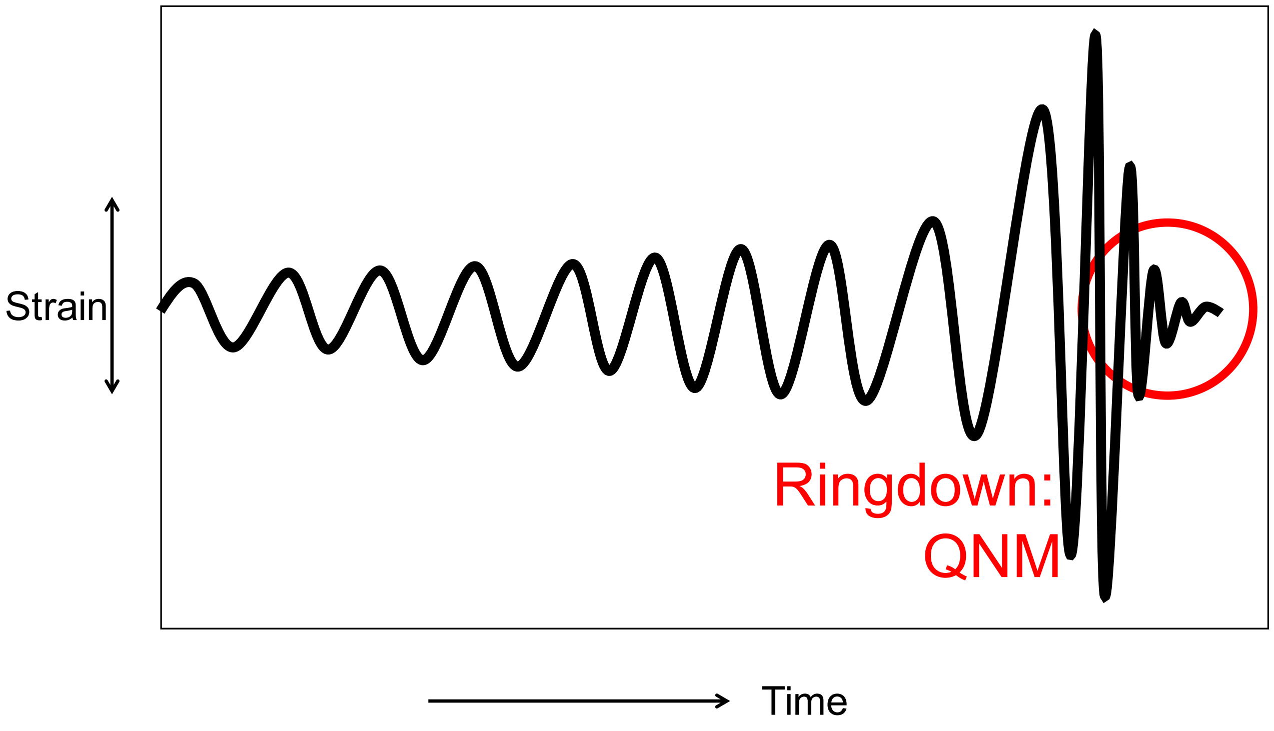


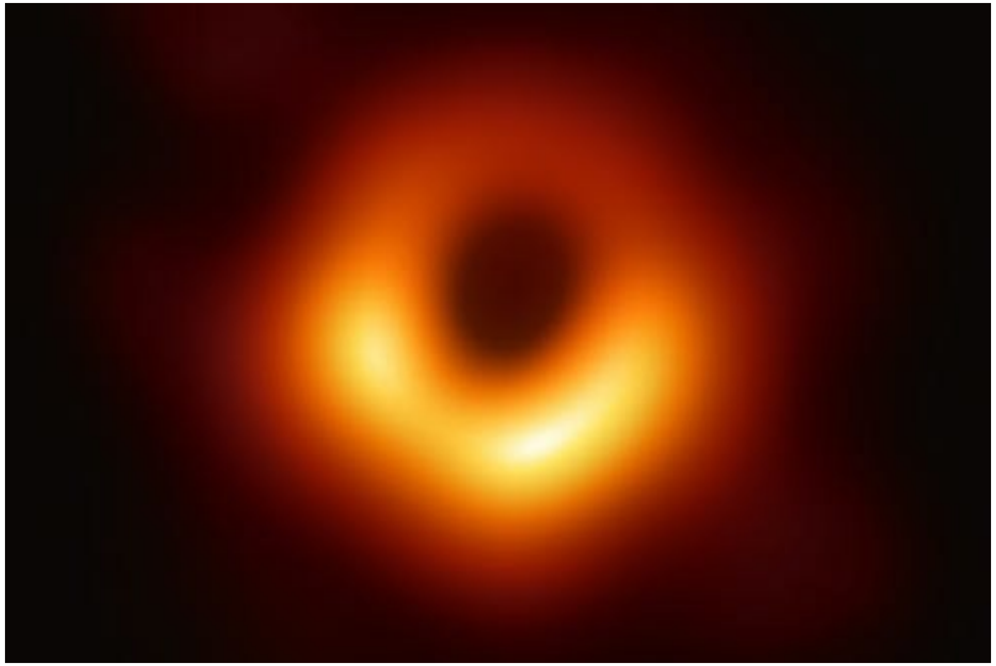


Field propagation in BH spacetimes

$$\nabla^{\alpha}\nabla_{\alpha}A = \cdots$$

Black Hole Images





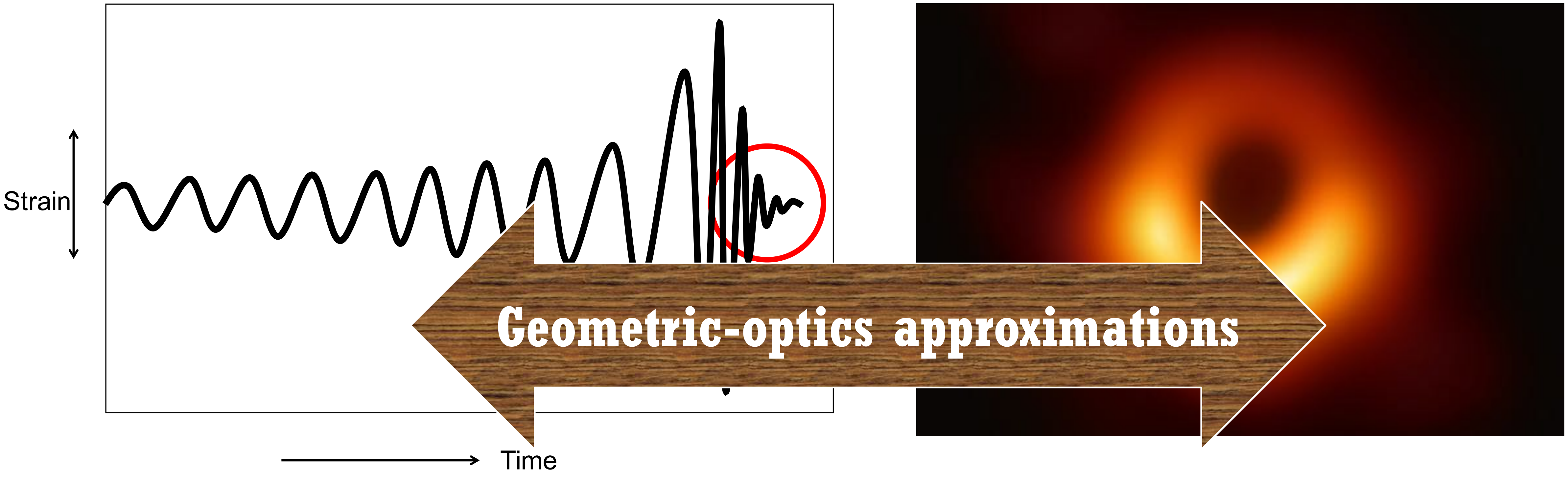
Field propagation in BH spacetimes

$$\nabla^\alpha \nabla_\alpha A = \cdots$$

