Continuously Controllable Non-Markovianity in Phase Relaxation

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Joint work with

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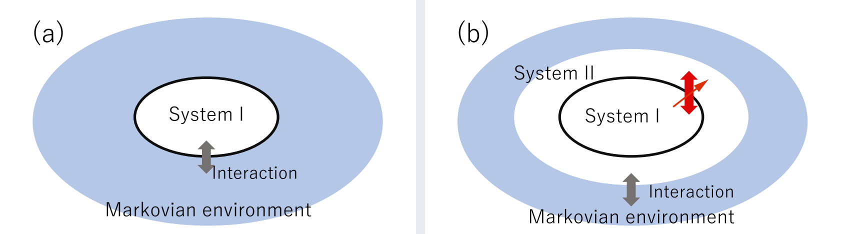
Reference; New J. Phys. **22**, 103048 (2020).

1. Introduction

1. Quantum system is always subject to noise from environment $\rightarrow$ Open quantum system
2. Quantum devices often require small size environment, ultra low temperature and/or ultra high vacuum.
3. Then the quantum dynamics can be non-Markovian.
4. Non-Markovian relaxation is studied in a controllable way here.
5. NMR is used to simulate such an open quantum system experimentally.
6. The Gorini–Kossakowski–Lindblad–Sudarshan (GKLS) master equation is solved exactly for interesting cases and the results are compared with NMR experiment.

2. Theory

2.1 Basic Idea



(a) is the conventional situation in which quantum information of System I "leaks" to the Markovian environment. We assume System I is made of a single qubit from now on.

We analyze the case (b), in which the information leaks to the environment through System II. Interaction between Systems I and II is treated rigorously. The coupling strength between Systems I and II is controllable.

2.2 Markovian Environment (Case (a))

Let ρ be a System I state. Find ρ at $t > 0$ by solving the GKLS eqn.

$$\frac{d\rho}{dt} = -i[H, \rho] + \mathcal{L}[\rho].$$

H is the Hamiltonian of System I and $\mathcal{L}$ is called the Lindbladian;

$$\mathcal{L}[\rho] := \sum_i \gamma_i (2 L_i \rho L_i^\dagger - \{L_i^\dagger L_i, \rho\}).$$

Consider a case in which the environment randomly flips System I qubit;

$$\mathcal{L}[\rho] := \sum_{\pm} \gamma_{\pm} \left(2 \frac{\sigma_{\pm} \rho \sigma_{\mp}}{4} - \left\{ \frac{\sigma_{\mp} \sigma_{\pm}}{4}, \rho \right\} \right).$$

$\sigma_{\pm} = (\sigma_x \pm i \sigma_y)/2$. $\gamma_{\pm}$ represents the flip-flop ($| \downarrow\rangle \leftrightarrow | \uparrow\rangle$) rate of the qubit. Assume they are symmetric; $\gamma_+ = \gamma_- := \gamma_{\mathrm{I}}$.

2. Theory

2.2 Markovian Environment (Case (a))

We take $H = 0$ for simplicity. Then the GKLS eqn. is

$$\frac{d\rho}{dt} = \sum_{\pm} \gamma_{\mathrm{I}} \left(2 \frac{\sigma_{\pm}\rho\sigma_{\mp}}{4} - \left\{ \frac{\sigma_{\mp}\sigma_{\pm}}{4}, \rho \right\} \right).$$

This is solved exactly leading to exponential relaxation with a characteristic time $2/\gamma_{\mathrm{I}}$.

Remark: When the System I qubit is under a magnetic field along the z-axis, the GKLN eqn. is given by

$$\frac{d\rho}{dt} = -i \left[\omega_0 \frac{\sigma_z}{2}, \rho \right] + \sum_{\pm} \gamma \left(2 \frac{\sigma_{\pm}\rho\sigma_{\mp}}{4} - \left\{ \frac{\sigma_{\mp}\sigma_{\pm}}{4}, \rho \right\} \right).$$

The first term can be dropped if we move to the rotating frame.

