Nonlinear Schrödinger models and control soliton collisions in optical waveguides

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Introduction to solitons

- What is a soliton?

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Recreation of Russell's 1834 observation on the Union Canal near Edinburgh in July 1995. (Photo from Nature, 1995.)

Introduction to solitons

- Korteweg-de Vries (KdV) equation (1895):

$$\psi_t + \psi_{xxx} + 6\psi\psi_x = 0,$$

where ψ is the elevation of the water surface.

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- We start with the ansatz: $\psi = \phi\left(y\right),\ y = x - ct.$

- This yields the ODE: $-c\phi' + \phi''' + 6\phi\phi' = 0.$

- Integrating this ODE and then multiplying the resulting equation by ϕ' and integrating again yields:

$$-\frac{c}{2}\phi^2 + \frac{1}{2}\left(\phi'\right)^2 + \phi^3 + C_1\phi + C_2 = 0.$$

- We want a solution in the form of a localized pulse, so we need ϕ, ϕ', and all higher derivatives to vanish as $y \to \pm\infty$. This implies $C_1 = C_2 = 0$:

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$$-\frac{c}{2}\phi^2 + \frac{1}{2}\left(\phi'\right)^2 + \phi^3 = 0.$$

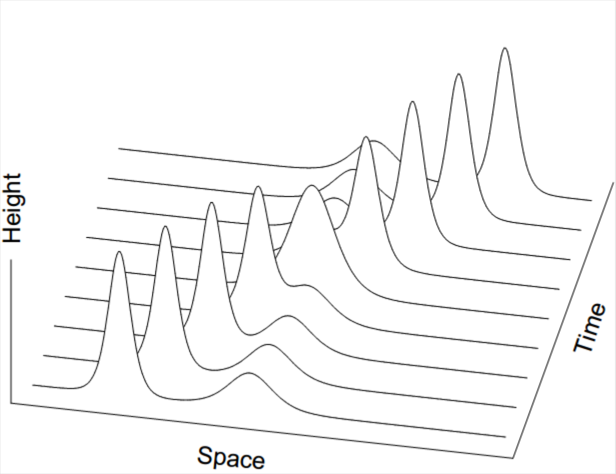
- The KdV equation thus admits traveling solitary waves

$$\psi\left(x,t\right) = \frac{1}{2}c\,\mathrm{sech}^2\,\frac{1}{2}\sqrt{c}\left(x - ct - x_0\right),$$

  where c is the wave speed.

- These solitary wave solutions correspond to the *wave of translation* in Russell's observations.

Introduction to solitons



Solitary wave collision: A larger and faster solitary wave overtakes a smaller and slower one.

Kadomtsev and Petviashvili (1970) derived a 2D-generalization of the KdV equation, the KP equation:

$$(\psi_t + \psi_{xxx} + \alpha\psi\psi_x)_x + \rho^2\psi_{yy} = 0.$$

Crossing swells, consisting of near-cnoidal wave trains. Photo taken by Michel Griffon from Phares des Baleines (Whale Lighthouse) France.

- Zabusky and Kruskal (1965) numerically discovered the elastic collision between KdV solitary waves, and then Gardner, et al. (1967) invented the inverse scattering transform method and solved the KdV equation analytically.
- This pioneering work initiated an unprecedented burst of research activities on nonlinear waves of integrable equations: Toda (1967), AKNS (1973), Ablowitz and Segur (1981), Zakharov et al. (1984), Newell (1985), (Hirota, 2004), ...
- Applications in: Optics, plasma, fluids, biological systems, condensed matter, astronomy,...

Transmission of information in fiber optics systems

- Mollenauer, Stolen, and Gordon (1980) reported the experimental observation of solitons in optical fibers.
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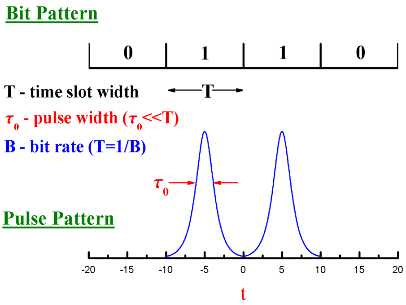
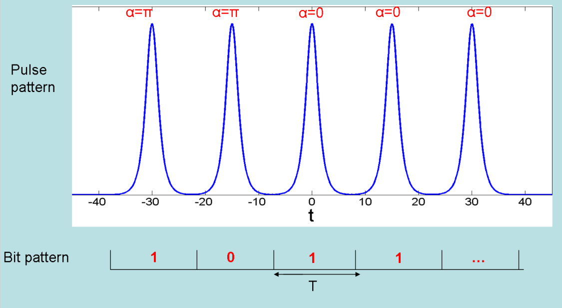
  $\rightarrow$ fiber-optic technology!
- The message is coded in binary by representing a one as pulselike modulation of a carrier wave.
- State 1/0 is assigned to a slot if the slot is occupied/empty (on-off keying)