Photon propagation in BH spacetimes

$$k^\alpha k_\alpha = 0$$

Geometric-Optics (Eikonal) Approximations



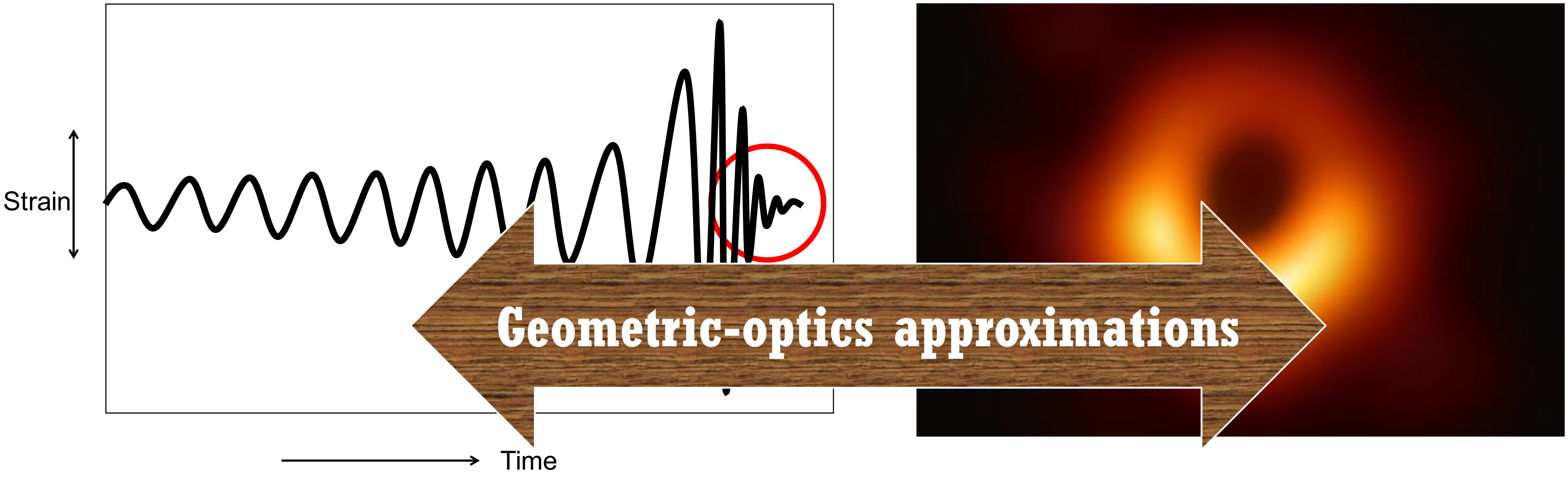
Field propagation in BH spacetimes

$$\nabla^\alpha \nabla_\alpha A = \cdots$$

Photon propagation in BH spacetimes

$$k^\alpha k_\alpha = 0$$

Geometric-Optics (Eikonal) Approximations