2.3 Non-Markovian Environment: $1+1$-Case (Case (b))

Assume both System I and System II are made of 1 qubit. Let

$$\rho^{(1)} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12}^* & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{13}^* & \rho_{23}^* & \rho_{33} & \rho_{34} \\ \rho_{14}^* & \rho_{24}^* & \rho_{34}^* & \rho_{44} \end{pmatrix}$$

be the state of the total system. Basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.
$|ab\rangle = |a\rangle_0 \otimes |b\rangle_1 := |a\rangle_{\mathrm{I}} |b\rangle_{\mathrm{II}}$.

$$\mathcal{L}[\rho^{(1)}] = \sum_{i=0,1} \sum_{\pm} \gamma_i \left(2 \frac{\sigma_{\pm}^{(i)} \rho^{(1)} \sigma_{\mp}^{(i)}}{4} - \left\{ \frac{\sigma_{\mp}^{(i)} \sigma_{\pm}^{(i)}}{4}, \rho^{(1)} \right\} \right) := \sum_{i=0,1} \mathcal{L}^{(i)}[\rho^{(1)}],$$

where $\sigma_\mu^{(0)} = \sigma_\mu \otimes \sigma_0$ etc. $\gamma_{\mathrm{I}} := \gamma_0, \gamma_{\mathrm{II}} := \gamma_1$.

2. Theory

2.3 Non-Markovian Environment: $1+1$-Case (Case (b))

Let's take

$$H^{(1)} = H_J^{(1)} + H_{\omega_1}^{(1)}, \quad H_J^{(1)} := J\frac{\sigma_z^{(0)} \cdot \sigma_z^{(1)}}{4}, H_{\omega_1}^{(1)} := \omega_1 \frac{\sigma_x^{(1)}}{2}.$$

It is shown that ω_1 controls the effective coupling between qubits and also non-Markovianity of relaxation.

GKLS Eqn.

$$\frac{d\rho^{(1)}}{dt} = -i[H^{(1)}, \rho^{(1)}] + \mathcal{L}[\rho^{(1)}] = \mathcal{D}^{(1)}[\rho^{(1)}] + \mathcal{L}^{(0)}[\rho^{(1)}],$$
$$\mathcal{D}^{(1)}[\bullet] := -i\left[H^{(1)}, \ \bullet \ \right] + \mathcal{L}^{(1)}[\bullet].$$

Solve this with the initial condition $\rho^{(1)}(0) = |+\rangle\langle+| \otimes \sigma_0/2, |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$:

$$\rho^{(1)}(0) = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

2. Theory

GKLS Eqn.

Let us expand ρ in terms of the generators of $\mathfrak{sl}(2;\mathbb{C})$ and the unit matrix of qubit-0 as

$$\rho^{(1)} = \frac{\sigma_0^{(0)}}{2}\cdot\frac{A_1^{(1)}+A_2^{(1)}}{2} + \frac{\sigma_z^{(0)}}{2}\cdot\frac{A_1^{(1)}-A_2^{(1)}}{2} + \frac{\sigma_+^{(0)}}{2}\cdot B^{(1)} + \frac{\sigma_-^{(0)}}{2}\cdot (B^{(1)})^\dagger,$$

$$A_1^{(1)} := \sigma_0 \otimes \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & \rho_{22} \end{pmatrix}, A_2^{(1)} := \sigma_0 \otimes \begin{pmatrix} \rho_{33} & \rho_{34} \\ \rho_{34}^* & \rho_{44} \end{pmatrix}, B^{(1)} := \sigma_0 \otimes \begin{pmatrix} \rho_{13} & \rho_{14} \\ \rho_{23} & \rho_{24} \end{pmatrix}.$$

Then the GLKS eqn. is decomposed as

$$\frac{dA_1^{(1)}}{dt} = f(A_1^{(1)}, A_2^{(1)}),\ \frac{dA_2^{(1)}}{dt} = g(A_1^{(1)}, A_2^{(1)}),\ \frac{dB^{(1)}}{dt} = h(B^{(1)}),$$