Transmission of information in fiber optics systems

- Each optical pulse is positioned at the center of a time slot.
- Information can be coded by the difference between successive phases (differential phase-shift keying).



Nonlinear Schrödinger equation in optical fibers: Derivation

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- Dispersion originates from the frequency dependence of the refractive index of the fiber.
- Fiber nonlinearity is due to the dependence of the refractive index on the intensity of the optical pulse (optical Kerr effect).

$$n\left(\omega, |E|^2\right) = n_0\left(\omega\right) + n_2 \left|E\right|^2,$$

E represents the slowly varying envelope of the electric field

$$\epsilon\left(z, t\right) = E\left(z, t\right) e^{i(k_0 z - \omega_0 t)} + c.c.,$$

Nonlinear Schrödinger equation in optical fibers: Derivation

- Nonlinear dispersion relation:

$$k\left(\omega, |E|^2\right) = \frac{\omega}{c}\left(n_0(\omega) + n_2|E|^2\right)$$

  where c denotes the speed of light.

- Taylor expansion of the wave number:

$$k - k_0 = k'(\omega_0)(\omega - \omega_0) + \frac{k''(\omega_0)}{2}(\omega - \omega_0)^2 + \frac{\partial k}{\partial |E|^2}|E|^2$$

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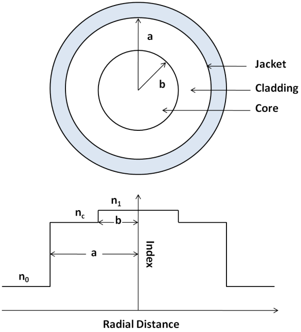
- Replacing $k - k_0$ and $\omega - \omega_0$ by their Fourier operator equivalents $-i\partial/\partial z$ and $i\partial/\partial t$ respectively, and operating on E:

$$i\left(\frac{\partial E}{\partial z} + k'(\omega_0)\frac{\partial E}{\partial t}\right) - \frac{k''(\omega_0)}{2}\frac{\partial^2 E}{\partial t^2} + \nu |E|^2 E = 0$$

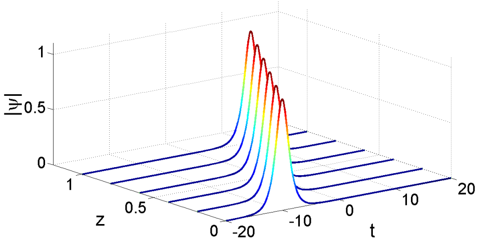
Nonlinear Schrödinger equation in optical fibers

- Hasegawa and Tappert (1973): pulse propagation in optical fibers can be described by the nonlinear Schrödinger (NLS) equation: $i\partial_z\psi - d_2\partial_t^2\psi + 2\kappa\psi\,|\psi|^2 = 0.$
- $\psi(t,z)$: electric field wave packet; z: distance along the fiber, t: time; d_2: second order dispersion coefficient; κ:Kerr nonlinearity coefficient.



Nonlinear Schrödinger equation in optical fibers

- In dimensionless form: $i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi = 0$.
- Soliton solution: $\psi_\beta(t,z) = \eta_\beta \frac{\exp(i\alpha_\beta + i\beta(t-y_\beta) + i(\eta_\beta^2 - \beta^2)z)}{\cosh(\eta_\beta(t - y_\beta - 2\beta z))}$, where η_β, α_β and y_β: the soliton amplitude, phase and position.



- Why using optical soliton?

