清華大學 交叉信息研究院
Tsinghua University Institute for Interdisciplinary Information Sciences

Fidelity measure of a multipartite state

Xiongfeng Ma
xma@tsinghua.edu.cn
Center for Quantum Information

Zhou, Guo, and Ma, PRA 99, 052324, (2019)
Zhou, Zhao, Yuan, and Ma, npj Quantum Information, 5, no. 1, pp. 1-8, (2019)
Zhang, Tang, Zhou, Ma, arxiv: 2012.07606

Outline

- Introduction
  - Genuine multipartite entanglement
  - Entanglement witness
- Permutation-invariant (symmetric) state
  - Symmetric subspace
  - Efficient decomposition to local measures
  - GHZ states, W states, Dick states
- Graph state
  - GHZ states, 1-D/2-D cluster states
- Conclusion

Zhou, Guo, and Ma, PRA 99, 052324, (2019)

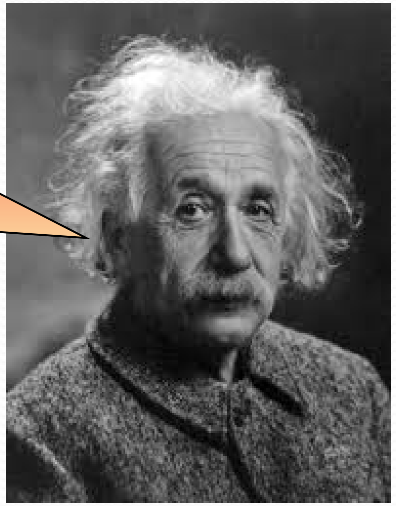
Zhou, Zhao, Yuan, and Ma, npj Quantum Information, 5, no. 1, pp. 1-8, (2019)

Introduction

Einstein-Podolsky-Rosen Paradox

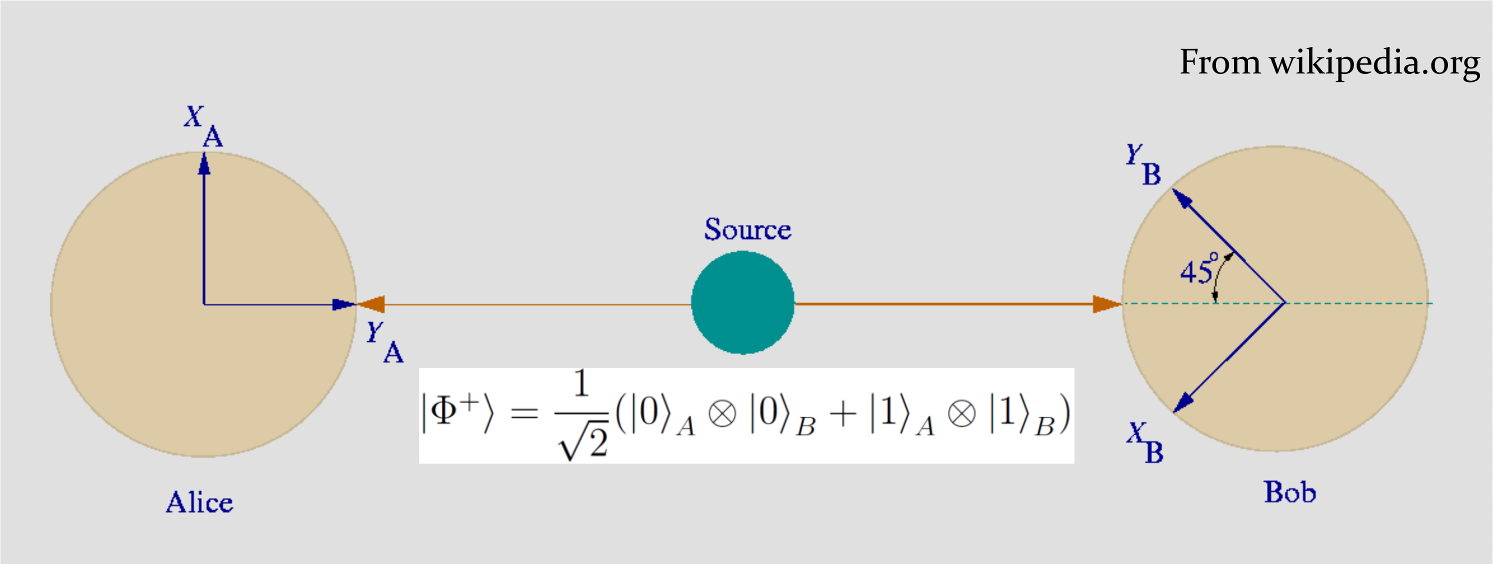
- Is Quantum Mechanics complete?
- Local hidden variable
- Entanglement
  - A pair of particles: measure on one particle would instantaneously affect the state of the other

Spooky action at a distance



Bell's inequality

- Quantum mechanics vs. local hidden variable



$$
\begin{aligned}
S &= -\langle x_a x_b \rangle + \langle x_a y_b \rangle + \langle y_a x_b \rangle + \langle y_a y_b \rangle \\
&= \langle (-x_a + y_a) x_b \rangle + \langle (x_a + y_a) y_b \rangle \\
&\leq |-x_a + y_a| + |x_a + y_a| \leq 2
\end{aligned}
$$

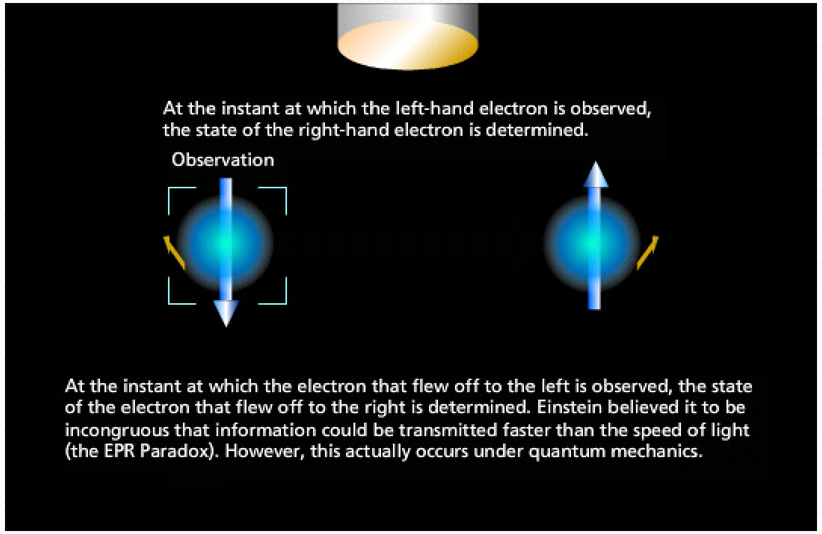
$$\langle x_a x_b \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle x_a y_b \rangle = \langle y_a x_b \rangle = \langle y_a y_b \rangle = \frac{1}{\sqrt{2}}$$

