# Entanglement Detection 2.0

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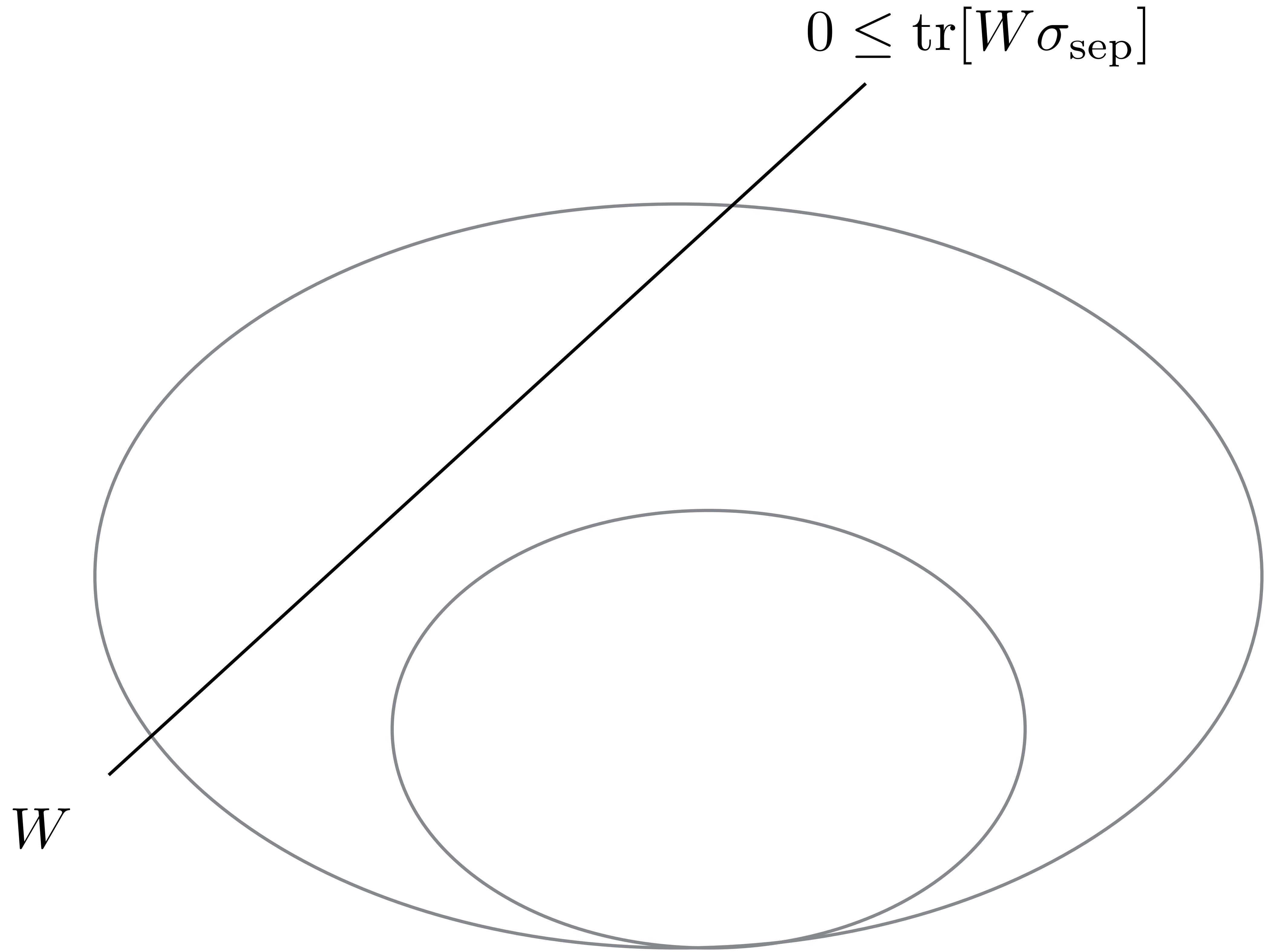
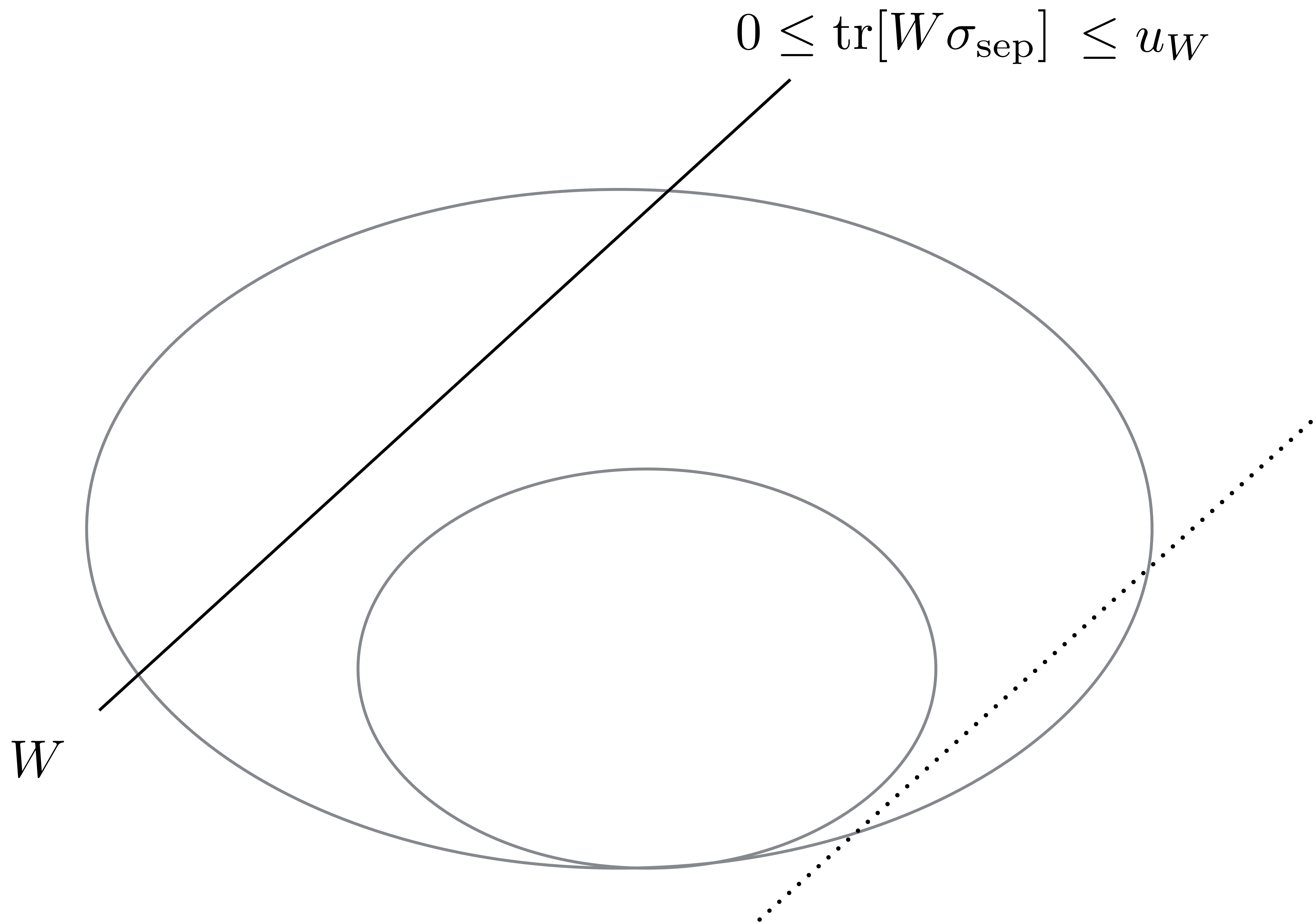
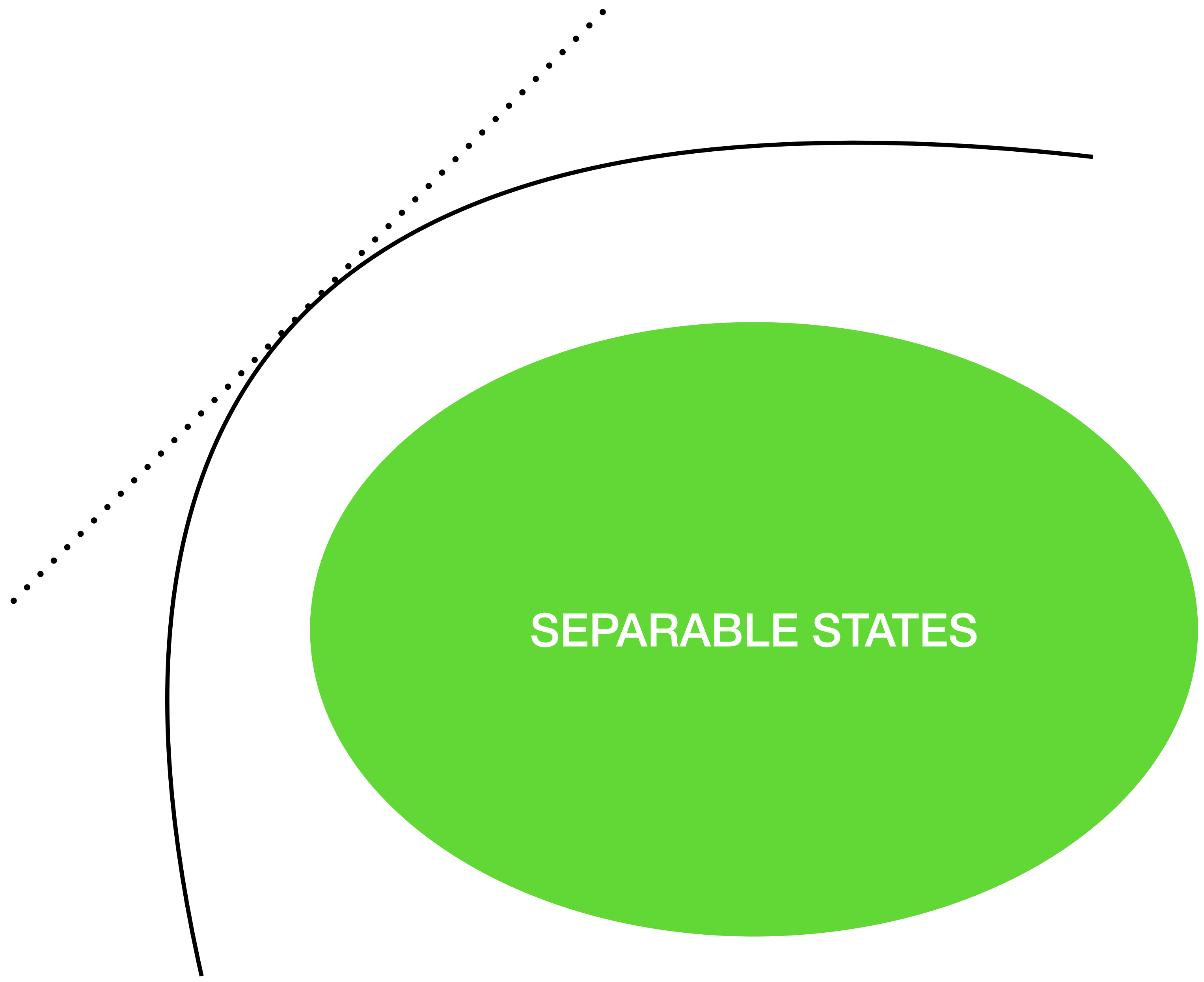
Mirrored Entanglement Witness, npj Quantum Information (2020)