Nonlinear Schrödinger equation in optical fibers

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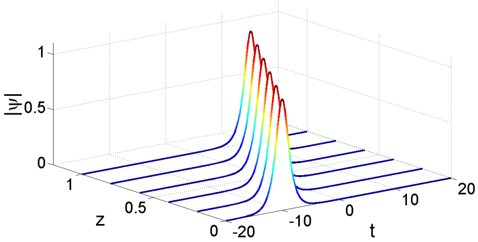
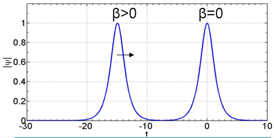
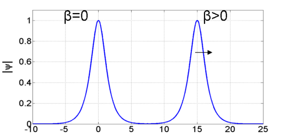


- Why using optical soliton?
  Stationarity: solitons propagate without any change in their parameters and without emitting any radiation (dispersion and nonlinearity exactly balance each other).
  $\Rightarrow$ Ideal candidates for transmission of information in fibers!

- In an ideal fiber, soliton collisions are elastic: the amplitude, frequency, and shape do not change as a result of the collision.

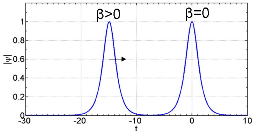


Before an ideal collision

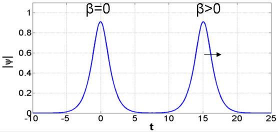


After an ideal collision

- Solition collisions in the presence of perturbations: emission of radiation, change in the soliton amplitude and group velocity, corruption of the shape, etc.



Before a non-ideal collision

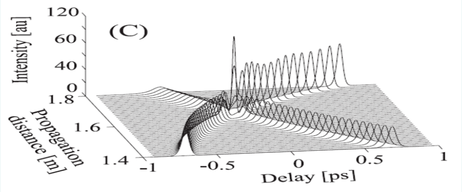


After a non-ideal collision

Effects of perturbations on optical solitions?

Important perturbations

- Important effects: Raman effect and nonlinear loss.
- Delayed Raman response is a nonlinear process affecting short or high-intensity pulses of light in optical fibers.
- An important phenomenon that is associated with delayed Raman response and nonlinear loss is energy exchange in inter-pulse collisions (crosstalk).



Soliton collisions in photonic crystal fiber, Luan et. al 2006

NLS with Raman effect

- Pulse propagation in the presence of delayed Raman response:

$$i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi = -\epsilon_R\psi\partial_t|\psi|^2$$

- The effect of delayed Raman response on a single pulse is a self frequency shift (Mitschke and Mollenauer 86, Gordon 86)

$$\frac{d\beta}{dz} = -\frac{8}{15}\epsilon_R\eta(z)^4.$$

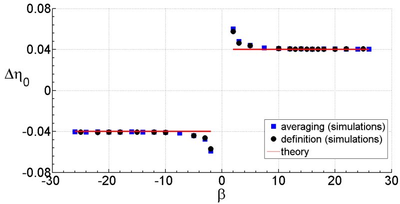
- Collision induced amplitude change (Raman crosstalk, Chung and Peleg 2005):

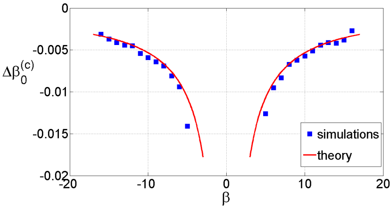
$$\Delta\eta_0 = 2\eta_0\eta_\beta \mathrm{sgn}(\beta)\epsilon_R$$

Effects of delayed Raman response on a single collision

- Collision induced frequency shift (Raman XFS, Chung and Peleg 05):

$$\Delta\beta_0^{(c)} = -(8\eta_0^2\eta_\beta\epsilon_R)/(3|\beta|)$$





(QN and Peleg, JOSA B, 2010)

The effects of nonlinear loss

- The waveguide's cubic loss can be a result of two-photon absorption (TPA).
- The subject TPA received attention in recent years due to the importance of TPA in silicon nanowaveguides (Foster et al. 06, Skryab et al. 08, Gaeta et al. 2010).
- The most important effect of a fast interchannel collision in the presence of cubic loss is a decrease in the energy of the colliding pulses (TPA-induced crosstalk).
- TPA-induced crosstalk can lead to relatively high values of the bit error rate (BER) for sufficiently high power levels of the optical pulses even in a two-channel system (Yoshitomo et al. 2010).

NLS with nonlinear loss

- Pulse propagation in the presence of generic nonlinear loss:

$$i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi = -i\epsilon_{2m+1}|\psi|^{2m}\psi,$$

where $0 \leq \epsilon_{2m+1} \ll 1$ for $m \geq 0$.