The dynamics of $B^{(1)}$ is decoupled from those of $A_1^{(1)}$, $A_2^{(1)}$ and $(B^{(1)})^\dagger$. The 1st and the 2nd eqns have no dynamics for our initial conditions:
$f(A_1^{(1)}(0), A_2^{(1)}(0)) = g(A_1^{(1)}(0), A_2^{(1)}(0)) = 0 \to dA_1^{(1)}/dt = dA_2^{(1)}/dt = 0.$

GKLS Eqn.

Dynamics of $B^{(1)}$; Initial condition $B^{(1)}(0) = \sigma_0 \otimes \sigma_0/2$.
Factorize $B^{(1)}(t)$ as $B^{(1)} = e^{-\gamma_{\mathrm{I}} t/2}\tilde{B}^{(1)}(t)$ and substitute it to GKLS eqn. to find

$$\frac{d}{dt}(b_0, b_x, b_y, b_z)^t = M_0(b_0, b_x, b_y, b_z)^t$$

where $\tilde{B}^{(1)} = \frac{1}{2}\sum_{\nu=0,x,y,z} b_\nu \sigma_\nu^{(1)}$ and $M_0 := \frac{1}{2}\begin{array}{c} \begin{array}{cccc} 0 & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & -iJ \\ 0 & -\gamma_{\mathrm{II}} & 0 & 0 \\ 0 & 0 & -\gamma_{\mathrm{II}} & 2\omega_1 \\ -iJ & 0 & -2\omega_1 & -2\gamma_{\mathrm{II}} \end{pmatrix} \end{array}.$

b_x is decoupled from the rest of components and $b_x(t) = 0$ with our initial condition $b_0 = 1, b_x = b_y = b_z = 0$ at $t = 0$. Then GKLS eqn is simplified as

$$\frac{d}{dt}\begin{pmatrix} b_0 \\ b_y \\ b_z \end{pmatrix} = M^t \begin{pmatrix} b_0 \\ b_y \\ b_z \end{pmatrix} \quad M := \frac{1}{2}\begin{pmatrix} 0 & 0 & -iJ \\ 0 & -\gamma_{\mathrm{II}} & 2\omega_1 \\ -iJ & -2\omega_1 & -2\gamma_{\mathrm{II}} \end{pmatrix}.$$

2. Theory

GKLS Eqn.

This equation is exactly solvable, see our NJP paper. The reduced density matrix is

$$\rho_{\mathrm{I}}^{(1)} = \mathrm{Tr}_{\mathrm{II}}(\rho^{(1)}) = \frac{1}{2}\begin{pmatrix} 1 & e^{-\gamma_{\mathrm{I}} t/2} b_0(t) \\ e^{-\gamma_{\mathrm{I}} t/2} b_0(t) & 1 \end{pmatrix}.$$

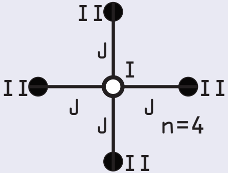
It can be shown that $b_0(t) \in \mathbb{R}$ for $\forall t > 0$. Note that $\rho_{\mathrm{I}}^{(1)}(0) = |+\rangle\langle+|$ and $\rho_{\mathrm{I}}^{(1)}(\infty) = \sigma_0/2$.

Remark:

$$b_0(t) = u_1 \exp(\lambda_1 t/2) + 2\exp(\lambda_2^R t/2)\big(u_2^R \cos(\lambda_2^I t/2) - u_2^I \sin(\lambda_2^I t/2)\big).$$

2.3 Non-Markovian Environment: $1+n$-Case (Case (b))

Let us generalize the previous analysis to $n \geq 2$ cases. Consider a star-shaped network in which the central qubit (System I) couples with other qubits (System II) with the same strength J. No interactions among System II qubits.



$$H = \sum_{i=1}^{n} \left(H_J^{(i)} + H_{\omega_1}^{(i)} \right), \quad H_J^{(i)} := J \frac{\sigma_z^{(0)} \cdot \sigma_z^{(i)}}{4}, \quad H_{\omega_1}^{(i)} := \omega_1 \frac{\sigma_x^{(i)}}{2},$$

Index 0 for System I while indices $1 \leq i \leq n$ for System II.