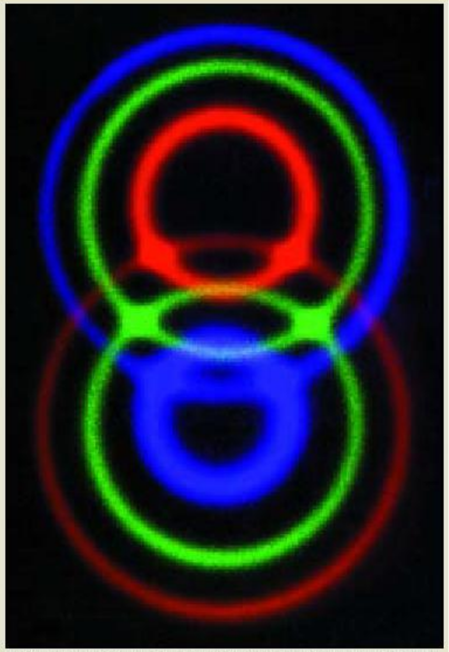
$$S = 2\sqrt{2}$$

Entanglement

- Nonlocal correlation
  - Why quantum mechanics is "weird"
  - Stronger than any classical correlation
- Unpredictable results
  - Any prediction will be served as hidden variables
- Not for instantaneous communication
  - Compatible with causality (relatives)
- Useful in many information processing tasks



1 Entangled photons



Observations of entanglement

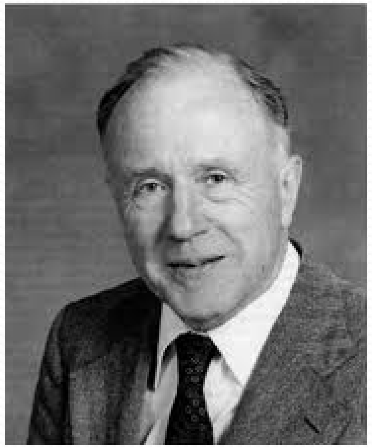
- Ann. New York Acad. Sci. 48, 219 (1946)
  - "Two quanta emitted in the annihilation of a positron-electron pair, with zero relative angular momentum, are polarized at right angles to each other"
- Phys. Rev. 77, 136 (1950)
  - Early observation of quantum entanglement

The Angular Correlation of Scattered Annihilation Radiation*

C. S. Wu and I. Shaknov

Pupin Physics Laboratories, Columbia University, New York, New York

November 21, 1949



John Wheeler

Chien-Shiung Wu

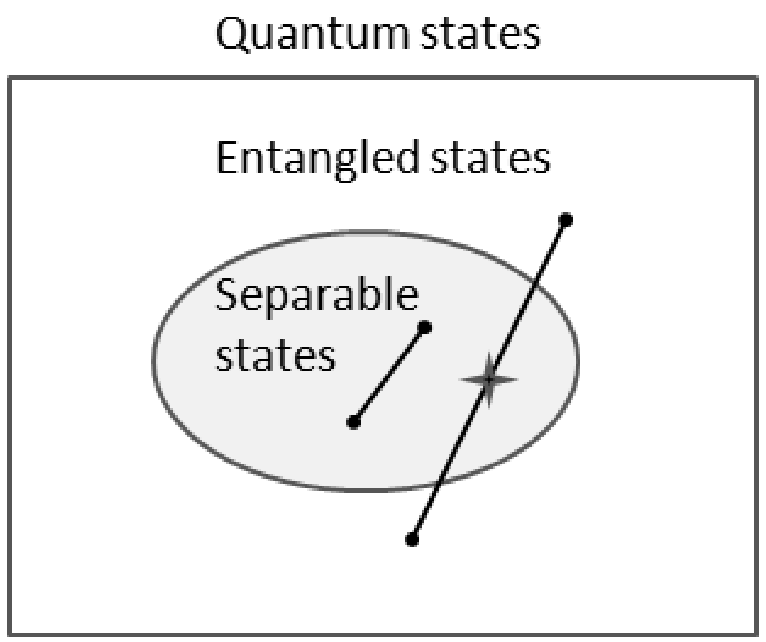
Entanglement witness

Definition of bipartite entangled state

- Separable state

$$\sigma_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i$$

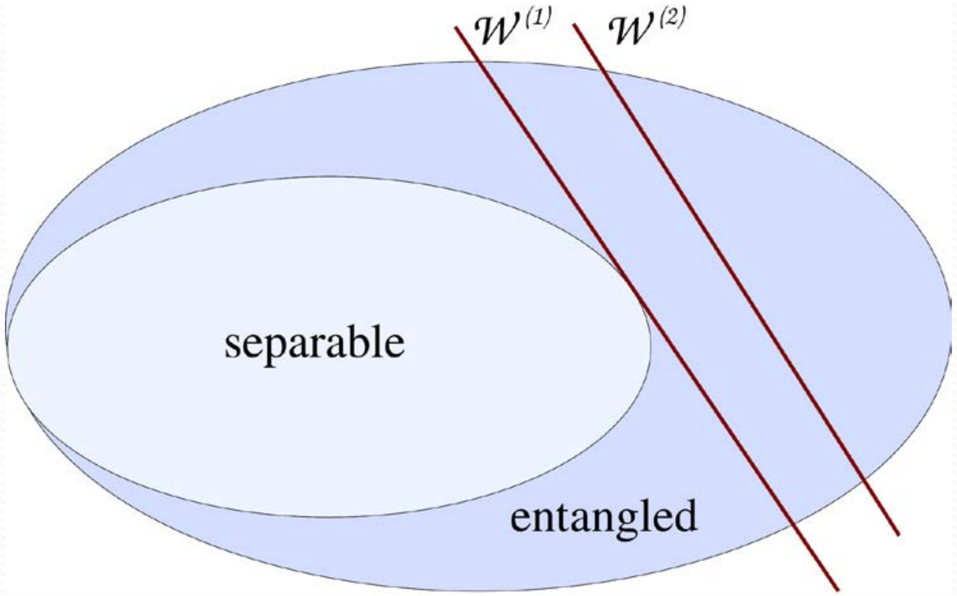
  - Forms a convex set
- Entangled state
  - Cannot be written in the above form
  - E.g., $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  - Not convex