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  where $0 \le \epsilon_{2m+1} \ll 1$ for $m \ge 0$.
- Pulse propagation in the presence of weak cubic loss:

$$i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi = -i\epsilon_3|\psi|^2\psi.$$

- Equation for the dynamics of its amplitude (Aceves and Moloney, 1992):

$$\frac{d\eta^{(s)}(z)}{dz} = -\frac{4}{3}\epsilon_3\eta^{(s)3}(z),$$

  where the superscript s denotes self-amplitude shift.

The effect of cubic loss

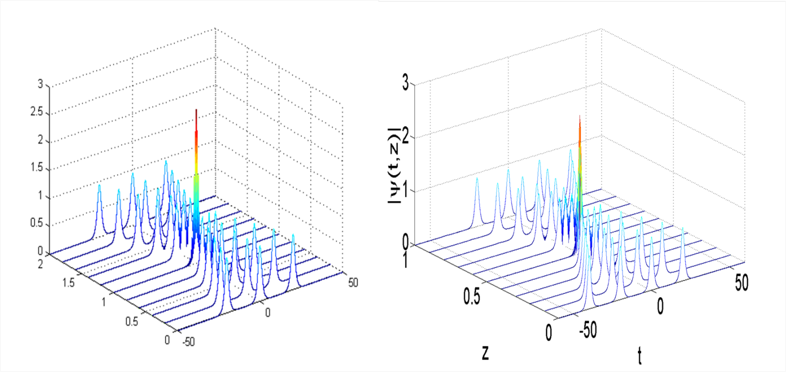
- The effect of cubic loss on a fast two-soliton collsion is an amplitude change : $\Delta\eta_0^{(2s)} = -4\epsilon_3\eta_0\eta_\beta/|\beta|$.

- The amplitude shift $\Delta\eta_0^{(3s)}$ in a fast three-soliton collision is given by a sum of the amplitude shifts due to two-soliton interaction:

$$\Delta\eta_0^{(3s)} = -4\epsilon_3\eta_0\left(\eta_\beta + \eta_{-\beta}\right)/|\beta|.$$

(Peleg, QN, Chung, PRA 2010)

A 3- and 4-soliton interaction.



Q: Can we measure collision-induced amplitude shift in fast collisions of N solitons?

The total contribution of n-pulse interaction to the amplitude shift in a fast full-overlap N-soliton collision in the presence of $(2m+1)$-order loss is

$$\Delta\eta_j^{(mn)} = \sum_{l_1=1}^{N} \sum_{l_2=l_1+1}^{N} \cdots \sum_{l_{n-1}=l_{n-2}+1}^{N} \Pi_{j'=1}^{n-1} \left(1 - \delta_{jl_{j'}}\right) \Delta\eta_{j(l_1 \ldots l_{n-1})}^{(mn)},$$

where

$$\Delta\eta_{j(l_1 \ldots l_{n-1})}^{(mn)} = -\epsilon_{2m+1} \sum_{k_{l_1}=1}^{m-(n-2)} \cdots \sum_{k_{l_{n-1}}=1}^{m-s_{n-2}} \frac{m!(m+1)!\eta_{l_1}^{2k_{l_1}} \ldots \eta_{l_{n-1}}^{2k_{l_{n-1}}} \eta_j^{2m-2s_{n-1}+1}}{(k_{l_1}! \ldots k_{l_{n-1}}!)^2 (m+1-s_{n-1})!(m-s_{n-1})!}$$

$$\times \int_{-\infty}^{\infty} dx_j [\cosh(x_j)]^{-(2m-2s_{n-1}+2)} \int_{-\infty}^{\infty} dz [\cosh(x_{l_1})]^{-2k_{l_1}} \ldots [\cosh(x_{l_{n-1}})]^{-2k_{l_{n-1}}}.$$

(Peleg, QN, Glenn, PRE 2014)

Numerical simulations: split-step method

- Numerical schemes: the finite-difference and pseudo-spectral methods.