Basis vectors $\{|00\ldots00\rangle, |00\ldots01\rangle, |00\ldots11\rangle, \ldots, |11\ldots10\rangle, |11\ldots11\rangle\}$,

$|ab\ldots cd\rangle = |a\rangle_0 \otimes |b\rangle_1 \otimes \ldots \otimes |c\rangle_{n-1} \otimes |d\rangle_n$.

2.3 Non-Markovian Environment: $1+n$-Case (Case (b))

Lindbladian is

$$\mathcal{L}[\rho] = \sum_{i=0}^{n} \sum_{\pm} \gamma_i \left(2\frac{\sigma_{\pm}^{(i)} \rho \sigma_{\mp}^{(i)}}{4} - \left\{ \frac{\sigma_{\mp}^{(i)} \sigma_{\pm}^{(i)}}{4}, \rho \right\} \right) := \sum_{i=0}^{n} \mathcal{L}^{(i)}[\rho]$$

where we assume $\gamma_1 = \gamma_2 = \dots \gamma_n =: \gamma_{\mathrm{II}}$. The GKLS eqn. is

$$\frac{d\rho^{(n)}}{dt} = -i[H, \rho^{(n)}] + \mathcal{L}[\rho^{(n)}] = \sum_{i=1}^{n} \mathcal{D}^{(i)}[\rho^{(n)}] + \mathcal{L}^{(0)}[\rho^{(n)}],$$

$$\mathcal{D}^{(i)}[\bullet] := -i \left[(H_J^{(i)} + H_{\omega_1}^{(i)}), \ \bullet \ \right] + \mathcal{L}^{(i)}[\bullet],$$

$$\rho^{(n)}(0) = |+\rangle\langle+| \otimes (\frac{1}{2}\sigma_0)^{\otimes n} = \frac{1}{2^{n+1}} \begin{pmatrix} \sigma_0^{\otimes n} & \sigma_0^{\otimes n} \\ \sigma_0^{\otimes n} & \sigma_0^{\otimes n} \end{pmatrix}$$

Decompose $\rho^{(n)}$ as $\rho^{(n)} =$

$$\frac{\sigma_0^{(0)}}{2} \cdot \frac{A_1^{(n)} + A_2^{(n)}}{2} + \frac{\sigma_z^{(0)}}{2} \cdot \frac{A_1^{(n)} - A_2^{(n)}}{2} + \frac{\sigma_+^{(0)}}{2} \cdot B^{(n)} + \frac{\sigma_-^{(0)}}{2} \cdot (B^{(n)})^\dagger = \begin{pmatrix} A_1'^{(n)} & B'^{(n)} \\ (B'^{(n)})^\dagger & A_2'^{(n)} \end{pmatrix}.$$

2.3 Non-Markovian Environment: $1+n$-Case (Case (b))

With initial condition $A_1^{(n)}(0) = A_2^{(n)}(0) = B^{(n)}(0) = (B^{(n)})^\dagger(0) = \sigma_0 \otimes \frac{1}{2^n}\sigma_0^{\otimes n}$, $A_1^{(n)}$ and $A_2^{(n)}$ are t-independent and $B^{(n)}$ decouples with $A_1^{(n)}, A_2^{(n)}$ and $B^{(n)\dagger}$ as before. By factoring $B^{(n)}$ as $e^{-\gamma_{\mathrm{I}} t/2}\tilde{B}^{(n)}$ as before, we get rid of the effect of $\mathcal{L}^{(0)}$ and GKLS eqn. is written as

$$\frac{d}{dt}\left(\frac{\sigma_+^{(0)}}{2}\cdot \tilde{B}^{(n)}(t)\right) = \sum_{i=1}^{n} \mathcal{D}^{(i)}\left[\frac{\sigma_+^{(0)}}{2}\cdot \tilde{B}^{(n)}(t)\right].$$