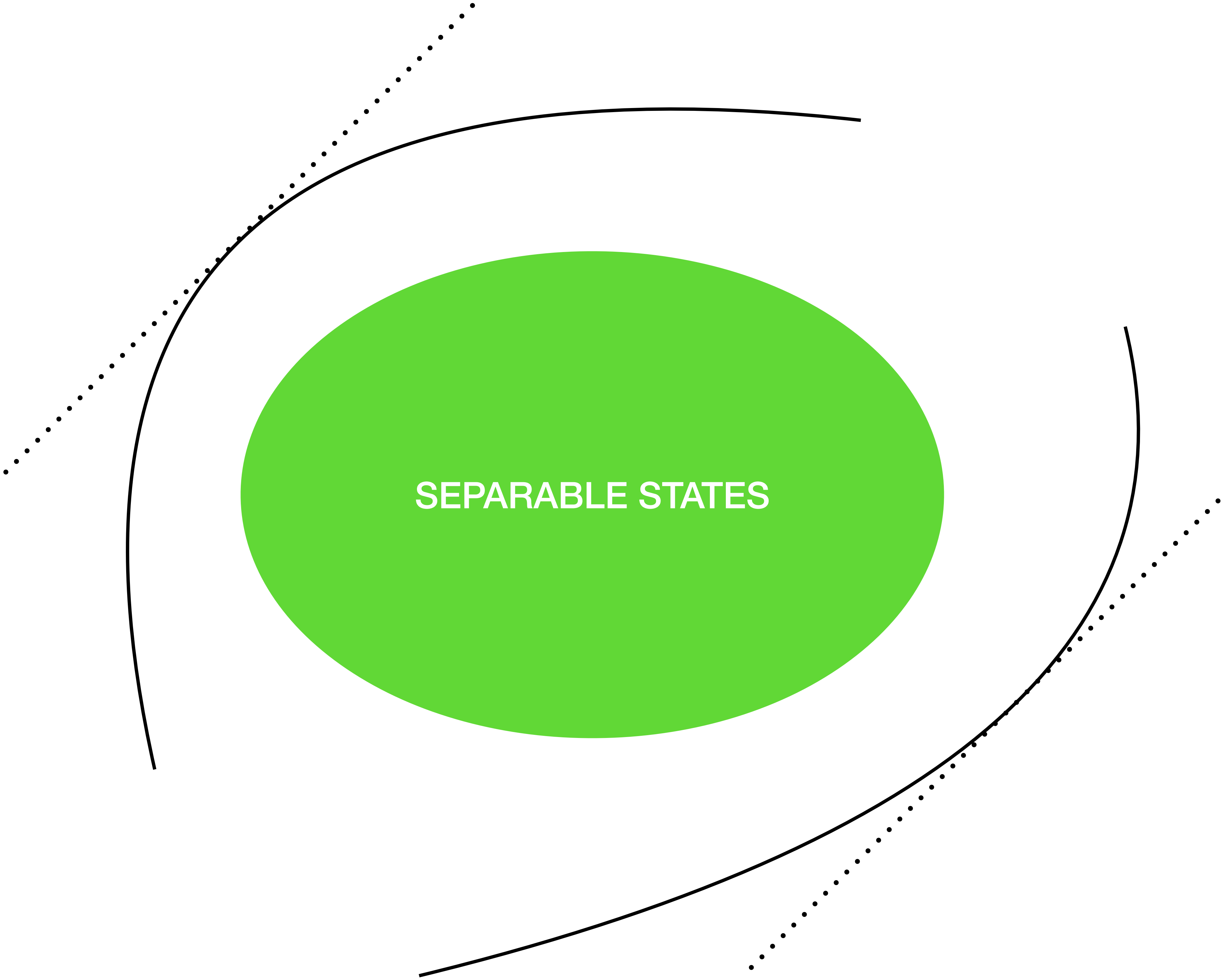
## Take-home message



## Take-home message




SEPARABLE STATES


SEPARABLE STATES

**Introduction: Entanglement Witnesses (EWs)**

**Main Question: Entanglement Detection vs. Quantum State Tomography**

**Our contribution : EWs are more useful than we thought**

Theoretical parts: Many hyperplanes can be generated

Experimental proposal: Entanglement detection can be tested many times

**Discussions: I no longer need Positive Maps to construct EWs.**

**Applications : MUBs, the conjecture in d=6, etc.**

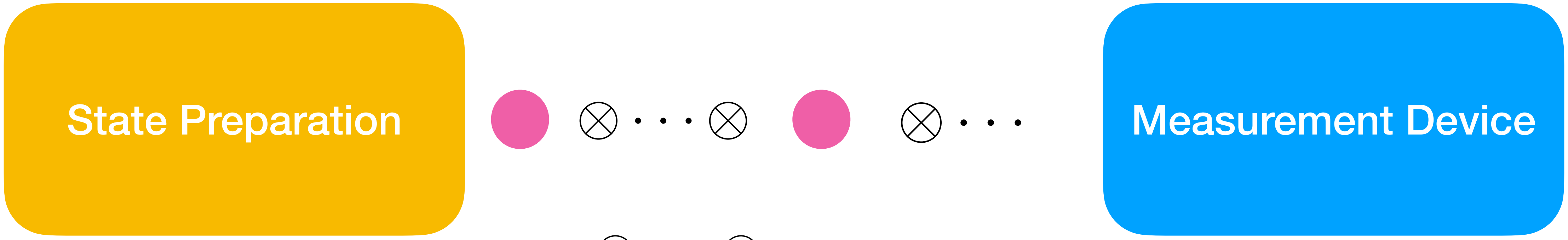
**On-going directions and questions**

# A scenario of detecting entangled states

**The Problem**

Entangled?

Separable?




$\rho_{AB} \otimes \cdots \otimes \rho_{AB}$

Measurement on systems A&B on $\rho_{AB}^{\otimes n}$

Strategy 1. Collective measurement : m copies together

quantum memory
+ joint measurement on A& B (entangling operation + local operations )

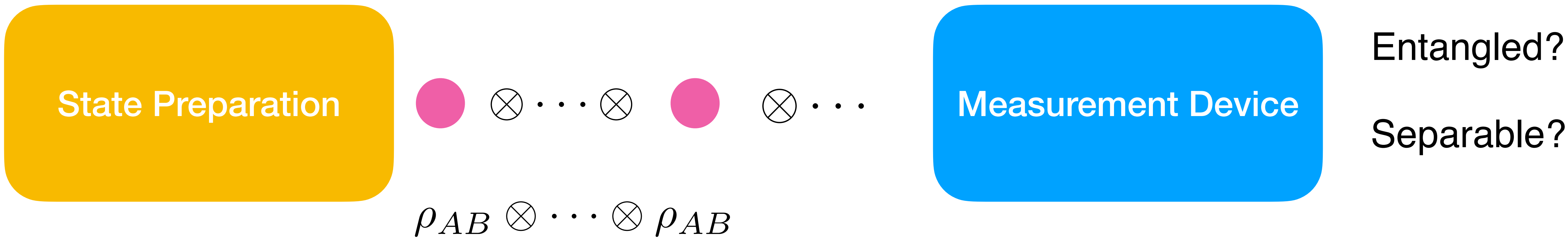
Strategy 2. Joint measurement on A&B : measurement on individual copies