O. Guhne and G. Toth, Physics Reports 474, 1 (2009)

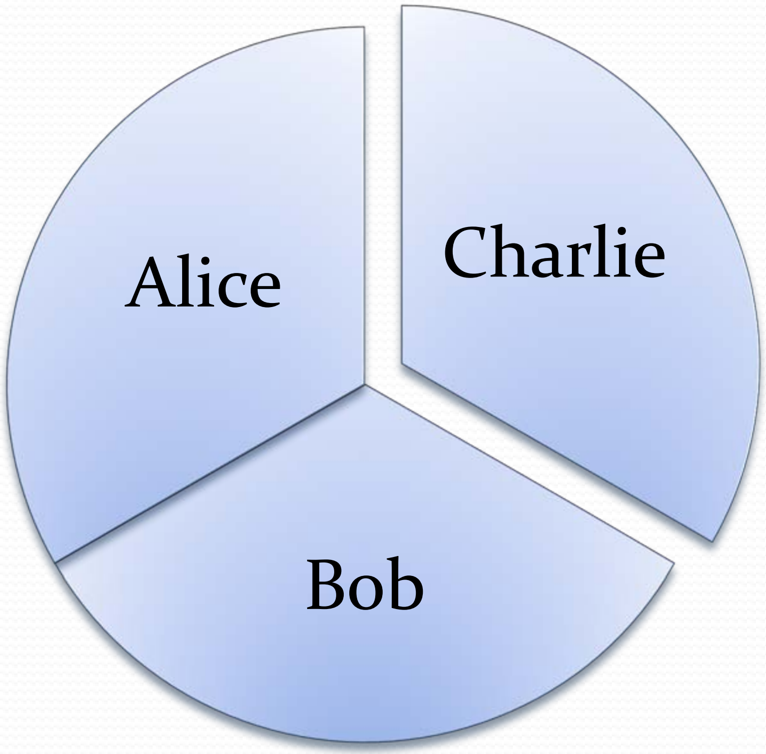
Entanglement witness

- Hermitian operator W
  - $\mathrm{tr}[W\sigma] \geq 0$ for all separable state σ
  - $\mathrm{tr}[W\rho] < 0$ for some entangled state ρ
- Decomposed as a linear combination of local observables
  - $W = \frac{1}{2}I - |\Psi^-\rangle\langle\Psi^-|$
  - $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$
  - $W = \frac{1}{4}(I + \sigma_x\sigma_x + \sigma_y\sigma_y + \sigma_z\sigma_z)$
  - 3 local measurement settings (LMSs)



Tripartite entanglement

- Entanglement of three qubits
  - $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$; $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$
- Fully separable state
  - $\sigma_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i \otimes \rho_C^i$
- Tripartite entangled state
  - Cannot be written in the above form
  - What about $|\Phi^+\rangle\langle\Phi^+|_{AB} \otimes \rho_C$?
- Rule out "bipartite" entangled states
  - $\rho_{AB} \otimes \rho_C$

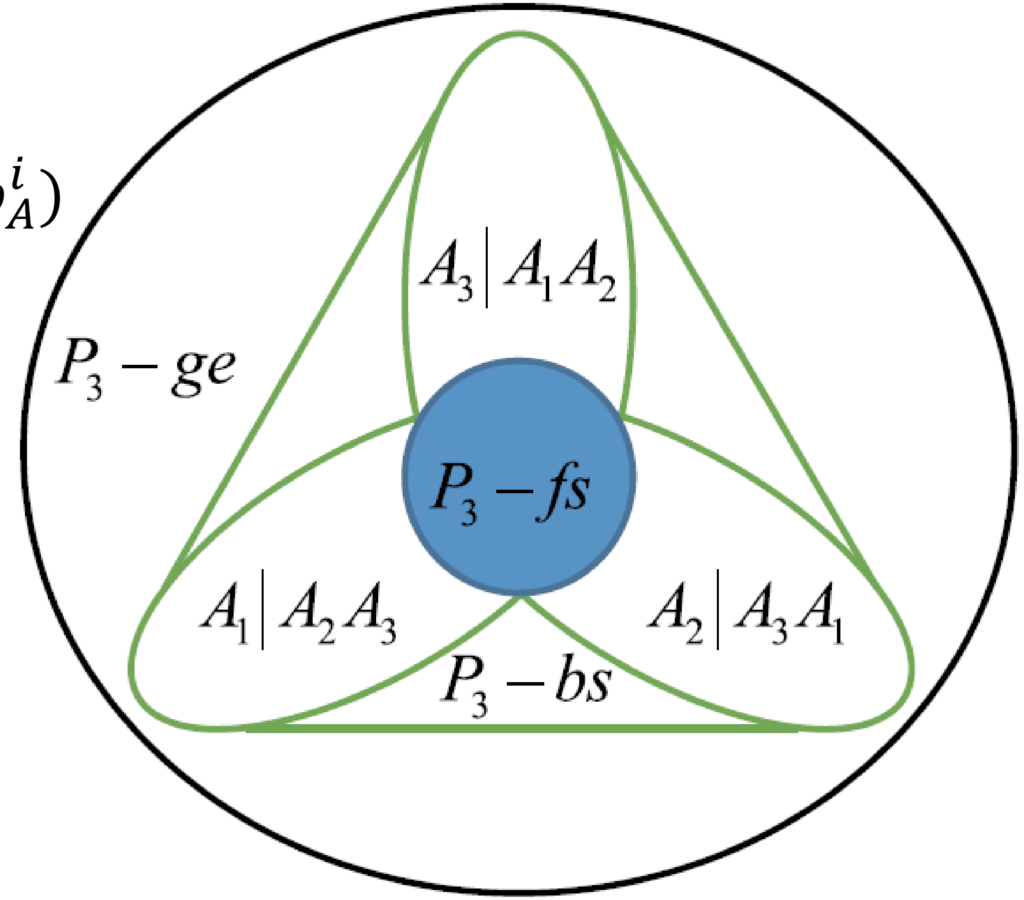


Genuine tripartite entanglement

- Bi-separable states

$$\sigma_{ABC} = \sum_i p_i(\rho^i_{AB} \otimes \rho^i_C + \rho^i_{AC} \otimes \rho^i_B + \rho^i_{BC} \otimes \rho^i_A)$$