Numerical simulations: split-step method

- Numerical schemes: the finite-difference and pseudo-spectral methods.
- It is useful to write the perturbed NLS

$$i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi = -i\epsilon_3|\psi|^2\psi$$

in the form

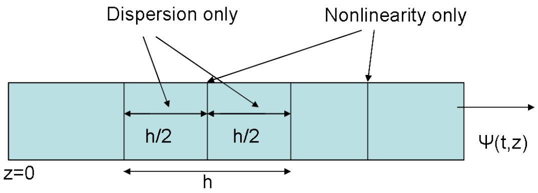
$$\frac{\partial\psi}{\partial z} = (D + N)\,\psi,$$

where D: dispersion (linear part) and N: fiber nonlinearities.

- Split-step Fourier method: Assuming that in propagating the optical field over a very small distance h, the dispersive and nonlinear effects can be assumed to act independently.

Numerical simulations: split-step method

- The linear part $i\psi_z = -\psi_{tt}$ was advanced efficiently via computation of the operator exponential in frequency domain (Fast Fourier Transform).
- The nonlinear part $i\partial_z\psi = -2\left|\psi\right|^2\psi - i\epsilon_3|\psi|^2\psi$ was advanced via a fourth order Runge-Kutta scheme.



Schematic illustration of the symmetrized split-step Fourier method.

Numerical simulations: split-step method

- The exact solution at the propagation distance $z+h$:
  $\psi(t,z+h)=\exp(h(D+N))\psi(t,z)$.

- The approximation solution at the propagation distance $z+h$

$$\psi(t,z+h)\approx FT^{-1}\exp\{hD(-i\omega)\}\,FT\left[\exp(hN)\,\psi(t,z)\right].$$

- If using the Baker-Campbell-Hausdorff formula for two non-commutative operators A,B where $A=hD, B=hN$

$$\exp(A)\exp(B)=\exp(A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A-B,[A,B]]+...),$$

  the error $E=|\exp(h(D+N))-\exp(hD)\exp(hN)|$ is found to result from $\frac{1}{2}h^2[N,D]$, i.e, second order only!

Numerical simulations

- Can we find a set of real numbers $(c_I, c_2...., c_k)$ and $(d_I, d_2, ..., d_k)$ such that

$$\exp\left(h\left(D+N\right)\right)=\prod_{i=1}^{n}\exp(c_i hD)\exp(d_i hN)+o\left(h^{n+1}\right),$$

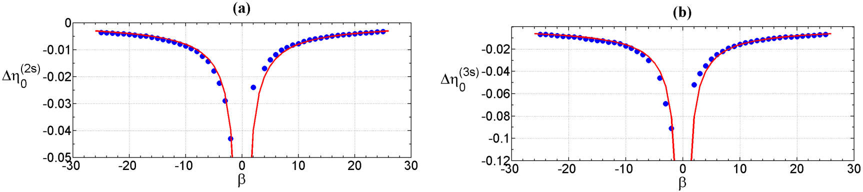
where D and N are non-commutative operators?

- Using Yoshida's result (PLA, 1990) for $n=4$: $d_1=d_3=x_1$, $d_2=x_0$, $d_4=0$, $c_1=c_4=1/2x_1$, $c_2=c_3=1/2\left(x_0+x_1\right)$, where $x_0=-2^{1/3}/(2-2^{1/3})$, $x_1=1/(2-2^{1/3})$.

- Condition for numerical stability (Yang 2009) for $SSFM_4$: $\dfrac{\Delta z}{\Delta t^2}<\dfrac{1}{\pi}$.

Effects of cubic loss: Numerical simulations

Collision-induced amplitude shift of the reference-channel soliton $\Delta\eta_0^{(2s)}$ for $\epsilon_3 = 0.02$ (a) and amplitude shifts of the 0-channel solitons $\Delta\eta_0^{(3s)}$ in a three-soliton collision for $\epsilon_3 = 0.02$ (b):



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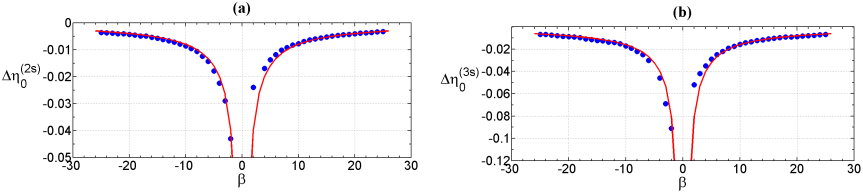
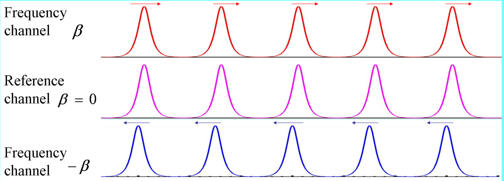


Q: Can we find the way to control the dynamics of energy loss in many soliton collisions in the presence of weak cubic loss?