entangling operation + local operations

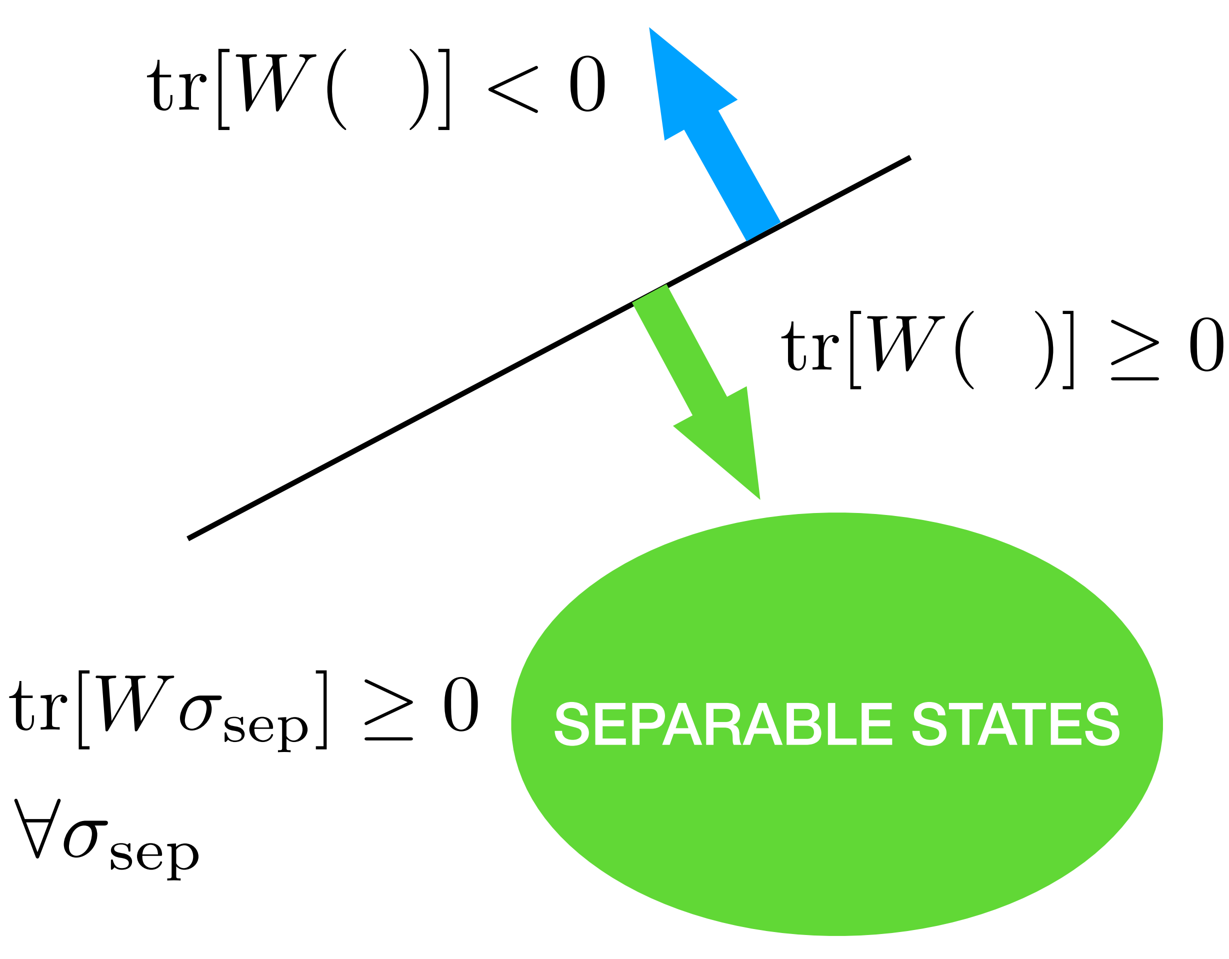
Strategy 3. Local measurements + classical communication (post-processing)

Strategy 3.1. State tomography + All theoretical tools (SDP, P not CP maps, etc.)
<u>Strategy 3.2. Local measurements + Classical communication</u>

# Entanglement detection by individual measurements



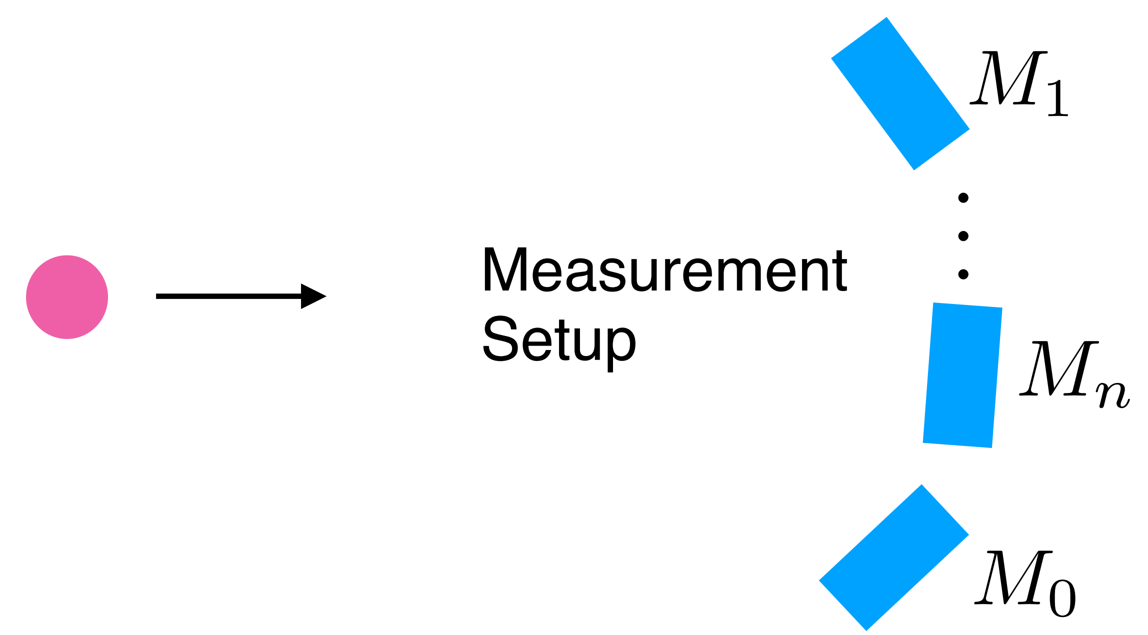

## Entanglement Witnesses




$$W = W^{\dagger} = \sum_k c_k M_k \qquad c_k \in \mathrm{R}$$

$$\mathrm{tr}[W(\ \ )] = \sum_k c_k \mathrm{tr}[M_k(\ \ )]$$

from detectors



Bipartite quantum states

$\rho_{AB}$ product if $\rho_{AB} = \rho_A \otimes \rho_B$

$\rho_{AB}$ separable if $\rho_{AB} = \sum_i q_i \rho_i^A \otimes \rho_i^B$

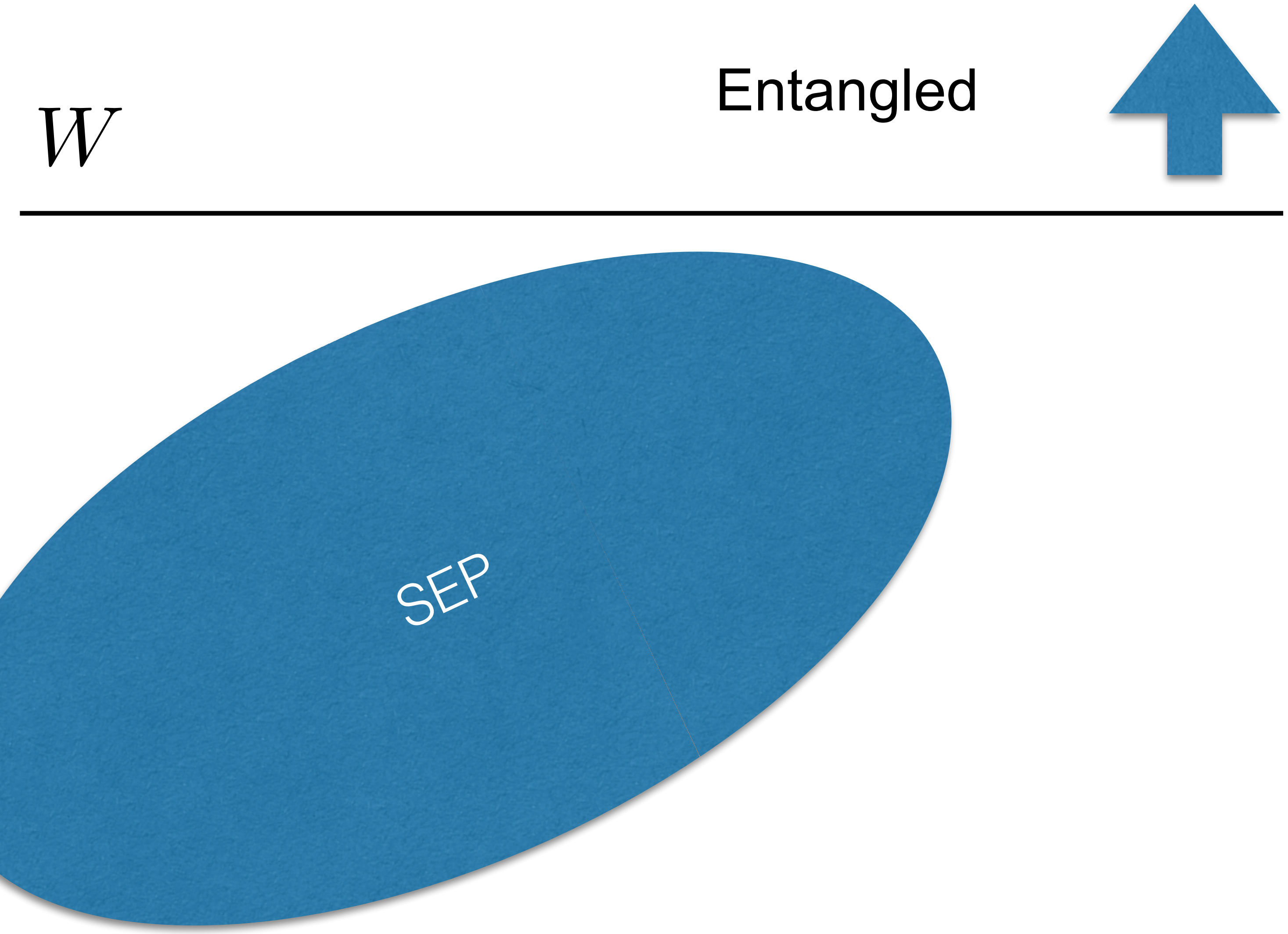
$\rho_{AB}$ entangled if it cannot be prepared by LOCC

$$\rho_{AB} \neq \sum_i q_i \rho_i^A \otimes \rho_i^B$$


The separability problem : border characterization

NP-Hard

## Entanglement Witness




$$\forall \sigma_{\mathrm{sep}} \quad \mathrm{tr}[W \sigma_{\mathrm{sep}}] \geq 0$$

$$\exists \rho \quad \mathrm{tr}[W \rho] < 0$$

$$W = \frac{1}{2} I \otimes I - |\psi^-\rangle\langle\psi^-|$$ is an entanglement witness