- Genuine multipartite entanglement (GME): cannot be written in the above form
- Pairwise entanglement doesn't mean GME!
  - E.g., $\frac{1}{3}(\Phi^+_{BC} \otimes \rho_A + \Phi^+_{\mathrm{AC}} \otimes \rho_B + \Phi^+_{\mathrm{AB}} \otimes \rho_C)$
- Stochastic LOCC
  - 3-qubit: two equivalent classes



Multipartite entanglement

- Fully separable state

$$\sigma_{A_1A_2\ldots A_n} = \sum_i p_i \bigotimes_{j=1}^{n} \rho_{A_j}^i$$

- Bi-separable state

$$\sigma_{A_1A_2\ldots A_n} = \sum_i p_i \rho_{\mathcal{A}}^i \otimes \rho_{\bar{\mathcal{A}}}^i$$

Genuine multipartite entanglement (GME): cannot be written in this form

  - where $\mathcal{A} + \bar{\mathcal{A}} = [n] = \{1,2,\ldots,n\}$
- GME: key resource for quantum advantage?
  - Competition among different physical systems

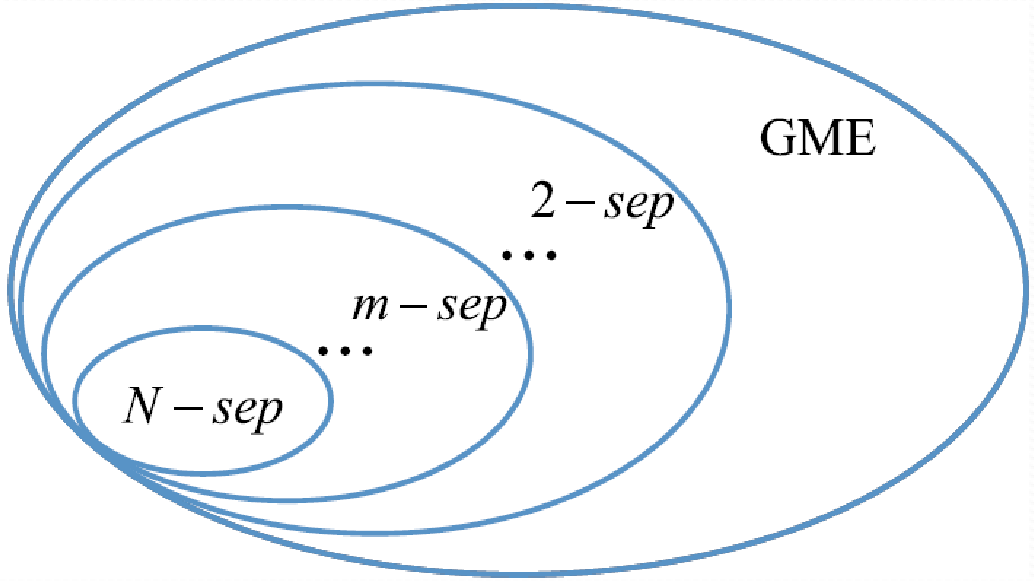
Deutsch, PRX QUANTUM 1, 020101 (2020)

Entanglement structure

- m-separable state

$$\sigma_{A_1A_2\ldots A_n} = \sum_i p_i \bigotimes_{j=1}^{m} \rho^i_{\mathcal{A}_j}$$

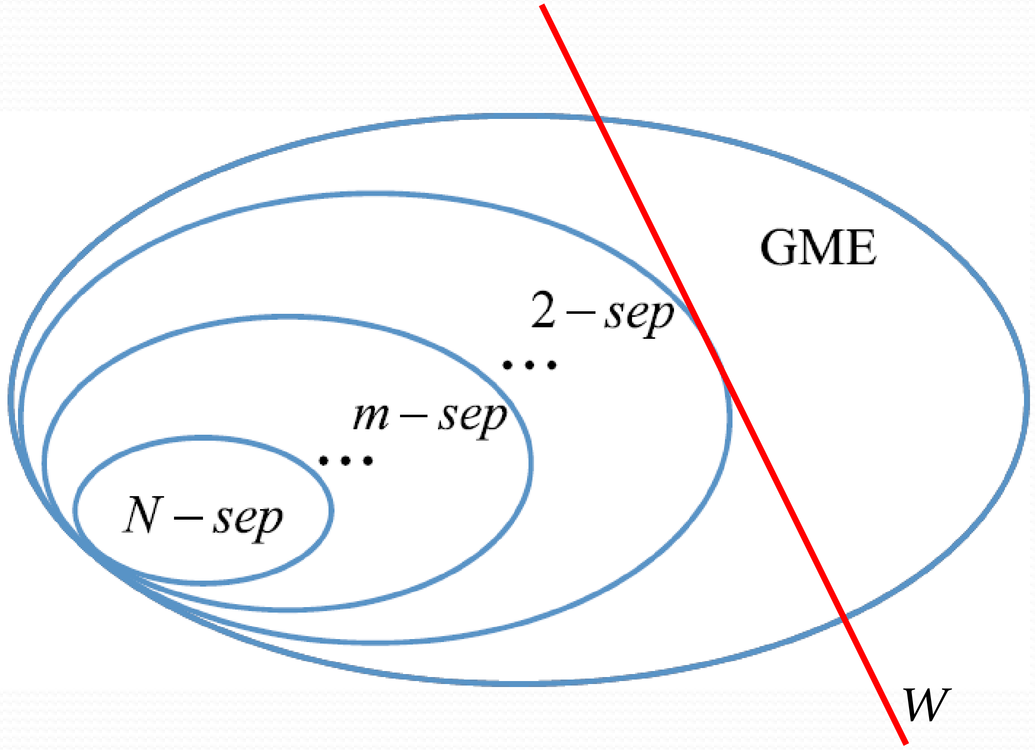
  - where $\mathcal{A}_1 + \mathcal{A}_2 + \cdots + \mathcal{A}_m = [n] = \{1,2,\ldots,n\}$