Field propagation in BH spacetimes

$$\nabla^\alpha \nabla_\alpha A = \boldsymbol{O(\lambda/L)} \sim \boldsymbol{0}$$

Photon propagation in BH spacetimes

$$k^\alpha k_\alpha = 0$$

How does the correspondence manifest in BH spacetimes?

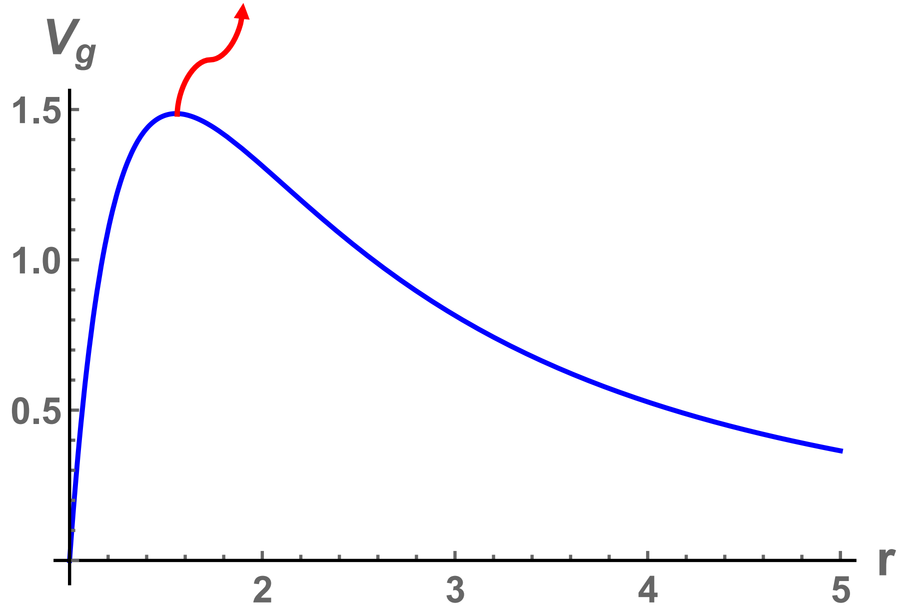
- <u>Spacetime symmetry is crucial</u>
- Non-rotating BH:

Static and spherically symmetric $ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2$

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\Psi = V_g \Psi$$

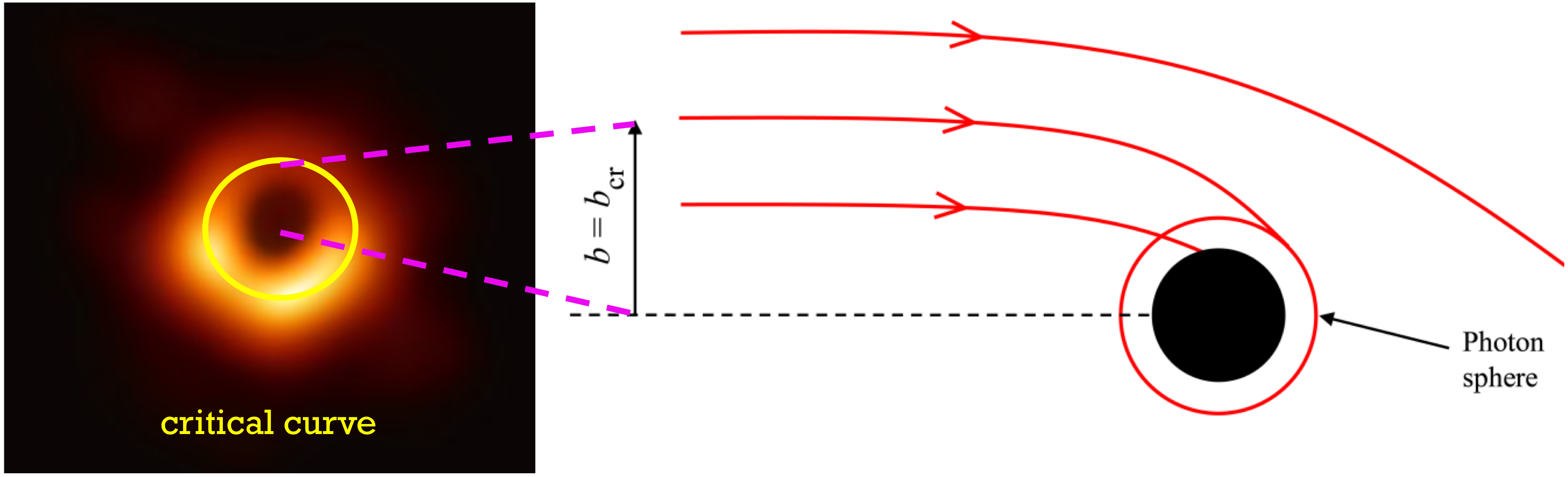
- The potential for eikonal ($l \to \infty$) QNMs: $V \approx \frac{A(r)}{r^2} l^2$
- The peak of the potential coincides with the photon sphere
  - Photon sphere equation: $\partial_r [A(r)/r^2] = 0$

approach photon sphere when ($l \to \infty$)



Eikonal QNMs Correspondence

- The eikonal QNMs ($l \to \infty$) and the photon sphere



$$\omega \approx \Omega_c l - i(n + 1/2)|\lambda_c|$$

- $Re(\omega)$: $\leftrightarrow$ Ω_c (orbital frequency of the photon sphere)
- $Im(\omega)$: $\leftrightarrow$ λ_c (Lyapunov exponent)

Cardoso, Miranda, Berti, Witek, Zanchin (2009)

Correspondence in Kerr Spacetime

- Separable geodesic equations (Carter constant), and separable wave equations

Wave Quantity	Ray Quantity	Interpretation
ω_R	$\mathscr{E}$	Wave frequency is same as energy of null ray (determined by spherical photon orbit).
m	L_z	Azimuthal quantum number corresponds to z angular momentum (quantized to get standing wave in ϕ direction).
A^R_{lm}	$\mathscr{Q}+L_z^2$	Real part of angular eigenvalue related to Carter constant (quantized to get standing wave in θ direction).
ω_I	$\gamma=-\mathscr{E}_I$	Wave decay rate is proportional to Lyapunov exponent of rays neighboring the light sphere.
A^I_{lm}	$\mathscr{Q}_I$	Nonzero because $\omega_I \neq 0$ (see Secs. II B 2 and III C 3 for further discussion).