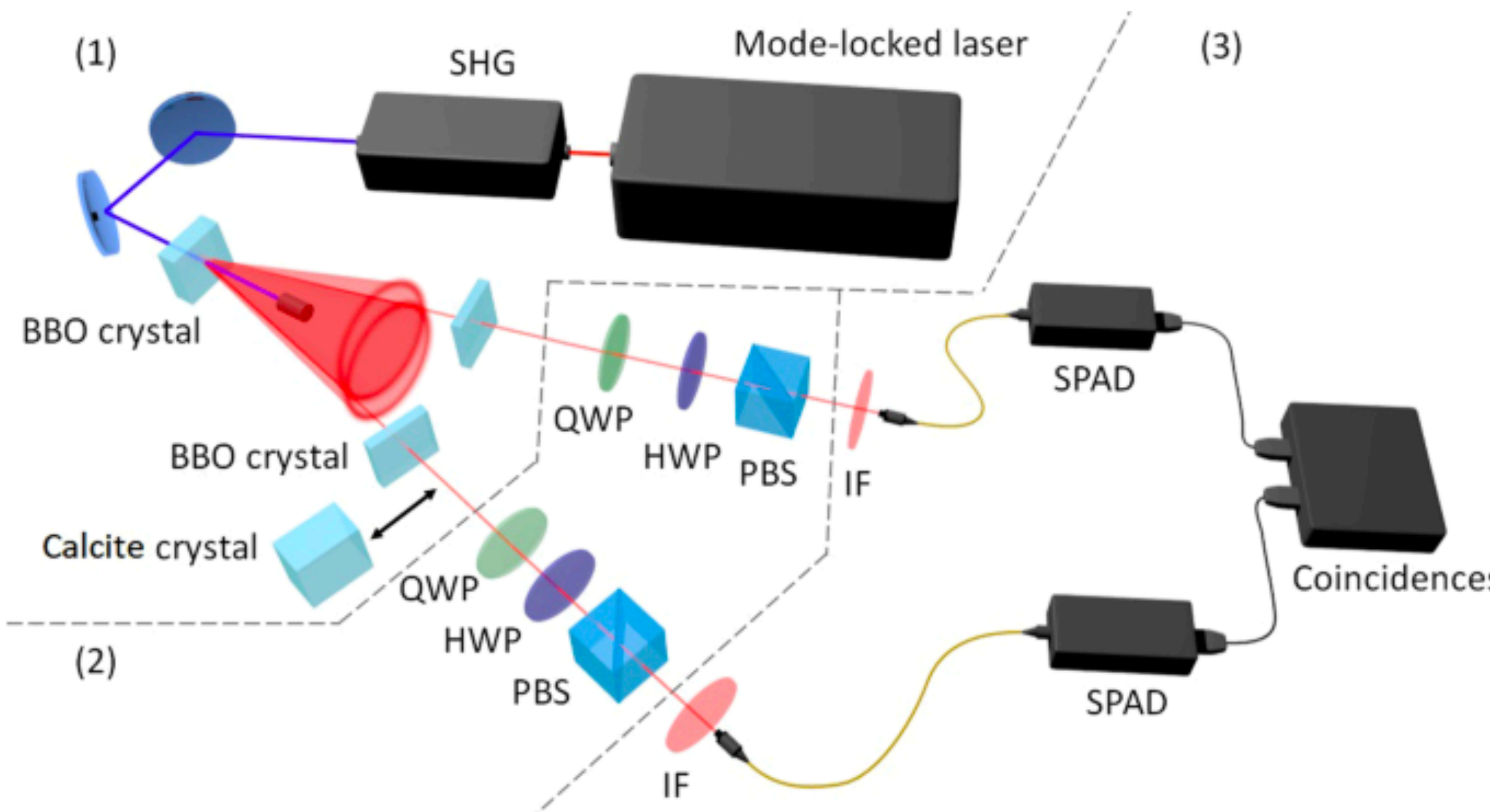
$$\max_{|a\rangle, |b\rangle} |\langle ab|\psi^-\rangle|^2 = \frac{1}{2}$$

$$\mathrm{tr}[W |\psi^-\rangle\langle\psi^-|] = -1/2 < 0$$

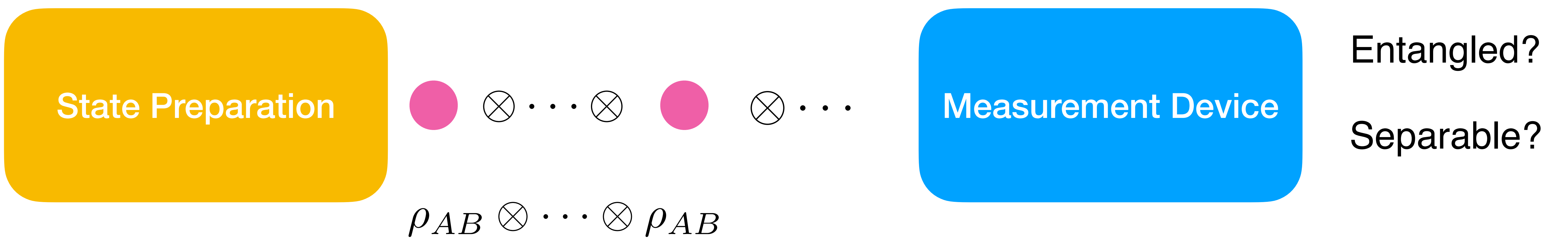
$$W = \frac{1}{2} I \otimes I - |\psi^-\rangle\langle\psi^-|$$ is an observable that can be realized in an experiment

$$W = \frac{1}{4}(I \otimes I + X \otimes X + Y \otimes Y + Z \otimes Z)$$

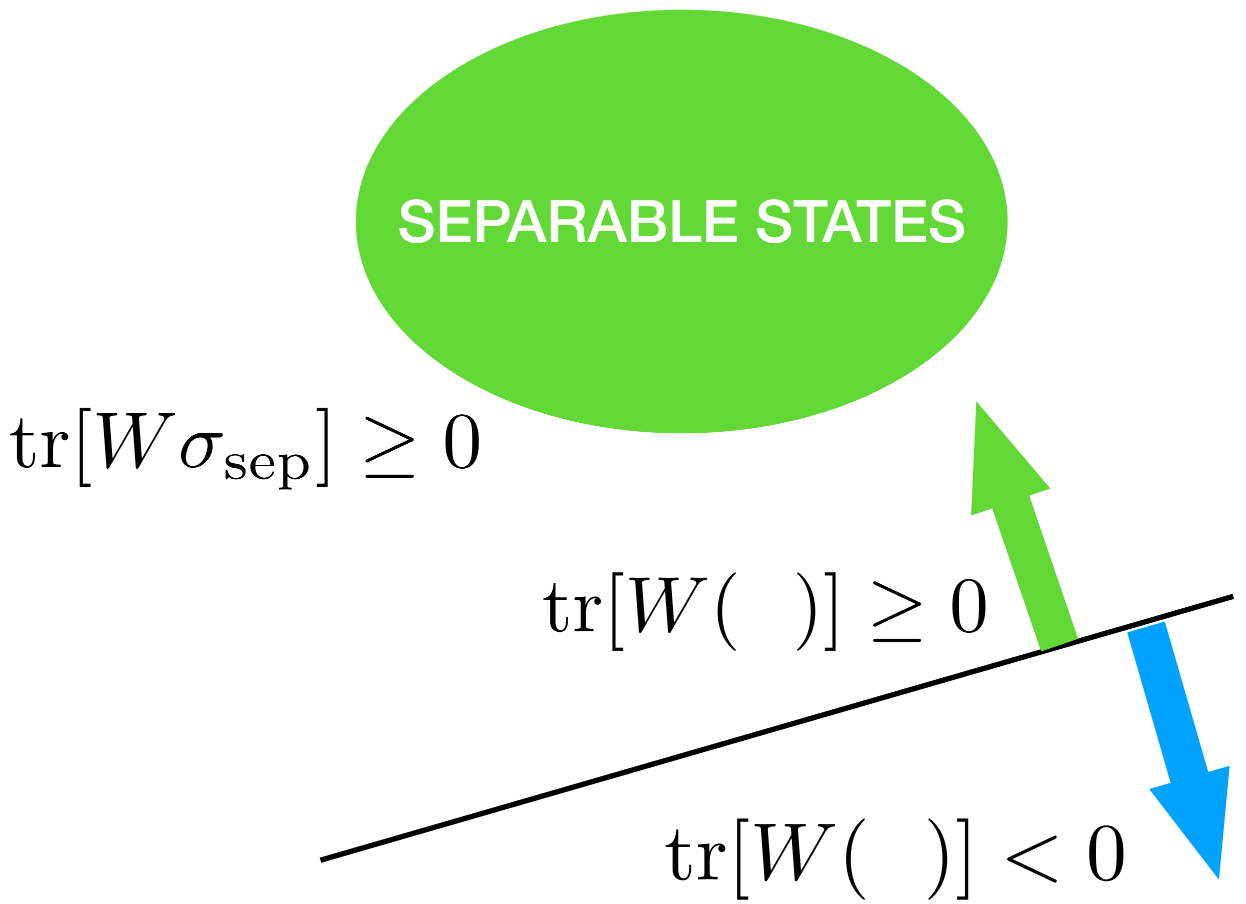
can be factorized into local observables