Entanglement hierarchy

GME witness

- Hermitian operator W
  - $\mathrm{tr}[W\sigma] \geq 0$ for all bi-separable state σ
  - $\mathrm{tr}[W\rho] < 0$ for some GME state ρ
- Typical way of construct GME witness
  - E.g., $|GHZ\rangle_n = \frac{1}{\sqrt{2}}(|00\ldots 0\rangle + |11\ldots 1\rangle)$
  - $W = \alpha I - |GHZ\rangle_n\langle GHZ|$
  - $\alpha = \max_{bisep}\langle GHZ|\sigma|GHZ\rangle$=0.5
- Decomposition to local observables
  - Feasibility: tomography (exponential in n)
  - Local measurement setting (LMS) complexity



Permutation-invariant state

Definition: permutation-invariant state

- Symmetric state S_s

$$|\psi_s\rangle = P(\pi)|\psi_s\rangle, \ \forall\pi$$

- Permutation-invariant state S_{PI}

$$\rho^{\mathrm{PI}} = P(\pi)\rho^{\mathrm{PI}}P(\pi), \ \forall\pi$$

- Obviously, $S_s \subset S_{PI}$
  - Consider a PI state that is not symmetric: $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$
- GHZ state: $|GHZ\rangle_n = \frac{1}{\sqrt{2}}(|00 \dots 0\rangle + |11 \dots 1\rangle)$
- W state: $|W\rangle_n = \frac{1}{\sqrt{n}}(|0 \dots 01\rangle + |0 \dots 10\rangle + \cdots + |10 \dots 0\rangle)$
- Dicke state: superposition of all the pure states with *m 1's*

Global operator decomposition

- Consider the GME witness
  - $W = \alpha I - |\Psi\rangle\langle\Psi|$, where $\alpha = \max_{bisep}\langle\Psi|\sigma|\Psi\rangle$
- Decompose the operator W into local observables

$$W = \sum_i O_1^i \otimes O_2^i \otimes \cdots \otimes O_N^i$$

  - LMS complexity: number of terms in the summation
- In general, the LMS complexity is in the order of 3^N, considering tomography
- For symmetric state projection
  - Number of free parameters is limited
  - Can we do it more efficiently?

Symmetric subspace

- Two equivalent definitions of symmetric subspace

$$\mathrm{Sym}_N(\mathcal{H}_d) = \{|\Psi\rangle \in \mathcal{H}_d^{\otimes N} : P_d(\pi)|\Psi\rangle = |\Psi\rangle, \quad \forall \pi \in S_N\}.$$

$$\mathrm{Sym}_N(\mathcal{H}_d) = \mathrm{span}\{|\phi\rangle^{\otimes N} : |\phi\rangle \in \mathcal{H}_d\}.$$

- An orthogonal basis:

$$\left\{ |\Psi_{\vec{i}}\rangle = \sum_{\pi} |0\rangle^{\otimes i_0} |1\rangle^{\otimes i_1} \cdots |d-1\rangle^{\otimes i_{d-1}} \middle| i_k \in \mathbb{Z}^+, \ \sum_{k=0}^{d-1} i_k = N \right\}$$

- Dimension of the symmetric subspace

$$\left\{ |\Psi_i\rangle = \sum_{\pi} |0\rangle^{\otimes N-i} |1\rangle^{\otimes i}, 0 \le i \le N \right\}$$

$$D_S = \binom{N+d-1}{N} = \frac{(N+d-1)!}{N!(d-1)!}$$

Decomposition of symmetric subspace

- Lemma: another (non)-orthogonal product form basis

$$\mathcal{B} = \left\{ \left| \Phi_{\vec{j}} \right\rangle = \left(a_{0,j_0}|0\rangle + a_{1,j_1}|1\rangle + \cdots + a_{d-1,j_{d-1}}|d-1\rangle \right)^{\otimes N} \middle| j_k \in \mathbb{Z}^+, \sum_{k=0}^{d-1} j_k = N \right\}$$

$$\mathcal{B} = \{|\Phi_j\rangle = (|0\rangle + a_j|1\rangle)^{\otimes N} | 0 \leqslant j \leqslant N\}$$

- Coefficient $d \times (N+1)$ matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_{1,0} & a_{1,1} & \cdots & a_{1,N} \\ a_{2,0} & a_{2,1} & \cdots & a_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{d-1,0} & a_{d-1,1} & \cdots & a_{d-1,N} \end{pmatrix}$$

Make sure the basis states are linearly independent:

$$a_{0,j} = 1, \forall j$$

$$a_{k,j} \neq a_{k,j'}, \forall j, j', 1 \leq k \leq d-1$$

A. W. Harrow, arXiv: 1308.6595 (2013); Zhou, Guo, and Ma, PRA 99, 052324, (2019)

Symmetric fidelity observable decomposition

- GME witness: N-qubit target PI state $|\Psi^{PI}\rangle$
  - Essentially measures the fidelity between a prepared state ρ and $|\Psi^{PI}\rangle$
- **Theorem**: $|\boldsymbol{\Psi}^{PI}\rangle\langle\boldsymbol{\Psi}^{PI}|$**can be decomposed to** $\binom{N+2}{2}$ **LMSs**
  - For some special cases, the number can be further reduced
- Decompose the projector $|\Psi^{PI}\rangle\langle\Psi^{PI}|$ into local operators
  - Think of tomography: linear combination of N-qubit Pauli group elements
  - Denote single-qubit Pauli group: $G_1 = \{I, \sigma_X, \sigma_Y, \sigma_Z\}$
  - The PI density operators form $Sym_N(G_1)$ subspace

$$\mathrm{Sym}_N(G_1) = \{M \in F_{G_N} : P(\pi) M P(\pi) = M, \forall \pi \in S_N\}$$

Proof of the theorem I

- Pauli operators form an orthogonal basis (in the sense of the Hilbert-Schmidt inner product) of $Sym_N(G_1)$

$$M_{i,j,k} = \sum_{\pi} \mathbb{I}^{\otimes i} \otimes \sigma_X^{\otimes j} \otimes \sigma_Y^{\otimes k} \otimes \sigma_Z^{\otimes (N-i-j-k)}$$