Since there is no correlations among System II qubits, we may introduce an Ansatz $\tilde{B}^{(n)}(t) = \prod_{i=1}^{n} \varsigma^{(i)}(t)$, where $\varsigma^{(i)} = \frac{1}{2}\sum_{\nu=0,x,y,z} b_\nu^{(i)}\sigma_\nu^{(i)}$. The dynamics of $b_x^{(i)}$ decouples from those of other $b_\nu^{(i)}$'s. The initial condition $b_0^{(i)}(0) = 1, b_x^{(i)}(0) = b_y^{(i)}(0) = b_z^{(i)}(0) = 0$ tells us that $b_x^{(i)}(t)$ vanishes identically. The action of $\mathcal{D}^{(i)}$ on $\frac{1}{2}\sigma_+^{(0)}\cdot(\sigma_\mu^{(i)}/2)$ is

$$\mathcal{D}^{(i)}\left[\frac{\sigma_+^{(0)}}{2}\cdot \varsigma^{(i)}\right] = \frac{\sigma_+^{(0)}}{2}\cdot \sum_{\nu,\mu=0,y,z} b_\nu^{(i)}(M)_{\nu\mu}\frac{\sigma_\mu^{(i)}}{2}, \quad M = \frac{1}{2}\begin{array}{c} \begin{array}{ccc} 0 & \quad y & \quad z \end{array} \\ \begin{pmatrix} 0 & 0 & -iJ \\ 0 & -\gamma_{\mathrm{II}} & 2\omega_1 \\ -iJ & -2\omega_1 & -2\gamma_{\mathrm{II}} \end{pmatrix} \end{array}$$

2.3 Non-Markovian Environment: $1+n$-Case (Case (b))

This M is exactly the same as M for $1+1$-case. We obtain the ODE for $b_\nu^{(i)}$ as

$$\frac{d}{dt}\begin{pmatrix} b_0^{(i)} \\ b_y^{(i)} \\ b_z^{(i)} \end{pmatrix} = M^t \begin{pmatrix} b_0^{(i)} \\ b_y^{(i)} \\ b_z^{(i)} \end{pmatrix}, \qquad 1 \le i \le n.$$

The solution is independent of i and easily found from the one for the $1+1$-case.

The reduced density matrix of System I is

$$\rho_{\mathrm{I}}^{(n)}(t) := \mathrm{Tr}_{\mathrm{II}}\,\rho^{(n)} = \frac{\sigma_0^{(0)}}{2}\cdot \mathrm{Tr}\left(\prod_{i=1}^{n}\frac{\sigma_0^{(i)}}{2}\right) + e^{-\gamma_{\mathrm{I}} t/2}\left[\frac{\sigma_+^{(0)}}{2}\cdot \mathrm{Tr}\left(\prod_{i=1}^{n}\varsigma^{(i)}\right) + \text{h.c.}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 1 & e^{-\gamma_{\mathrm{I}} t/2}\,(b_0(t))^n \\ e^{-\gamma_{\mathrm{I}} t/2}\,(b_0(t))^n & 1 \end{pmatrix}.$$

Let us define $\beta_n(t) := e^{-\gamma_{\mathrm{I}} t/2}(b_0(t))^n$ for later convenience. The first factor represents Markovian relaxation due to direct interaction with environment while the second factor represents non-Markovian relaxation through System II.