Yang *et al*. (2012)

Recently extended to Kerr-Newman by Li *et al*. (2021)

Eikonal QNMs and BH Shadows

- A simple relation formula (non-rotating BH)

$$\omega_R = \lim_{l \gg 1} \frac{l}{R_S}$$

Jusufi (2020), Cuadros-Melgar *et al.* (2020)

- A general mapping exists for Kerr BHs

Jusufi (2020), Yang (2021)

- <u>What if the black hole spacetime has less symmetry?</u>

- Geometric-optics approximations for BHs
- Black hole spacetimes with less symmetry
  - Deformed Schwarzschild spacetime
- Conclusions

Deformed Schwarzschild Spacetime

$$g_{tt} = -\left(1 - \frac{2M}{r}\right)\left(1 + \epsilon A_j(r)\cos^j\theta\right)\ ,$$

$$g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1}\left(1 + \epsilon B_j(r)\cos^j\theta\right)\ ,$$

$$g_{\theta\theta} = r^2\left(1 + \epsilon C_j(r)\cos^j\theta\right)\ ,$$

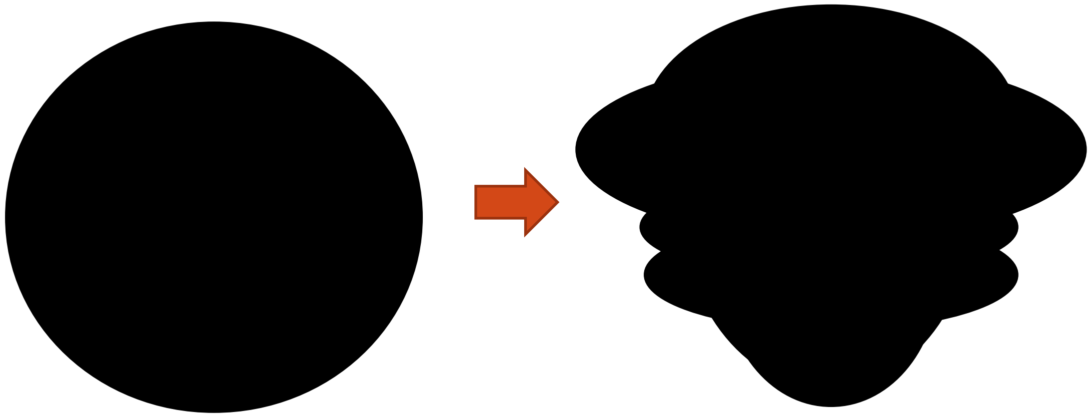
$$g_{\varphi\varphi} = r^2\sin^2\theta\left(1 + \epsilon D_j(r)\cos^j\theta\right)\ ,$$

$$g_{tr} = \epsilon a_j(r)\cos^j\theta\,, \qquad g_{t\theta} = \epsilon b_j(r)\cos^j\theta\,,$$

$$g_{r\theta} = \epsilon c_j(r)\cos^j\theta \qquad g_{r\varphi} = \epsilon d_j(r)\cos^j\theta\,,$$

$$g_{\theta\varphi} = \epsilon e_j(r)\cos^j\theta\,.$$

- A general axisymmetric deformation which excludes frame-dragging effects



- Small deformation: $|\epsilon| \ll 1$

Separability Issue of Master Wave Equation

- How to deal with the separability issue?
- Up to 1st order of ϵ, one can use the projection method and obtain the ϵ-corrections on the zeroth order equation

Cano, Fransen, Hertog (2020)

- A Schrödinger-like master equation is attainable (up to 1st order of ϵ)

$$(\partial_{r_*}^2 + \omega^2)\Psi_{lm} = V_{\mathrm{eff}}(r, l, m)\Psi_{\mathrm{lm}}$$

$|m| = l \gg 1$ Circular Photon Orbits

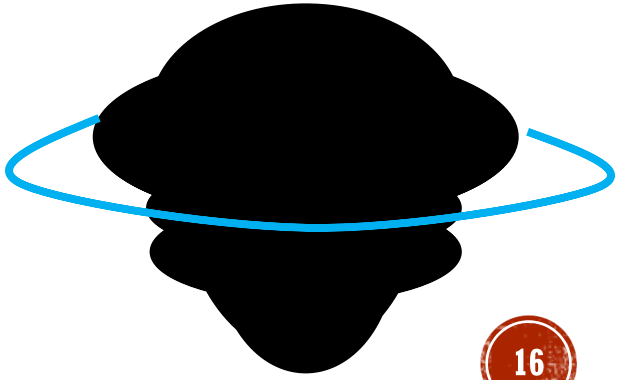
$$V_{\text{eff}}(r) \approx \frac{l^2}{r^2}\left(1-\frac{2M}{r}\right)[1+\epsilon(A_0(r)-D_0(r))]$$