- These product operators can span the symmetric subspace

$$\mathrm{Sym}_N(G_1) = \mathrm{span}\{A^{\otimes N} : A = a\mathbb{I} + b\sigma_X + c\sigma_Y + d\sigma_Z\}$$

- Consider Pauli basis $\mathrm{G}_1 = \{\mathrm{I}, \sigma_x, \sigma_y, \sigma_z\}$, then $Sym_N(G_1)$ is isomorphic to $Sym_N(\mathcal{R}_4)$, which has the dimension of $\binom{N+3}{3}$

$$F_{G_N} \simeq (\mathbb{R}^4)^{\otimes N}$$

Proof of the theorem II

- From the Lemma, we can construct a basis set for the subspace with $\binom{N+3}{3}$ basis states

$$\mathcal{B}_o = \left\{ (a_{1,j_1}\mathbb{I} + a_{2,j_2}\sigma_X + a_{3,j_3}\sigma_Y + a_{0,j_0}\sigma_Z)^{\otimes N} \mid \sum_{k=0}^{3} j_k = N \right\}$$

- Some of the product operators can be measured by the same LMS

$$\mathcal{B}_o = \{(a_i\mathbb{I} + b_j\sigma_X + c_k\sigma_Y + \sigma_Z)^{\otimes N} | 0 \leqslant i, j, k \leqslant N, \ i + j + k \leqslant N\}, \longrightarrow (b_j\sigma_X + c_k\sigma_Y + \sigma_Z)^{\otimes N} \longrightarrow \binom{N+2}{2}$$

Measurement complexity of some typical PI states

- Further reduction of LMS complexity is possible with additional information of the target states

$$F = \langle \Psi^{PI} | \rho | \Psi^{PI} \rangle = \mathrm{Tr}(\rho | \Psi^{PI} \rangle \langle \Psi^{PI} |)$$

TABLE I. Measurement complexity of some typical PI states. For these states, the measurement complexity is linear, $\Theta(N)$.

State	Upper bound	Lower bound
W	$2N - 1$ [29]	$N - 1$
Dicke	$m(2m + 3)N + 1$	$N - 2m + 1$
GHZ	$N + 1$ [29]	$\lceil \frac{N+1}{2} \rceil$

Zhou, Guo, and Ma, PRA 99, 052324, (2019); [29] Gühne et al, PRA 76, 030305 (2007)

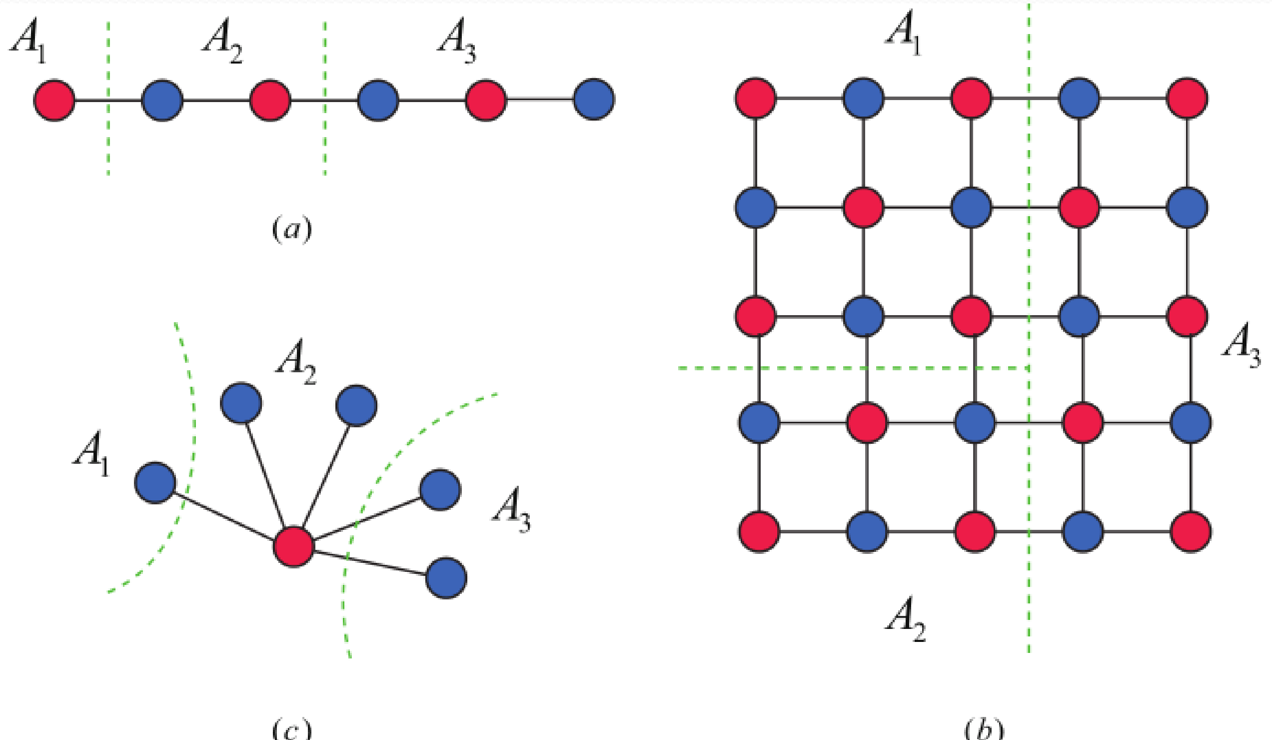
Graph state

Briegel and Raussendorf, PRL 86,910 (2001)
Hein et al, arXiv e-prints , quantph/0602096 (2006)

Definition of graph states

- Initialize all the qubits in $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$
  - Vertices
- Apply Control-Z operation between some of the qubits
  - Edges

$$|G\rangle = \prod_{(i,j)\in E} \mathrm{CZ}^{\{i,j\}} |+\rangle^{\otimes N}$$



Very convenient to work with; generated by Ising interaction

Toth and Guhne, PPL 94, 060501 (2005)
Zhou, Zhao, Yuan, and Ma, npj Quantum Information, 5, no. 1, pp. 1-8, (2019)