2. Theory

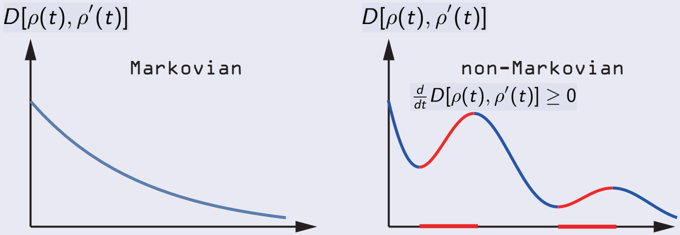
2.4 Non-Markovianity Measure

It is possible to control non-Markovianity by controlling ω_1. To quantify non-Markovianity, we introduce the trace distance $D[\rho(t), \rho'(t)] = \mathrm{Tr}|\rho(t) - \rho'(t)|/2$ of ρ and ρ' and define the measure

$$\mathcal{N} := \max_{\rho(0), \rho'(0)} \int_{\Omega_+} \frac{d}{dt} D[\rho(t), \rho'(t)] dt,$$

where $\Omega_+ := \{t \in [0, \infty) | \frac{d}{dt} D[\rho(t), \rho'(t)] \geq 0\}$.

[H.P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett., **103**, 210401 (2009).]



2. Theory

2.4 Non-Markovianity Measure

Consider an initial state

$$\rho^{(n)}(t=0,\theta) := \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & 1 \end{pmatrix} \otimes \Big(\prod_{i=1}^{n} \frac{\sigma_0^{(i)}}{2}\Big).$$

Then at a later time t, the reduced density matrix is

$$\rho_{\mathrm{I}}^{(n)}(t,\theta) = \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta}\beta_n(t) \\ e^{-i\theta}\beta_n(t) & 1 \end{pmatrix}.$$

The trace distance of two such states is

$$D[\rho_{\mathrm{I}}^{(n)}(t,\theta_1), \rho_{\mathrm{I}}^{(n)}(t,\theta_2)] = \left|\beta_n(t)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right|.$$

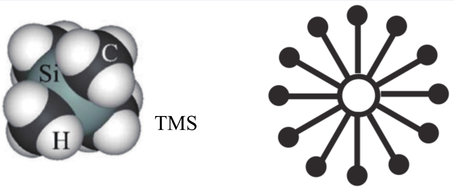
A pair of pure states in System I with antipodal initial Bloch vectors, $\rho_{\mathrm{I}}^{(n)}(0,\theta)$ and $\rho_{\mathrm{I}}^{(n)}(0,\theta+\pi)$, maximizes the integrand $dD[\rho_{\mathrm{I}}^{(n)}(t,\theta), \rho_{\mathrm{I}}^{(n)}(t,\theta+\pi)]/dt$ of $\mathcal{N}$ at $\forall t>0$.

$$\mathcal{N} = \int_{\Omega_+} dt \frac{dD[\rho_{\mathrm{I}}^{(n)}(t,\theta), \rho_{\mathrm{I}}^{(n)}(t,\theta+\pi)]}{dt} = \int_{\Omega_+} dt \frac{d|\beta_n(t)|}{dt}.$$

3. Experiment

3.1 NMR Setup

- Qubit = Spin-1/2 nucleus.



- TMS molecule: System I (^{29}Si nucleus) and System II ($n = 12$ H nuclei). The star-shaped TMS has the common J.
- Oxygen molecules in the solvent (acetone-d6) act as the magnetic impurities producing finite γ: Environment.
- The Zeeman energy $\omega_0^{(0)}\sigma_z^{(0)}/2 + \omega_0^{(1)}\sum_{i=1}^{n}\omega_z^{(i)}/2$ can be eliminated by employing the rotating frame.
- γ_{I} is measured first by "decoupling" System II spins.

3.2 NMR Hamiltonian

- The Hamiltonian of the TMS molecule is

$$H = J \sum_{i=1}^{12} \frac{\sigma_z^{(0)} \cdot \sigma_z^{(i)}}{4} + \omega_1 \sum_{i=1}^{12} \frac{\sigma_x^{(i)}}{2}.$$

  when the external RF field is in resonance with the Larmor frequency of the System II spins. $J = 2\pi \times 6.6$ rad s^{-1}.