# Standard Entanglement Witnesses (Local Measurements + CC)




## Entanglement Witnesses

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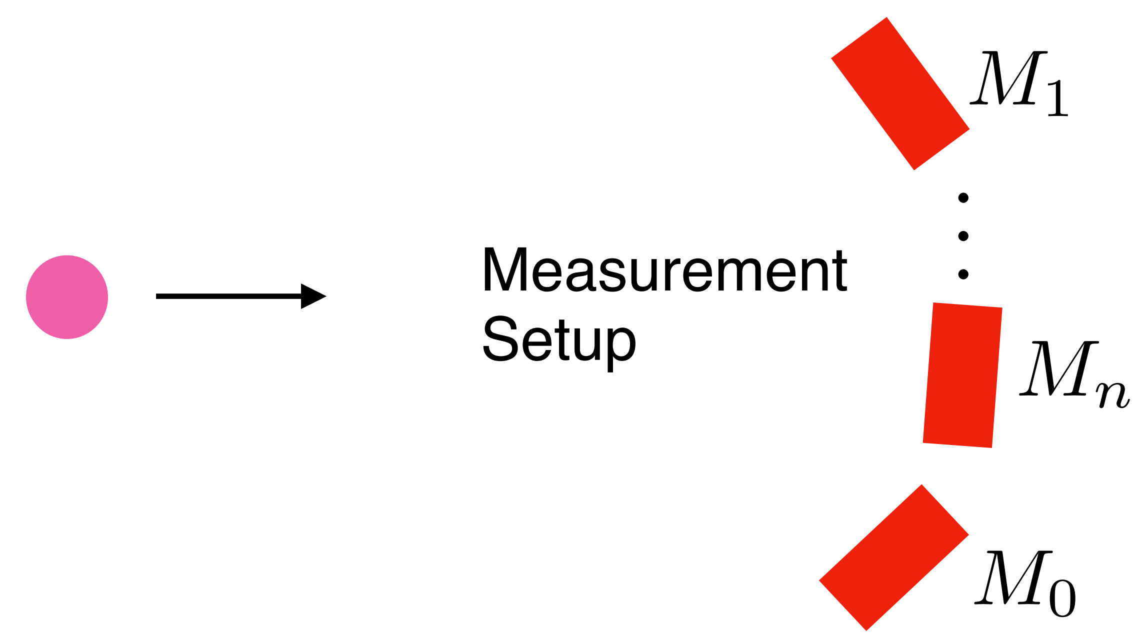
$$\mathrm{tr}[W(\ \ )] = \sum_k c_k \mathrm{tr}[M_k(\ \ )]$$

from detectors



## ***Entanglement Theory***

SEP $\rho_{12} = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)}$

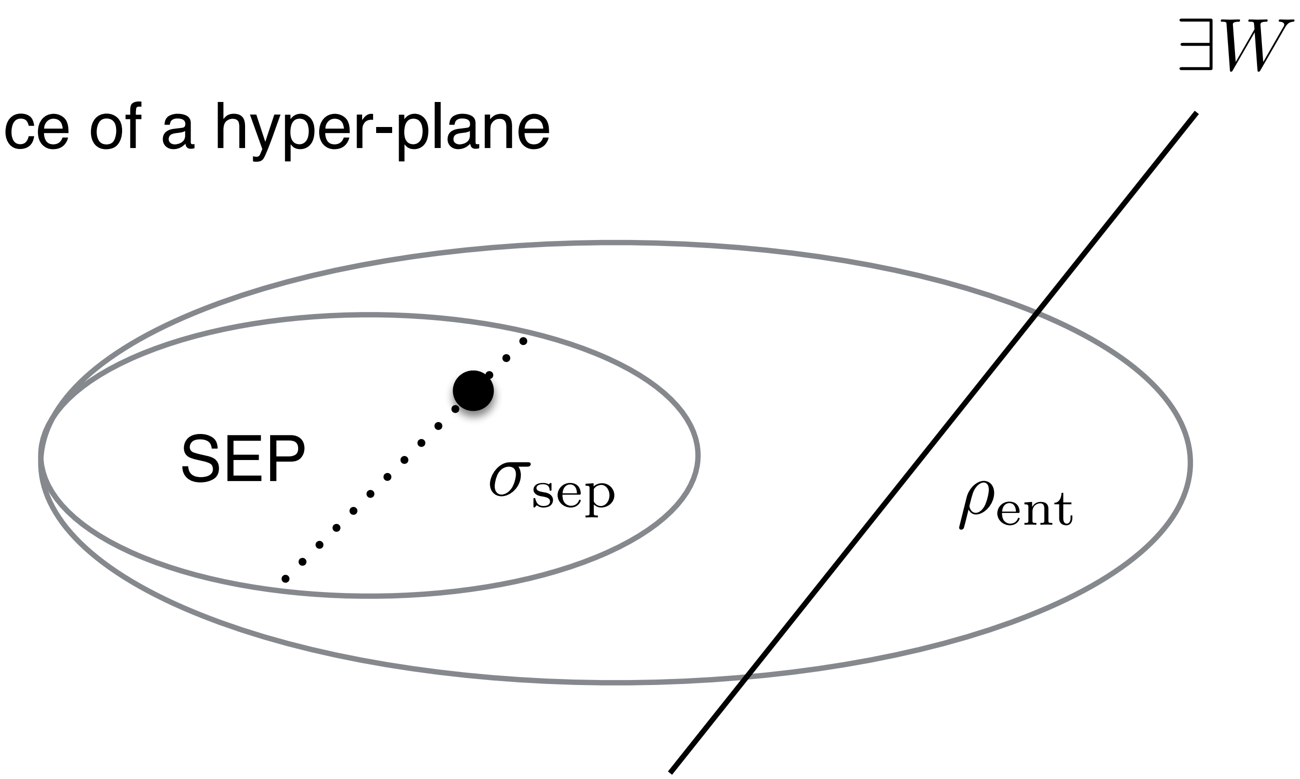
ENT $\rho_{12} \neq \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)}$

Entanglement Witnesses (EWs)

$$\mathrm{tr}[W \rho_{\mathrm{ent}}] < 0$$

$$\mathrm{tr}[W \sigma_{\mathrm{sep}}] \geq 0 \quad \forall \sigma_{\mathrm{sep}}$$

Hahn-Banach theorem: existence of a hyper-plane




Entanglement Witnesses (EWs): Hermitian Operators, non-positive $W = W^\dagger \ngeq 0$

$\rho$ must be entangled if $\mathrm{tr}[W \rho] < 0$

## The theoretical detection box

$$\rho_?$$

⇓

$$f(\rho_?) = \mathrm{tr}[W\rho_?]$$

⇓

$\mathrm{tr}[W\rho_?] < 0$ ⇒ Entangled!

$\mathrm{tr}[W\rho_?] \geq 0$ ⇒ Don't know

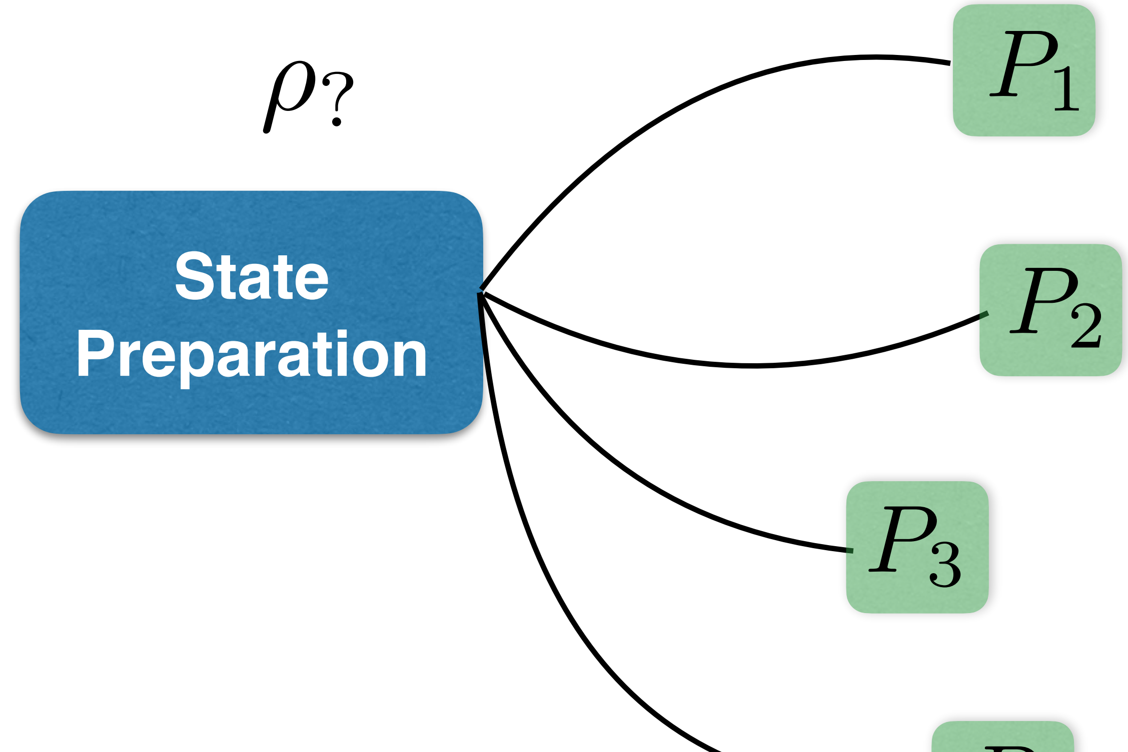
## The detection box in reality

$$W = W^\dagger$$

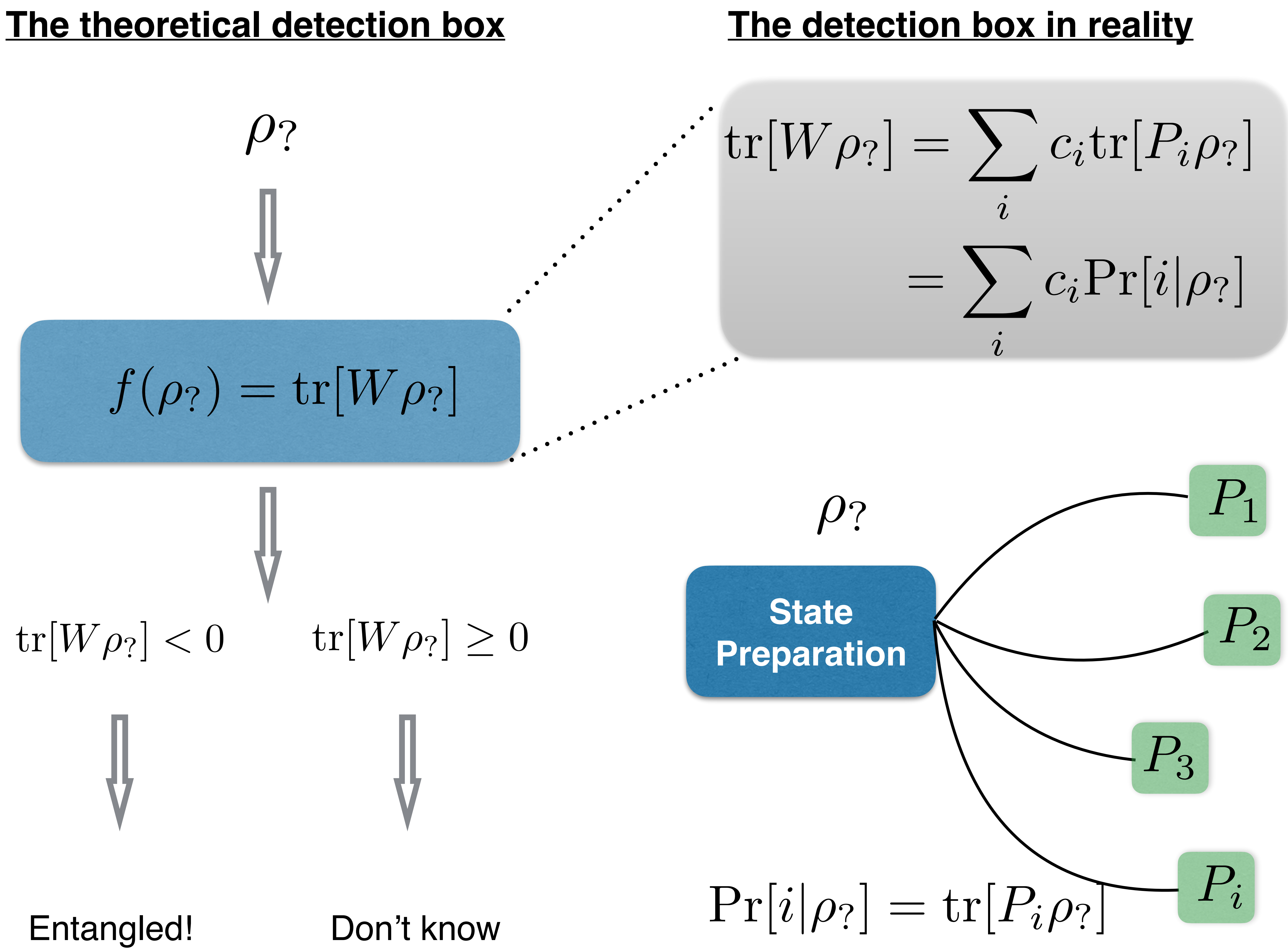
Positive decomposition $W = \sum_{i=1}^{n} c_i P_i$