EW based on graph states

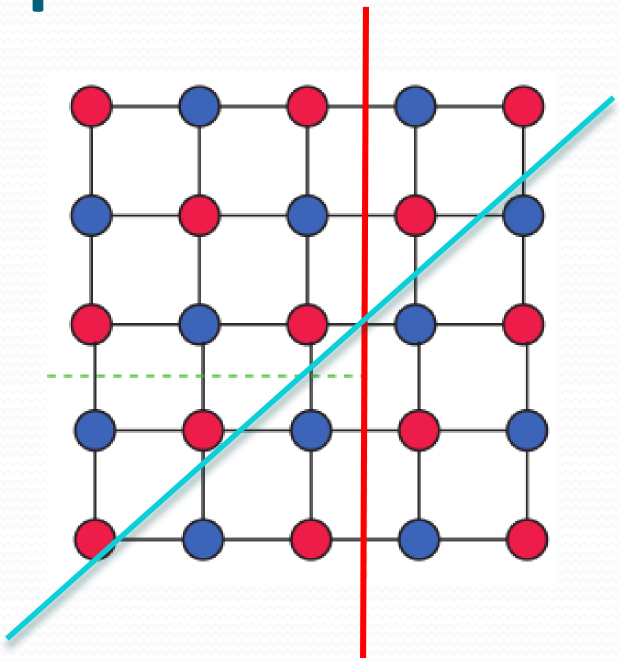
- Consider the witness
  - $W = \alpha I - |\mathrm{G}\rangle\langle \mathrm{G}|$, where $\alpha = \max_{sep}\langle \mathrm{G}|\sigma|\mathrm{G}\rangle$
- Similar to the symmetric state case, we need to measure the fidelity
  - $\langle \mathrm{G}|\rho|\mathrm{G}\rangle$
- Decompose the operator W into local observables

$$W = \sum_i O_1^i \otimes O_2^i \otimes \cdots \otimes O_N^i$$

- In this problem, we do not require the fidelity measure to be exact, a lower bound would do the job of witness

Schmidt decomposition for graph state

- $\alpha = \max_{sep}\langle G|\sigma|G\rangle = \lambda_{max}$
  - $|G\rangle = \sum_i \sqrt{\lambda_i}|\psi_i\rangle_A\, |\phi_i\rangle_{\bar{A}}$
- Fact: entanglement spectrum of graph state is flat
  - $\lambda_1 = \lambda_2 = \ldots\ \lambda_r = \frac{1}{r}$, where r is the rank
- $\alpha = \frac{1}{r} = 2^{-S(\rho_A)}$
  - Here $S(\rho_A)$ is the entanglement entropy (EE), can be defined on any Renyi-entropy.



Zhou, Zhao, Yuan, and Ma, npj Quantum Information, 5, no. 1, pp. 1-8, (2019)

Theoretical upper bounds for SEP states

- Bi-sep states according to the bipartition $\{A, \bar{A}\}$

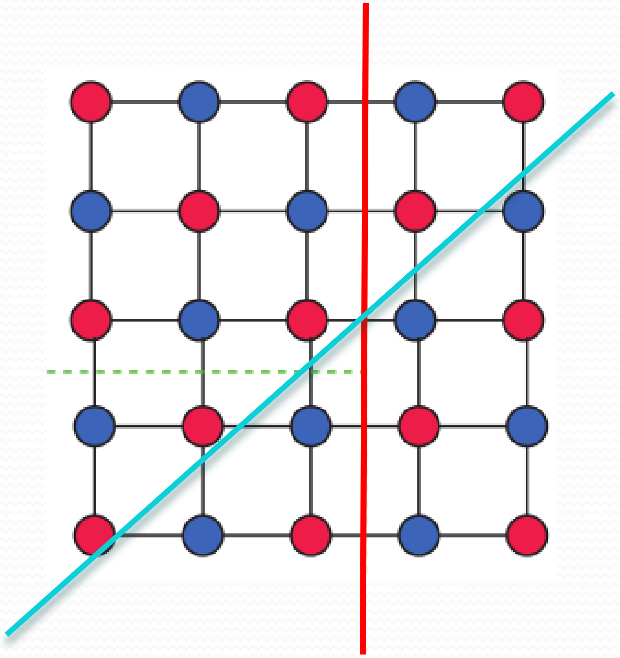
$$\mathrm{Tr}(|G\rangle\langle G|\rho_b) \leq \max_{\{A,\bar{A}\}} 2^{-S(\rho_A)}$$

  - Fact: entanglement spectrum of graph state is flat
  - Run all possible bipartition (tight)

- Fully-sep states according to $\cup_{i=1}^{m} A_i$

$$\mathrm{Tr}\left(|G\rangle\langle G|\rho_f\right) \leq \min_{\{A,\bar{A}\}} 2^{-S(\rho_A)}$$

  - $\{A, \bar{A}\}$ the bipartition of $\cup_{i=1}^{m} A_i$
  - Fully-Sep is bi-sep under any bipartition, thus a minimization here.



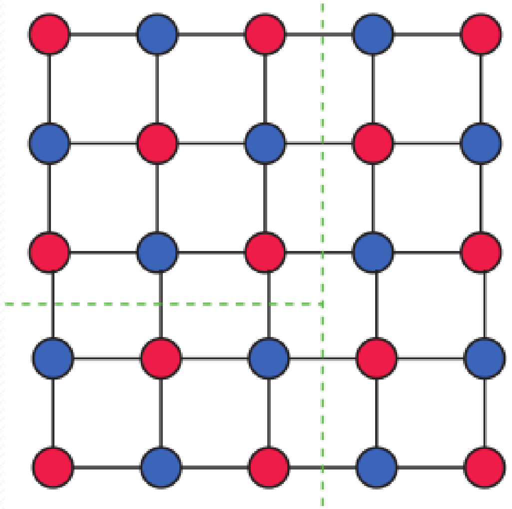
Stabilizers of graph state

- Sufficient way to detect GME for stabilizer states
  - # of LMSs = # of stabilizer operators
  - Still exponential #
- Further reduction (from stabilizer operator generator)
  - Color the graph with k indexes $V_l, l = 1,2 \dots k$
  - Stabilizer operators with the same color can be implemented in one setting
  - # of colors: only depends on the graph

$$S_i = X_i \otimes \prod_{j \in N_i} Z_j$$

$$|G\rangle\langle G| = \prod_{i=1}^{N} \frac{S_i + \mathbb{I}}{2}$$

$$P_l = \prod_{i \in V_l} \frac{S_i + \mathbb{I}}{2}$$



Lower bound for fidelity measure

- Similar to the symmetric state case, we need to measure the fidelity
  - $\langle \mathrm{G}|\rho|\mathrm{G}\rangle$
- Lower bound with less settings based on stabilizer formula

$$|G\rangle\langle G| = \prod_i \frac{S_i + I}{2} = \prod_l P_l$$

$$|G\rangle\langle G| \geq \sum_{l=1}^{k} P_l - (k-1)\mathbb{I} \qquad P_l = \prod_{i \in V_l} \frac{S_i + \mathbb{I}}{2}$$