- ω_1 is a static parameter proportional to the applied RF field strength.
- The Hamiltonian physically implements our theoretical model with $n = 12$.
- Non-Markovianity is controllable by adjusting ω_1.
- Measured values of γ's are $(\gamma_{\mathrm{I}}, \gamma_{\mathrm{II}}) = (0.41, 0.20)$ rad s^{-1}.

3. Experiment

3.3 FID Signals

- Set $\rho_{\rm I} = |+\rangle\langle+|$ at $t = 0$ and measure $M_x(t) :\propto \mathrm{Tr}(\rho_{\rm I}(t)\sigma_x)$. The signal decays as time goes (Free Induction Decay).
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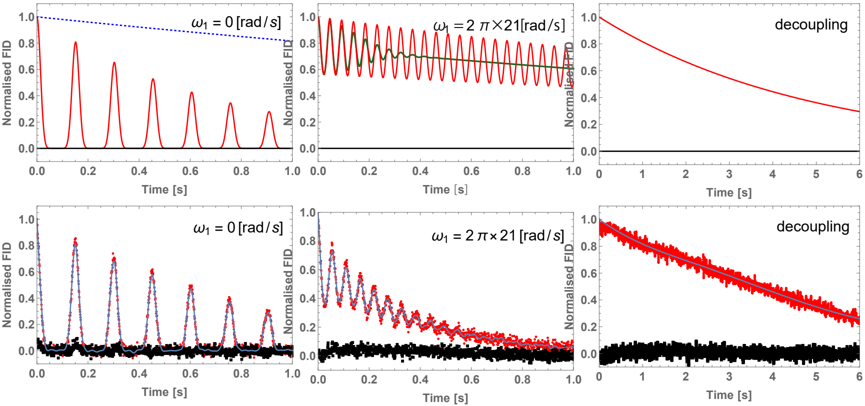
$$\mathrm{Tr}(\rho_{\rm I}(t)\sigma_x) = \frac{1}{2}\mathrm{Tr}\left(\begin{array}{cc} 1 & \beta_n(t) \\ \beta_n(t) & 1 \end{array}\right)\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) = \beta_n(t) = e^{-\gamma_{\rm I} t/2} b_0(t)^n.$$

- When ω_1 is very large, System II spins precess rapidly, which averages out the coupling J (decoupling). System I spin polarization decays as $\propto e^{-i\gamma_{\rm I} t/2}$ in this limit and relaxation is Markovian.
- When $\omega_1 = 0$, in contrast, we have non-Markovian limit.
- Let us look at experimental data keeping the above observation in our mind.

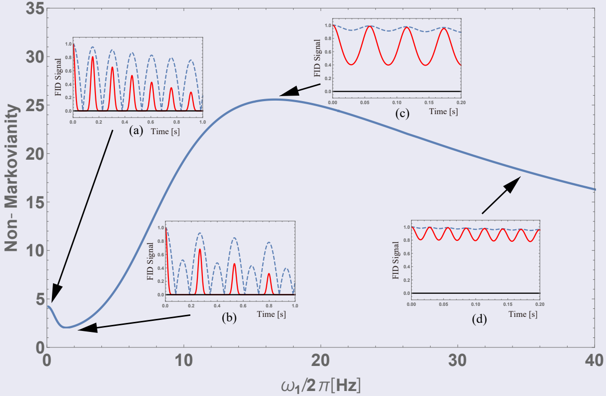
3. Experiment



- Red/black curves are real/imaginary parts of normalized FIDs.
- Blue curves in experiment are obtained by moving averaging.
- Dotted curve in top left shows $e^{-\gamma_I t/2}$.
- Green curve with spatial inhomogeneity $f(\omega_1) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\omega_1-\omega_c)^2}{2\sigma^2}\right)$.