$$\sum_{i=1}^{n} P_i = I \quad P_i \geq 0 \quad P_i : \ \mathrm{POVM}$$

corresponding to a description of a detecto




$$\Pr[i|\rho_?] = \mathrm{tr}[P_i\rho_?]$$



The theoretical detection box
The detection box in reality
$\rho_?$
$f(\rho_?) = \mathrm{tr}[W\rho_?]$
$\mathrm{tr}[W\rho_?] < 0$
$\mathrm{tr}[W\rho_?] \geq 0$
Entangled!
Don’t know
$\mathrm{tr}[W\rho_?] = \sum_i c_i \mathrm{tr}[P_i\rho_?]$
$= \sum_i c_i \mathrm{Pr}[i|\rho_?]$
$\rho_?$
State Preparation
$P_1$
$P_2$
$P_3$
$P_i$
$\mathrm{Pr}[i|\rho_?] = \mathrm{tr}[P_i\rho_?]$

**Comparisons to quantum state tomography**

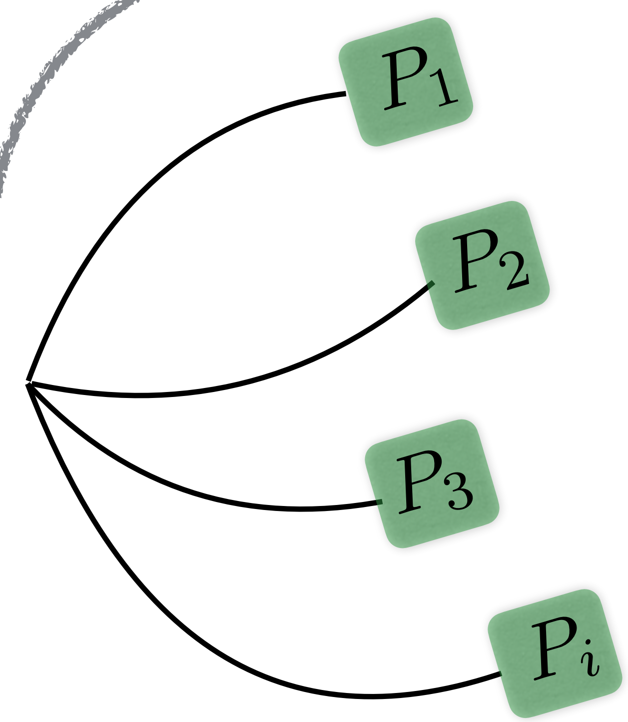
$$\rho_?$$

$$\mathrm{tr}[W\rho_?] = \sum_i c_i \mathrm{Pr}[i|\rho_?]$$

$\mathrm{tr}[W\rho_?] < 0$ → Entangled!

$\mathrm{tr}[W\rho_?] \geq 0$ → Don't know

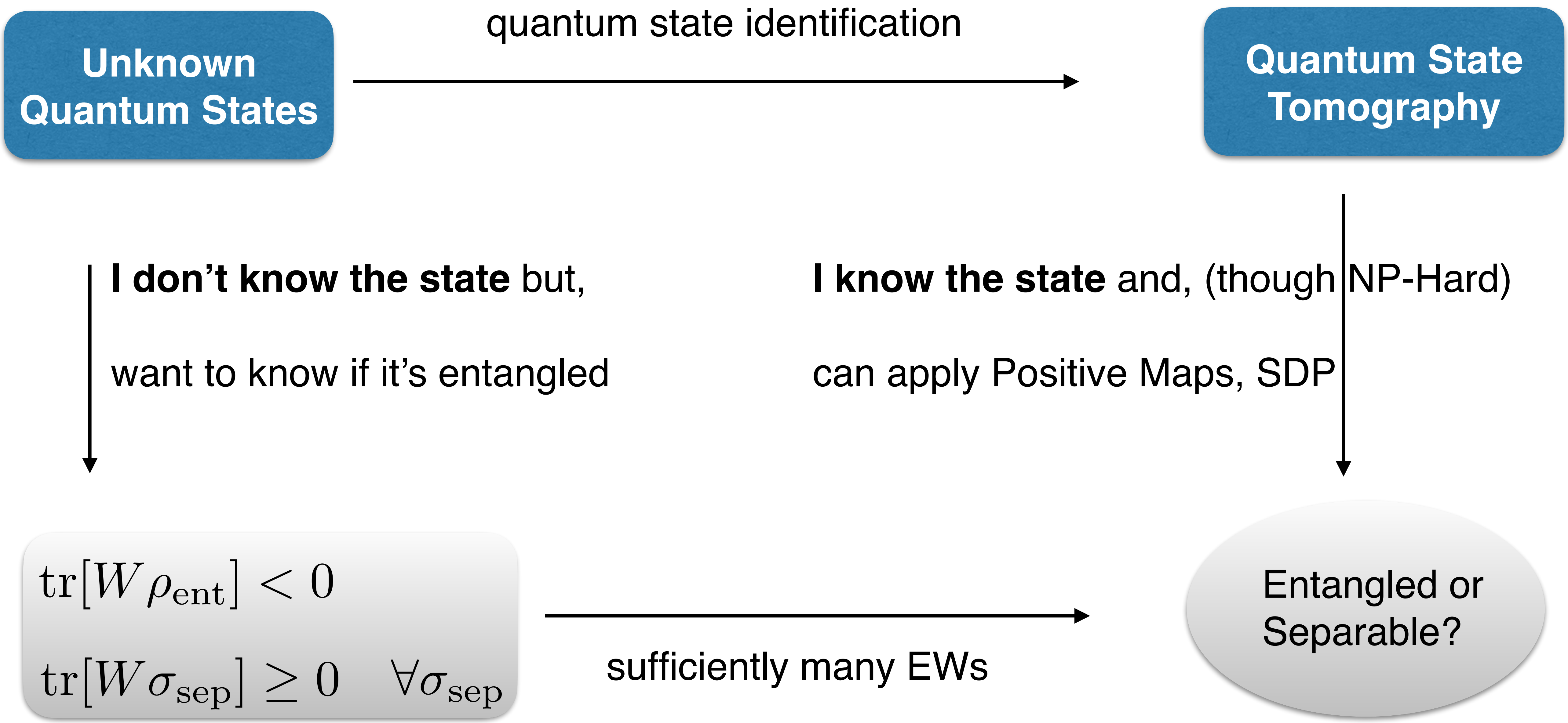
**Resources : classical post processing is free**