Amplitude dynamics at many soliton collisions

Optical pulses in Multi-channel transmission



A schematic description of multi-channel transmission

- Multi-channel transmission: transmit many pulse sequences through the same fiber.
- In each frequency channel (pulse sequence) the pulses propagate with the same group velocity, but the group velocity is different for different channels.
- Collisions between pulses from different channels are very frequent, which reduce the transmission quality.

Dynamics at many collisions and Lotka-Volterra model

- Recall the amplitude change on a single two-soliton collsion: $\Delta\eta_0^{(c)} = -4\epsilon_3\eta_0\eta_\beta/|\beta|$.
- Denote: $\Delta z_c^{(1)} = T/(2\Delta\beta)$-the distance traveled by the soliton while passing two successive time slots, g_j-the net gain/loss coefficient for the jth channel.
- Adding gain/loss g_j to the Eq. of change in soliton amplitude, summing over all collisions occurring in $\Delta z_c^{(1)}$:

$$\eta_j(z_{l-1} + \Delta z_c^{(1)}) = \eta_j(z_{l-1}) + g_j\eta_j(z_{l-1})\Delta z_c^{(1)} - \frac{4\epsilon_3}{3}\eta_j^3(z_{l-1})\Delta z_c^{(1)}$$

$$-\frac{4\epsilon_3}{\Delta\beta}\sum_{k=1}^{N}(1-\delta_{jk})\eta_j(z_{l-1})\eta_k(z_{l-1}),$$

Dynamics at many collisions and Lotka-Volterra model

- Going to the continuum limit:

$$\frac{d\eta_j}{dz} = \eta_j \left[g_j - \frac{4\epsilon_3}{3}\eta_j^2 - \frac{8\epsilon_3}{T}\sum_{k=1}^{N}(1-\delta_{jk})\eta_k \right].$$

- Look for a stationary state in the form $\eta_j^{(eq)} = \eta > 0$ for $j = 1, \ldots, N$
- This yields the following expression for g_j: $g_j = \frac{4\epsilon_3}{3}\eta^2 + \frac{8\epsilon_3}{T}(N-1)\eta$.
- The model describing the dynamics of soliton amplitudes in an N-channel transmission line

$$\frac{d\eta_j}{dz} = 4\epsilon_3\eta_j \left[\frac{1}{3}(\eta^2 - \eta_j^2) + \frac{2}{T}\sum_{k=1}^{N}(1-\delta_{jk})(\eta - \eta_k) \right].$$

- This is the Lotka-Volterra model for N competing species!!

Dynamics at many collisions and Lotka-Volterra model: Example in a two-channel transmission system

- Consider an example in a two-channel transmission system:

$$\frac{d\eta_1}{dz} = 4\epsilon_3\eta_1\left[(\eta^2 - \eta_1^2)/3 + 2(\eta - \eta_2)/T\right],$$

$$\frac{d\eta_2}{dz} = 4\epsilon_3\eta_2\left[(\eta^2 - \eta_2^2)/3 + 2(\eta - \eta_1)/T\right].$$

- The equivalent coupled-NLS model:

$$i\partial_z\psi_1+\partial_t^2\psi_1+2|\psi_1|^2\psi_1+4|\psi_2|^2\psi_1 = ig_1\psi_1/2-i\epsilon_3|\psi_1|^2\psi_1-2i\epsilon_3|\psi_2|^2\psi_1,$$

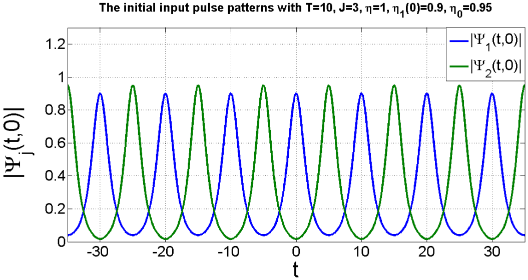
$$i\partial_z\psi_2+\partial_t^2\psi_2+2|\psi_2|^2\psi_2+4|\psi_1|^2\psi_2 = ig_2\psi_2/2-i\epsilon_3|\psi_2|^2\psi_2-2i\epsilon_3|\psi_1|^2\psi_2.$$