- Only deformations in g_{tt} and $g_{\varphi\varphi}$ components appear

Photon motion

- Planar circular photon orbits still exist, but not on equatorial plane (due to odd power of j in $\cos^{j}\theta$)
- Up to 1st of in ϵ, the radius of orbits is determined by

$$\partial_r\left[\frac{\left(1-\frac{2M}{r}\right)}{r^2}(1+\epsilon A_0-\epsilon D_0)\right]=0$$



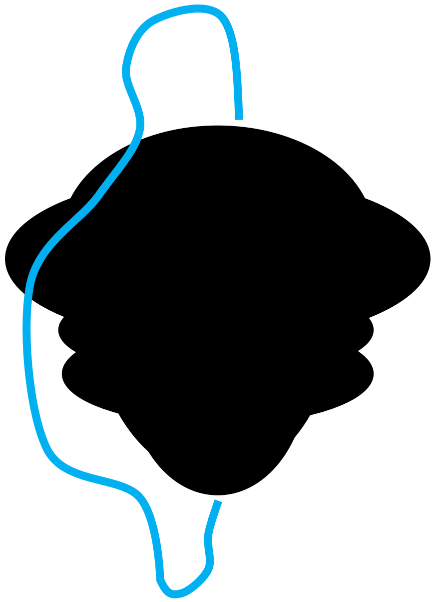
$m = 0, \quad l \gg 1$ Polar Orbits

$$V_{\text{eff}}(r) \approx \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right)\left[1 + \epsilon \sum_{k=0}^{\infty} \frac{1}{4^k} C_k^{2k} (A_{2k} - C_{2k})\right]$$

- Only deformations in g_{tt} and $g_{\theta\theta}$ components appear

<u>Photon motion</u>

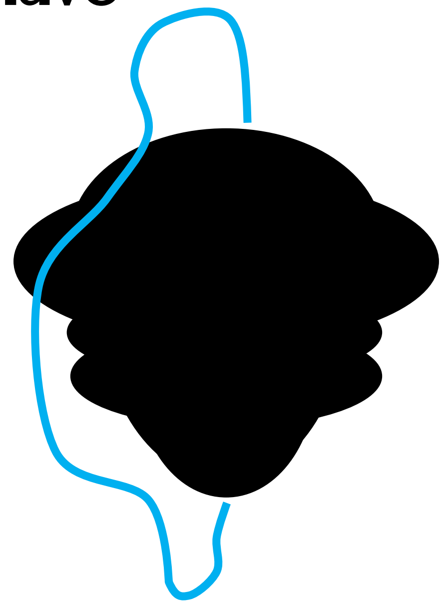
- Radial and latitudinal sectors are not separable
- Photon orbits $r(\theta)$ do not have constant r



$m = 0, \quad l \gg 1$ Polar Orbits

- These photon orbits should
  - be periodic
  - form a class of limit cycles
- We can integrate the orbits along full periods: $\oint d\lambda = \oint \dot{\theta}^{-1} d\theta$
- Integrating the radial component of geodesic eqs., we have

$$
\begin{aligned}
o(\epsilon) &\propto \int_0^{2\pi} d\theta \partial_r \ln \left| \frac{g_{tt}}{g_{\theta\theta}} \right| \\
&= \int_0^{2\pi} d\theta \partial_r \left[\frac{f}{r^2} \left(1 + \epsilon \left(A_j - C_j\right) \cos^j \theta\right) \right] \\
&\propto \partial_r \left[\frac{f}{r^2} \left(1 + \epsilon \sum_{k=0}^{\infty} \frac{1}{4^k} C_k^{2k} (A_{2k} - C_{2k}) \right) \right]
\end{aligned}
$$