$$\mathrm{Pr}[i|\rho_?] = \mathrm{tr}[P_i\rho_?]$$

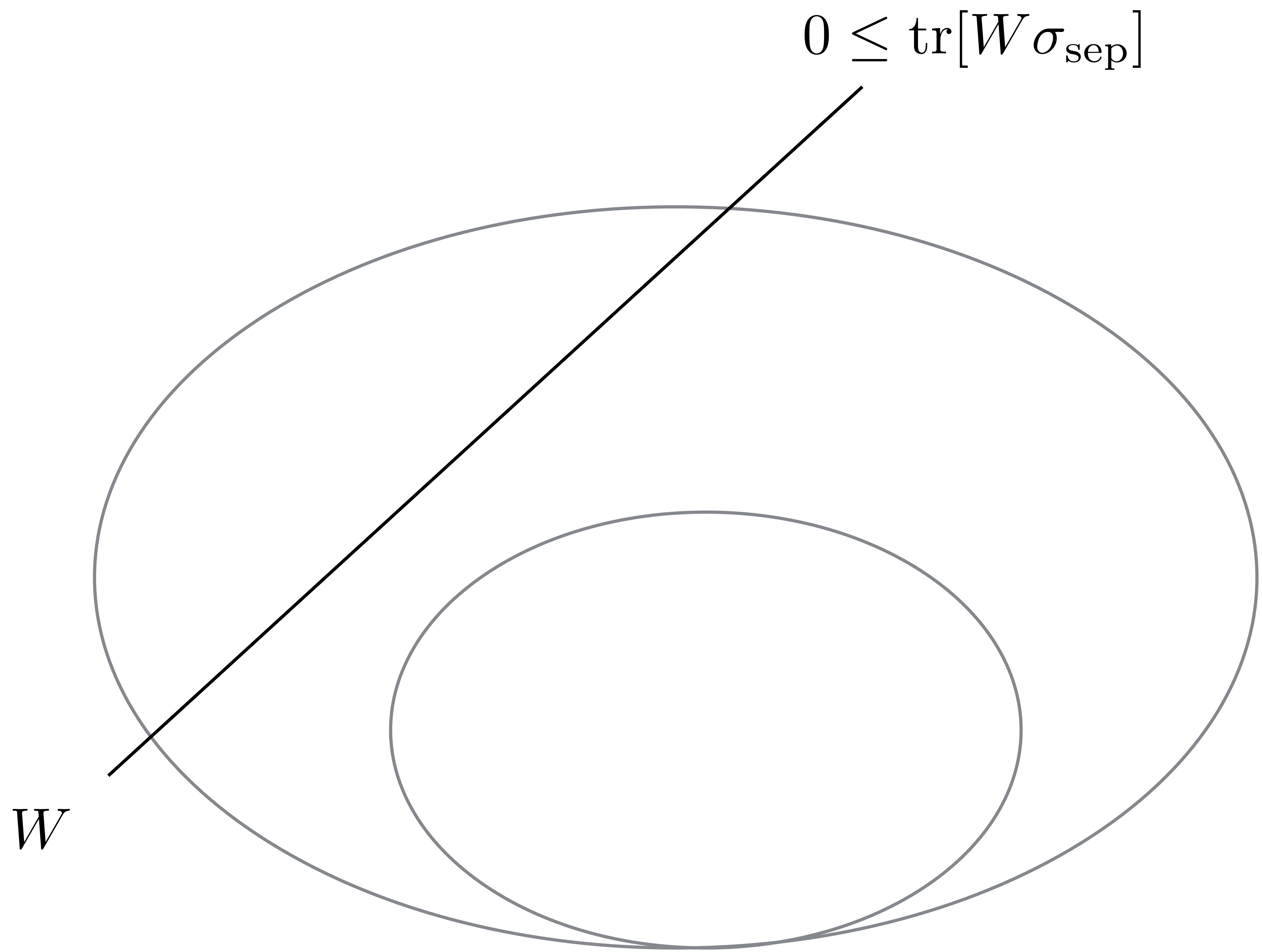
$$\sum_i c_i \mathrm{Pr}[i|\rho_?]$$

# General Picture of Entanglement Detection




**Advantage of EWs: Entanglement of unknown states can be detected**

**Is an EW really useful?**



## Result 1 : EWs to EWs to EWs …

**EXPERIMENT FOR ESTIMATION**

$$p^*/d^2 \leq \mathrm{tr}[\widetilde{W}\sigma_{\mathrm{sep}}] \leq q^*/d^2$$

$\mathrm{tr}[W^{(-)}\sigma_{\mathrm{sep}}] \geq 0$

$\mathrm{tr}[W^{(+)}\sigma_{\mathrm{sep}}] \geq 0$

## Structural Physical Approximation (SPA)

$$\widetilde{W} = (1-p^*)W + p^* \frac{I \otimes I}{d^2} \qquad p^* = \min_{\widetilde{W} \geq 0} p \qquad p \in [0,1]$$

**Detection scheme** $\mathrm{tr}[W\sigma_{\mathrm{sep}}] \geq 0 \iff \mathrm{tr}[\widetilde{W}\sigma_{\mathrm{sep}}] \geq p^*/d^2$

Proof.
$$\mathrm{tr}[\widetilde{W}\sigma_{\mathrm{sep}}] = (1-p^*)\underbrace{\mathrm{tr}[W\sigma_{\mathrm{sep}}]}_{\geq 0} + p^*\mathrm{tr}[\frac{I \otimes I}{d^2}\sigma_{\mathrm{sep}}]$$

$$\geq p^*\mathrm{tr}[\frac{I \otimes I}{d^2}\sigma_{\mathrm{sep}}] \quad = p^*/d^2$$

# positive and negative Structural Physical Approximation (SPA)

**pSPA**

$$\widetilde{W}^{(+)} = \underbrace{(1-p^*)}_{\geq 0} W + p^* \frac{I \otimes I}{d^2}$$

$$p^* = \min_{\widetilde{W}^{(+)} \geq 0} p$$

$p \in [0,1]$

**Detection scheme** $\mathrm{tr}[W\sigma_{\mathrm{sep}}] \geq 0 \iff \mathrm{tr}[\widetilde{W}^{(+)}\sigma_{\mathrm{sep}}] \geq p^*/d^2$

**nSPA**

$$\widetilde{W}^{(-)} = \underbrace{(1-q^*)}_{\leq 0} W + q^* \frac{I \otimes I}{d^2}$$

$$q^* = \max_{\widetilde{W}^{(-)} \geq 0} q$$

$q > 1$

**Detection scheme** $\mathrm{tr}[W\sigma_{\mathrm{sep}}] \geq 0 \iff \mathrm{tr}[\widetilde{W}^{(-)}\sigma_{\mathrm{sep}}] \leq q^*/d^2$

Proof.

$$\mathrm{tr}[\widetilde{W}^{(-)}\sigma_{\mathrm{sep}}] = \underbrace{(1-q^*)}_{\leq 0}\underbrace{\mathrm{tr}[W\sigma_{\mathrm{sep}}]}_{\geq 0} + q^*\mathrm{tr}[\frac{I \otimes I}{d^2}\sigma_{\mathrm{sep}}]$$

$$\leq q^*\mathrm{tr}[\frac{I \otimes I}{d^2}\sigma_{\mathrm{sep}}] \;\; = q^*/d^2$$

## positive and negative Structural Physical Approximation (SPA)

**pSPA** $\widetilde{W}^{(+)} = \underline{(1-p^*)}W + p^* \frac{I \otimes I}{d^2}$ $\quad p^* = \min_{\widetilde{W}^{(+)} \geq 0} p \quad p \in [0,1]$

$\geq 0$

**nSPA** $\widetilde{W}^{(-)} = \underline{(1-q^*)}W + q^* \frac{I \otimes I}{d^2}$ $\quad q^* = \max_{\widetilde{W}^{(-)} \geq 0} q \quad q > 1$

$\leq 0$

**Detection scheme** $\mathrm{tr}[W\sigma_{\mathrm{sep}}] \geq 0 \Leftrightarrow \mathrm{tr}[\widetilde{W}^{(-)}\sigma_{\mathrm{sep}}] \leq q^*/d^2$

$\Updownarrow$

$\mathrm{tr}[\widetilde{W}^{(+)}\sigma_{\mathrm{sep}}] \geq p^*/d^2$

**positive and negative Structural Physical Approximation (SPA)**

**pSPA** $$\widetilde{W}^{(+)} = \underbrace{(1-p^*)}_{\geq 0}W + p^*\frac{I\otimes I}{d^2} \qquad p^* = \min_{\widetilde{W}^{(+)}\geq 0} p \qquad p\in[0,1]$$

**nSPA** $$\widetilde{W}^{(-)} = \underbrace{(1-q^*)}_{\leq 0}W + q^*\frac{I\otimes I}{d^2} \qquad q^* = \max_{\widetilde{W}^{(-)}\geq 0} q \qquad q>1$$

**Given** $W$ → $\widetilde{W}^{(+)}$, $\widetilde{W}^{(-)}$

$$\mathrm{tr}[W\sigma_{\mathrm{sep}}]\geq 0 \iff \mathrm{tr}[\widetilde{W}^{(-)}\sigma_{\mathrm{sep}}]\leq q^*/d^2$$

$$\mathrm{tr}[W\sigma_{\mathrm{sep}}]\geq 0 \iff \mathrm{tr}[\widetilde{W}^{(+)}\sigma_{\mathrm{sep}}]\geq p^*/d^2$$

**Question: does the converse work?** $\widetilde{W} = \widetilde{W}^{(+)} = \widetilde{W}^{(-)}$ → $W^{(+)}$, $W^{(-)}$

## positive and negative Structural Physical Approximation (SPA)

**Conversely!** $\widetilde{W} = \widetilde{W}^{(+)} = \widetilde{W}^{(-)}$ → $W^{(+)}$, $W^{(-)}$

**pSPA**

$$p^* = \min_{\widetilde{W}^{(+)} \geq 0} p$$

$$p \in [0, 1]$$

$$\widetilde{W}^{(+)} = \underbrace{(1 - p^*)}_{\geq 0} W + p^* \frac{I \otimes I}{d^2}$$

$$\widetilde{W} = \widetilde{W}^{(+)} = (1 - p^*) W^{(+)} + p^* \frac{I \otimes I}{d^2}$$

$$\mathrm{tr}[W^{(+)} \sigma_{\mathrm{sep}}] \geq 0 \Longleftrightarrow \mathrm{tr}[\widetilde{W} \sigma_{\mathrm{sep}}] \geq p^*/d^2$$

**nSPA**

$$q^* = \max_{\widetilde{W}^{(-)} \geq 0} q$$

$$q > 1$$

$$\widetilde{W}^{(-)} = \underbrace{(1 - q^*)}_{\leq 0} W + q^* \frac{I \otimes I}{d^2}$$

$$\widetilde{W} = \widetilde{W}^{(-)} = (1 - q^*) W^{(-)} + q^* \frac{I \otimes I}{d^2}$$

$$\mathrm{tr}[W^{(-)} \sigma_{\mathrm{sep}}] \geq 0 \Longleftrightarrow \mathrm{tr}[\widetilde{W} \sigma_{\mathrm{sep}}] \leq q^*/d^2$$

**positive and negative Structural Physical Approximation (SPA)**

$$\mathrm{tr}[W^{(+)}\sigma_{\mathrm{sep}}] \geq 0 \iff \mathrm{tr}[\widetilde{W}\sigma_{\mathrm{sep}}] \geq p^*/d^2$$

$$\mathrm{tr}[W^{(-)}\sigma_{\mathrm{sep}}] \geq 0 \iff \mathrm{tr}[\widetilde{W}\sigma_{\mathrm{sep}}] \leq q^*/d^2$$

**Detection scheme : Detecting entanglement TWICE**

**EXPERIMENT FOR ESTIMATION**

$$p^*/d^2 \leq \mathrm{tr}[\widetilde{W}\sigma_{\mathrm{sep}}] \leq q^*/d^2$$

$\mathrm{tr}[W^{(-)}\sigma_{\mathrm{sep}}] \geq 0$

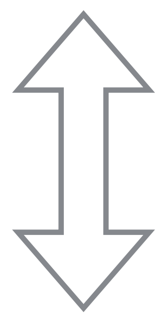
$\mathrm{tr}[W^{(+)}\sigma_{\mathrm{sep}}] \geq 0$