  - Color the graph with k indexes
  - Stabilizer in the same color share one setting, leads to total k settings

Combing the two bounds

Theorem 1. *Given a partition $\mathcal{P} = \{A_i\}$, the operator $W_f^{\mathcal{P}}$ can witness non-$\mathcal{P}$-fully separability (entanglement),*

$$W_f^{\mathcal{P}} = \left(k - 1 + \min_{\{A,\bar{A}\}} 2^{-S(\rho_A)}\right)\mathbb{I} - \sum_{l=1}^{k} P_l, \tag{12}$$

with $\langle W_f^{\mathcal{P}}\rangle \geq 0$ for all $\mathcal{P}$-fully-separable states; and the operator $W_b^{\mathcal{P}}$ can witness $\mathcal{P}$-genuine entanglement,

$$W_b^{\mathcal{P}} = \left(k - 1 + \max_{\{A,\bar{A}\}} 2^{-S(\rho_A)}\right)\mathbb{I} - \sum_{l=1}^{k} P_l, \tag{13}$$

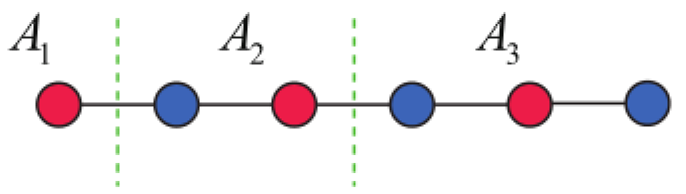
with $\langle W_b^{\mathcal{P}}\rangle \geq 0$ for all $\mathcal{P}$-bi-separable states, where $\{A, \bar{A}\}$ is a bipartition of $\{A_i\}$, $\rho_A = \mathrm{Tr}_{\bar{A}}(|G\rangle\langle G|)$, and the projectors P_l is defined in Eq. (25).

Example: 1-D cluster state

- Entanglement given geometry

$$W_{f,C_1}^{\mathcal{P}_3} = \frac{5}{4}\mathbb{I} - (P_1 + P_2)$$

$$W_{b,C_1}^{\mathcal{P}_3} = \frac{3}{2}\mathbb{I} - (P_1 + P_2)$$



- Area law of EE(counting # of boundary in the 1-D case)

$$S(\rho_{A_1}) = S(\rho_{A_3}) = 1 \qquad S(\rho_{A_2}) = 2$$

- Here color k=2, thus leads to 5/4 and 3/2, respectively.

Applications: without geometry

- GME detection

$$W_b^{\mathcal{P}_N} = \left(k - \frac{1}{2}\right)\mathbb{I} - \sum_{l=1}^{k} P_l$$

  - Minimal entropy is 1 for a connected graph state, taking one qubit as A
- Entanglement structure: separability hierarchy '*m*'
  - Non-*m*-separable witness

$$W_m = \left(k - 1 + \max_{\mathcal{P}_m} \min_{\{A,\bar{A}\}} 2^{-S(\rho_A)}\right)\mathbb{I} - \sum_{l=1}^{k} P_l$$

- Additional maximization on all possible *m*-partition $\cup_{i=1}^{m} A_i$

Example: cluster states

- Fully(N)-Sep (1-D,2-D)

$$W_{f,C}^{\mathcal{P}_N} = (1 + 2^{-\lfloor \frac{N}{2} \rfloor})\mathbb{I} - (P_1 + P_2)$$

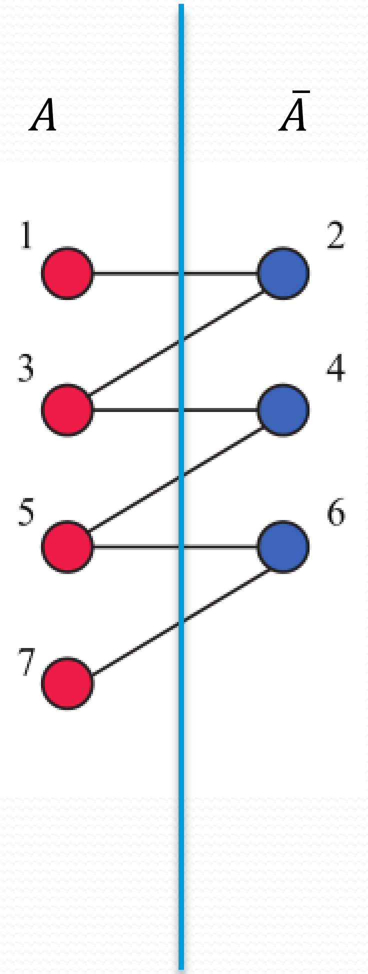
  - EE is upper bounded by the qubit number

$$S(\rho_A) \leq \min\{|A|, |\bar{A}|\} \quad \lfloor \frac{N}{2} \rfloor$$

  - Saturated by choosing the partition based on the color.

- General m-Sep (1-D)

$$W_{m,C_1} = (1 + 2^{-\lfloor \frac{m}{2} \rfloor})\mathbb{I} - (P_1 + P_2)$$



Conclusion

- GME witness
  - Measure the fidelity of a GME pure state $\langle\Psi|\rho|\Psi\rangle$
  - Decompose to LMSs
- Permutation-invariant state
  - State subspace is small
  - LMS complexity: $\binom{N+2}{2}$
  - Further improvement with special cases like GHZ, W, Dicke states
- Graph state
  - Small parameter space implies low LMS complexity
  - Entanglement structure can be very rich

Thank you!

- Welcome to visit