3.4 Non-Markovianity

$$\mathcal{N} := \max_{\rho(0),\rho'(0)} \int_{\Omega_+} \frac{d}{dt} D[\rho(t), \rho'(t)] dt.$$

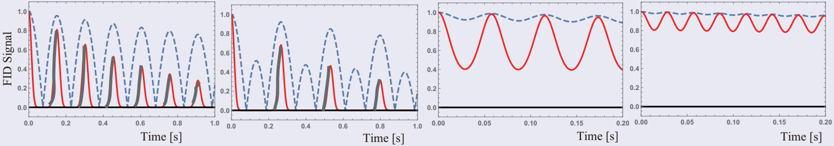


Solid red curves are $\beta_{12}(t)$ while dashed blue curves are $|\beta_1(t)|$.
$\beta_1(t) = e^{-\gamma_I t/2}[u_1 \exp(\lambda_1 t/2) + 2\exp(\lambda_2^R t/2)(u_2^R \cos(\lambda_2^I t/2) - u_2^I \sin(\lambda_2^I t/2))]$.

3. Experiment

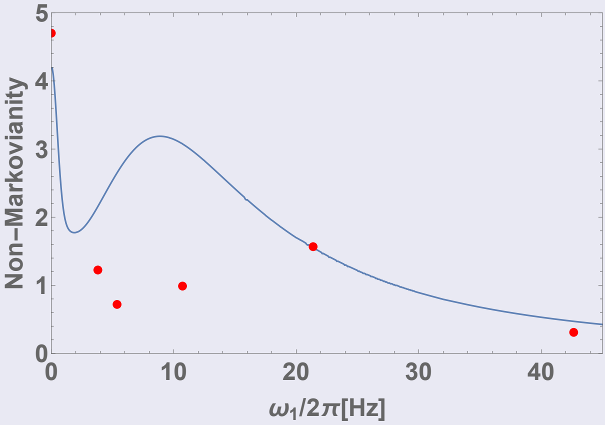
3.4 Non-Markovianity

- Why dip and peak instead of monotonic decrease?
- Note that ω_1 does not directly control $\mathcal{N}$. It controls $\mathcal{N}$ by modulating the effective coupling strength between Systems I and II.
- There are two time scales $2\pi/J$ and $2\pi/\omega_1$. They compete each other when $\omega_1/2\pi \sim J/2\pi \sim 6.6$.
- When ω_1 is small, the oscillation center of $\beta_1(t)$ gradually shifts up as ω_1 increases, which makes contribution of $\beta_{12} \sim \beta_1^{12}$ to $\mathcal{N}$ less and less as ω_1 increases.



- When β_1 is lifted up totally above 0, $n = 12$ enhances non-Markovianity since the power amplifies the derivative $\rightarrow$ the dip.
- $\mathcal{N}$ gradually decreases as ω_1 increases since the oscillation gradually disappears.

3.4 Non-Markovianity



Theory curve: Spatial inhomogeneity of ω_1 taken into account.
Qualitative agreement with $\mathcal{N}_{\mathrm{exp}}$. No fitting parameters in theory!

4. Summary and Outlook

- We proposed a theoretical model that interpolates between Markovian relaxation and non-Markovian relaxation.
- The total system is made of System I (principal system), System II (a part of environment interacting with System I) and environment.
- Interaction between Systems I and II is controllable by adjusting the external field applied to System II, by which the relaxation changes from Markovian to non-Markovian.
- Non-Markovianity is measured by $\mathcal{N}$.
- We implemented the theoretical model faithfully with NMR. FID signals and $\mathcal{N}$ show qualitative/quantitative agreement between theory and experiment.
- $\mathcal{N}(\omega_1)$ shows a peculiar behavior, which can be explained by analyzing FID signals.
- Is it possible to replace n spin-1/2 nuclei by a big spin? The Majorana representation is a technique to visualize a higher dimensional complex vector in terms of multiple Bloch vectors. How about (Majorana representation)$^{-1}$?
  $\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline\end{array}$
- Is there any practical application of our work?

Thank you very much for your attention. 謝謝