- The gain required to maintain the equal non-zero amplitudes steady state: $g_1 = g_2 = 4\epsilon_3\eta(\eta/3 + 2/T)$

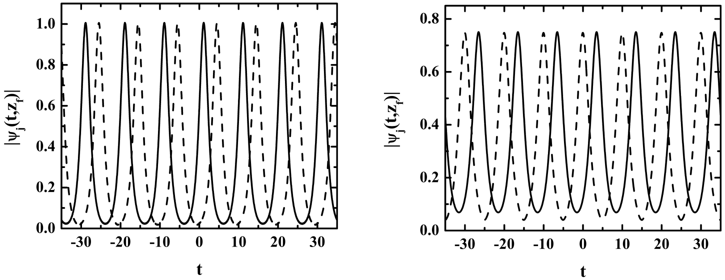
- The initial condition:

$$\psi_1(t,0) = \sum_{j=-J}^{J} \frac{\eta_1(0)}{\cosh[\eta_1(0)(t-jT)]},$$

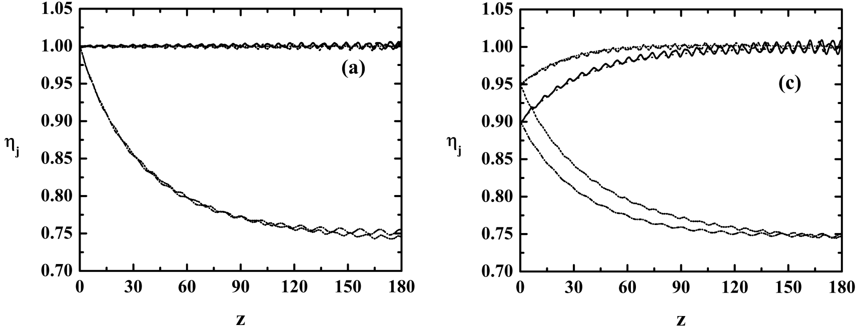
$$\psi_2(t,0) = \sum_{j=-J}^{J} \frac{\eta_2(0)\exp[i\beta_2(t-jT+T/2)]}{\cosh[\eta_2(0)(t-jT+T/2)]},$$



The final pulse patterns obtained by numerical integration of the coupled-NLS with compensation of collision-induced loss ($g_1 = g_2 = 4\epsilon_3\eta(\eta/3 + 2/T)$, left) and without compensation of collision-induced loss ($g_1 = g_2 = 4\epsilon_3\eta^2/3$, right):



Dynamics at many collisions and Lotka-Volterra model



Pulse dynamics with the IC $\eta_1(0) = \eta_2(0) = 1$ (a), and $\eta_1(0) = 0.90$ and $\eta_2(0) = 0.95$ (c).

Summary

- Solitary waves from water waves to nonlinear optics: history and general discussions.
- Nonlinear effects (delayed Raman response and nonlinear loss) lead to energy exchange in inter-pulse collisions (crosstalk): Raman effect leads to transfer energy, while nonlinear loss leads to a decreasing of energy.
- Amplitude dynamics in an N-channel waveguide system in the presence of weak cubic loss can be described by a Lotka-Volterra model for N competing species.
- Stability analysis of the steady states of the LV model was used to guide the choice of physical parameters values, which leads to a drastic enhancement in transmission stability.

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References

- G.P. Agrawal, Nonlinear Fiber Optics, Academic, San Diego, CA, (2001).
- Q. Nguyen, A. Peleg, and T.P. Tran, Phys. Rev. A, **91**, 013839 (2015).
- A. Peleg, Q. Nguyen, and P. Glenn, Phys. Rev. E, **89**, 043201 (2014).
- Gaeta et al., IEEE Photon. Technol. Lett. **24**, 185-187 (2012).
- Q. Nguyen and A. Peleg, J. Opt. Soc. Am. B **27**, 1985 (2010).
- Y. Chung and A. Peleg, Nonlinearity **18**, 1555 (2005).
- A.B. Aceves and J.V. Moloney, Opt. Lett. **17**, 1488 (1992).
- D. J. Kaup, Phys. Rev. A **42**, 5689 (1990).
- S. Chi and S. Wen, Opt. Lett. **14**, 1216 (1989).

Thank you!