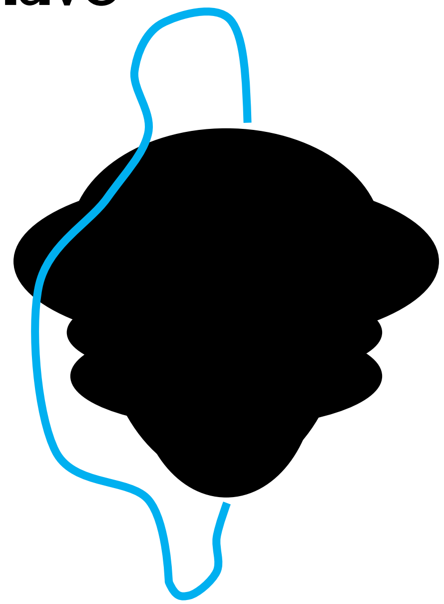
$m = 0, \quad l \gg 1$ Polar Orbits

- These photon orbits should
  - be periodic
  - form a class of limit cycles
- We can integrate the orbits along full periods: $\oint d\lambda = \oint \dot{\theta}^{-1} d\theta$
- Integrating the radial component of geodesic eqs., we have

$$\begin{aligned}\left\langle \frac{d}{d\lambda}\left(g_{rr}\dot{r}\right)\right\rangle &= \langle \partial_r F(r^*, \theta)(r - r^*)\rangle \\ &= \partial_r F_0(r_P)\,\langle r - r^*\rangle + o(\epsilon) \\ &= 0\end{aligned}$$

definition of limit cycle

Lyapunov exponent is $O(1)$



$m = 0, \quad l \gg 1$ Polar Orbits

- These photon orbits should
  - be periodic
  - form a class of limit cycles
- We can integrate the orbits along full periods: $\oint d\lambda = \oint \dot{\theta}^{-1} d\theta$
- Integrating the radial component of geodesic eqs., we have

$$\left\langle \frac{d}{d\lambda}\left(g_{rr}\dot{r}\right)\right\rangle = \langle \partial_r F(r^*, \theta)(r - r^*)\rangle$$

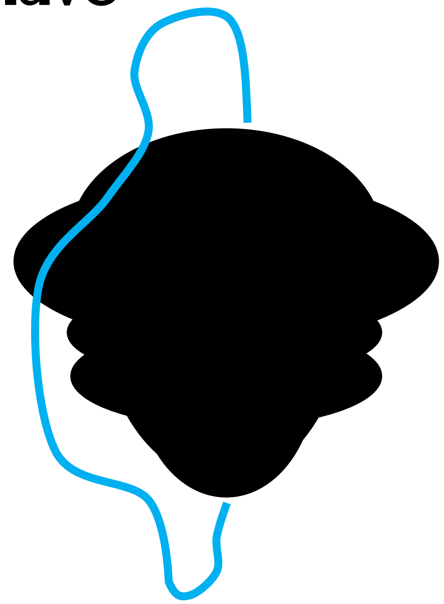
definition of limit cycle

$$= \partial_r F_0(r_P)\langle r - r^*\rangle + o(\epsilon)$$

Lyapunov exponent is $O(1)$

$$= 0$$

averaged radius along one period



Connect QNMs to photon geodesics

- The peak of the effective potential corresponds to the averaged radius of these orbits along one period

- This is true also for generic orbits, which corresponds to generic m

$$V_{\text{eff}}(r) \approx l^2 \frac{f}{r^2} \left\{ 1 + \epsilon \sum_{k=0}^{\infty} (1-\alpha^2)^k \, 4^{-k} C_k^{2k} \left[\alpha^2 \, {}_2F_1\left(1, k+\frac{1}{2}; k+1; 1-\alpha^2\right) (A_{2k} - D_{2k}) \right.\right.$$
$$\left.\left. + \frac{1-\alpha^2}{2k+2} \, {}_2F_1\left(1, k+\frac{1}{2}; k+2; 1-\alpha^2\right) (A_{2k} - C_{2k}) \right] \right\} ,$$

- Geometric-optics approximations for BHs

- Black hole spacetimes with less symmetry
  - Deformed Schwarzschild spacetime

- Conclusions

Conclusions

- Geometric-optics approximation adopted in BH spacetime
  - Correspondence between eikonal QNMs and photon orbits
- Schwarzschild and Kerr: Using their symmetries
- What if the black hole spacetime has less symmetry?
- Eikonal correspondence through the definition of averaged radius along full closed photon orbits
- Future:
  - Non-axisymmetric deformations
  - Deformed Kerr
  - Observational implications

Conclusions

Thank you for your attention!