## On the level of standard EWs

**EXPERIMENT FOR ESTIMATION**

$$p^*/d^2 \leq \mathrm{tr}[\widetilde{W}\sigma_{\mathrm{sep}}] \leq q^*/d^2$$

$\mathrm{tr}[W^{(+)}\sigma_{\mathrm{sep}}] \geq 0$ $\mathrm{tr}[W^{(-)}\sigma_{\mathrm{sep}}] \geq 0$



**EXPERIMENT FOR ESTIMATION**

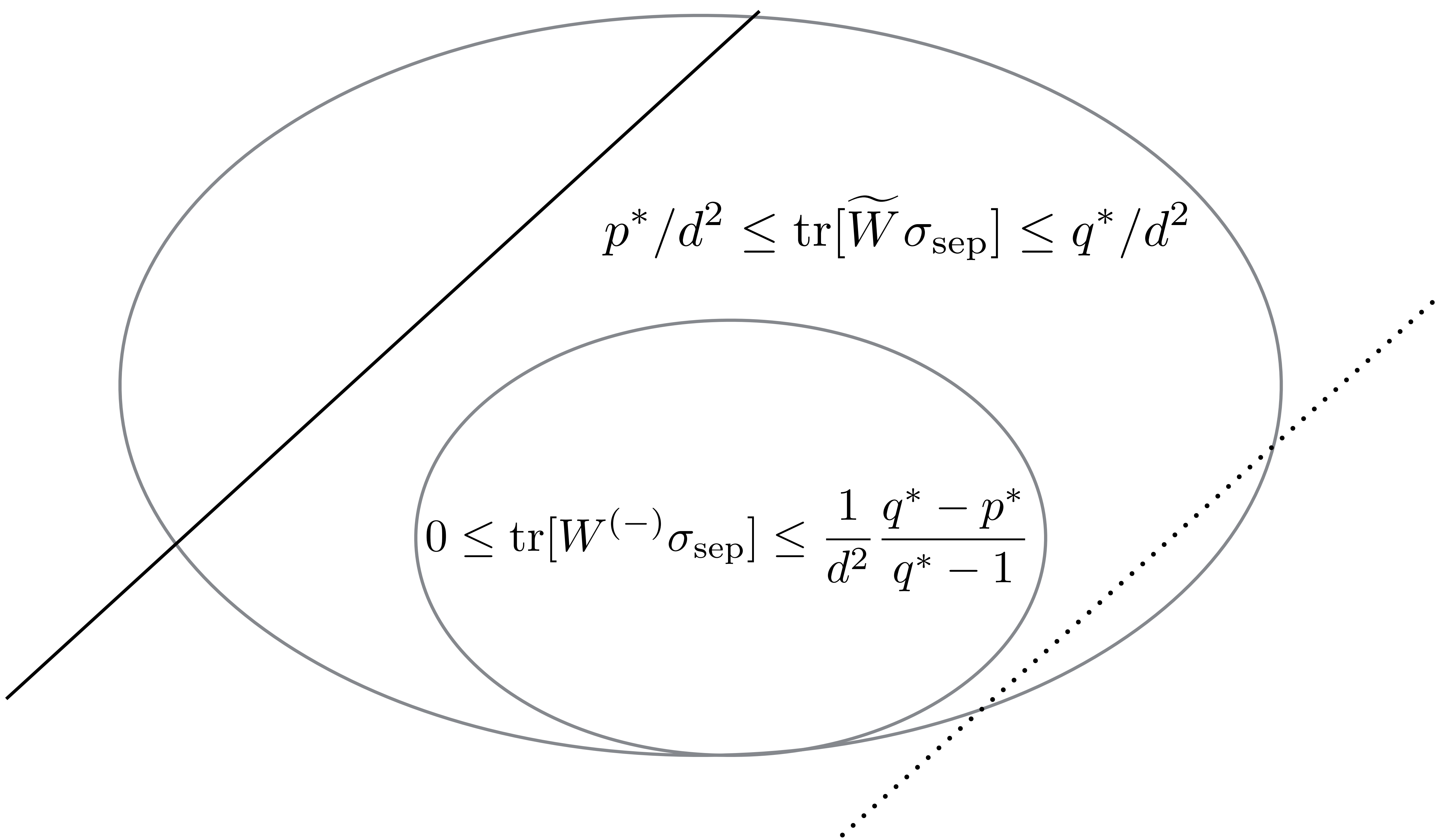
$$0 \leq \mathrm{tr}[W^{(-)}\sigma_{\mathrm{sep}}] \leq \frac{1}{d^2}\frac{q^* - p^*}{q^* - 1}$$

$\mathrm{tr}[W^{(+)}\sigma_{\mathrm{sep}}] \geq 0$

**Remarks.**

**1. p &n SPA are valid in multipartite systems**

**2. To detect entangled states, I don’t need EWs nor positive operators**



# Mirrored Entanglement Witnesses (Local Measurements + CC), EW 2.0…

Summary of the results $nW + n^{'}W^{'} = cI_A \otimes I_B$

There exists an operator, $A \in \mathcal{B}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})$ $A \geq 0$

$$n = d_A d_B U(A) - 1$$

$$n^{'} = 1 - d_A d_B L(A)$$

$$\min_{\sigma \in \mathrm{SEP}} \mathrm{tr}[A\sigma] = L(A) \leq \mathrm{tr}[A\sigma_{\mathrm{sep}}] \leq U(A) = \max_{\sigma \in \mathrm{SEP}} \mathrm{tr}[A\sigma]$$

Interpretation : $\mathrm{tr}[W\rho]$ → $\mathrm{tr}[W^{'}\rho]$ & $\mathrm{tr}[W^{'}\rho]$ → $\mathrm{tr}[W\rho]$

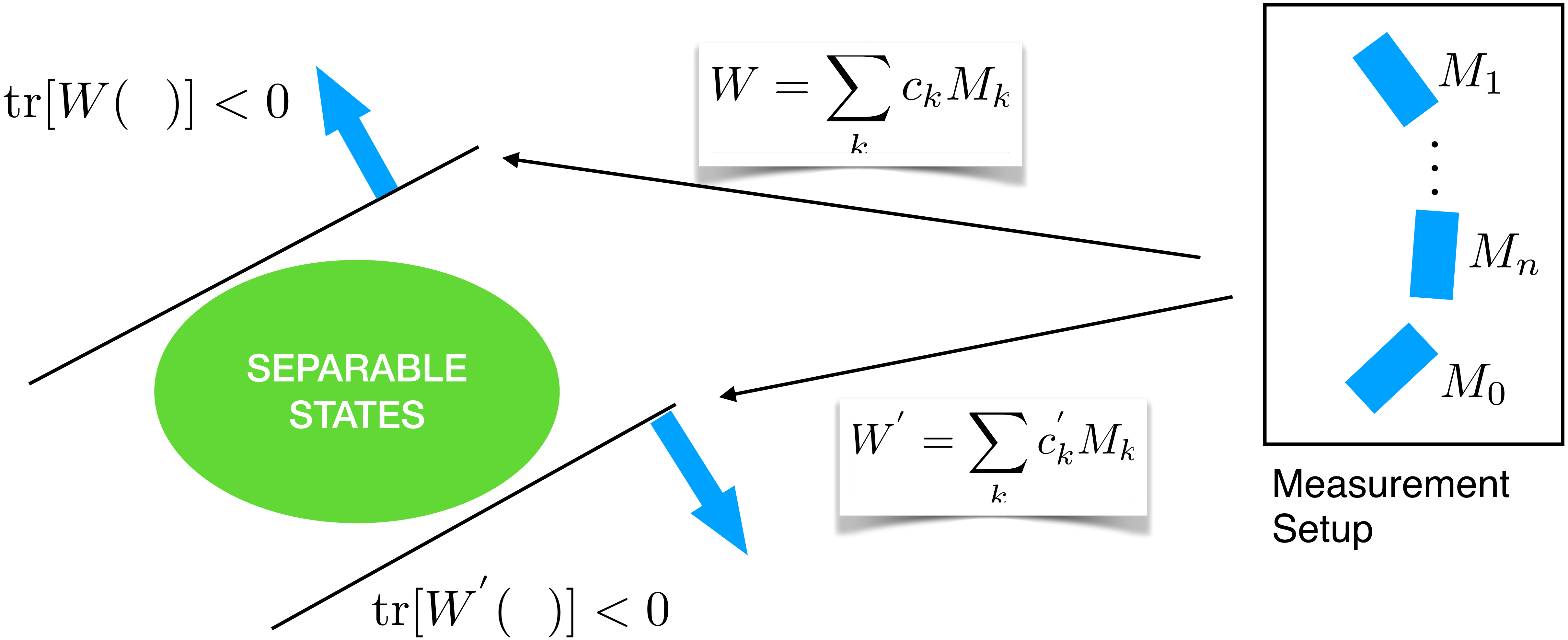
Note 1. This shares the same structure of Structural Physical Approximations (SPA)

$$P_W = (1-p)W^{(+)} + p\frac{I_A \otimes I_B}{d_A d_B} \qquad p < 1$$

$$Q_W = (1-q)W^{(-)} + q\frac{I_A \otimes I_B}{d_A d_B} \qquad q > 1$$

Note 2. The structure holds true for high-dimensional and multipartite systems.

# Mirrored Entanglement Witnesses (Local Measurements + CC), EW 2.0…

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$$n^{'} = 1 - d_A d_B L(A)$$

Note. This shares the same structure of Structural Physical Approximations (SPA)

# Mirrored Entanglement Witnesses (Local Measurements + CC), EW 2.0…

Example

$$W^{(+)} = \frac{1}{8}\left(\begin{array}{cc|cc} 1 & 0 & 0 & -4 \\ 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ -4 & 0 & 0 & 1 \end{array}\right), \quad W^{(-)} = \frac{1}{8}\left(\begin{array}{cc|cc} 3 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 3 \end{array}\right),$$

$$\mathrm{tr}[W^{(+)}\rho_\alpha] < 0 \qquad\qquad \mathrm{tr}[W^{(-)}\rho_\beta] < 0$$

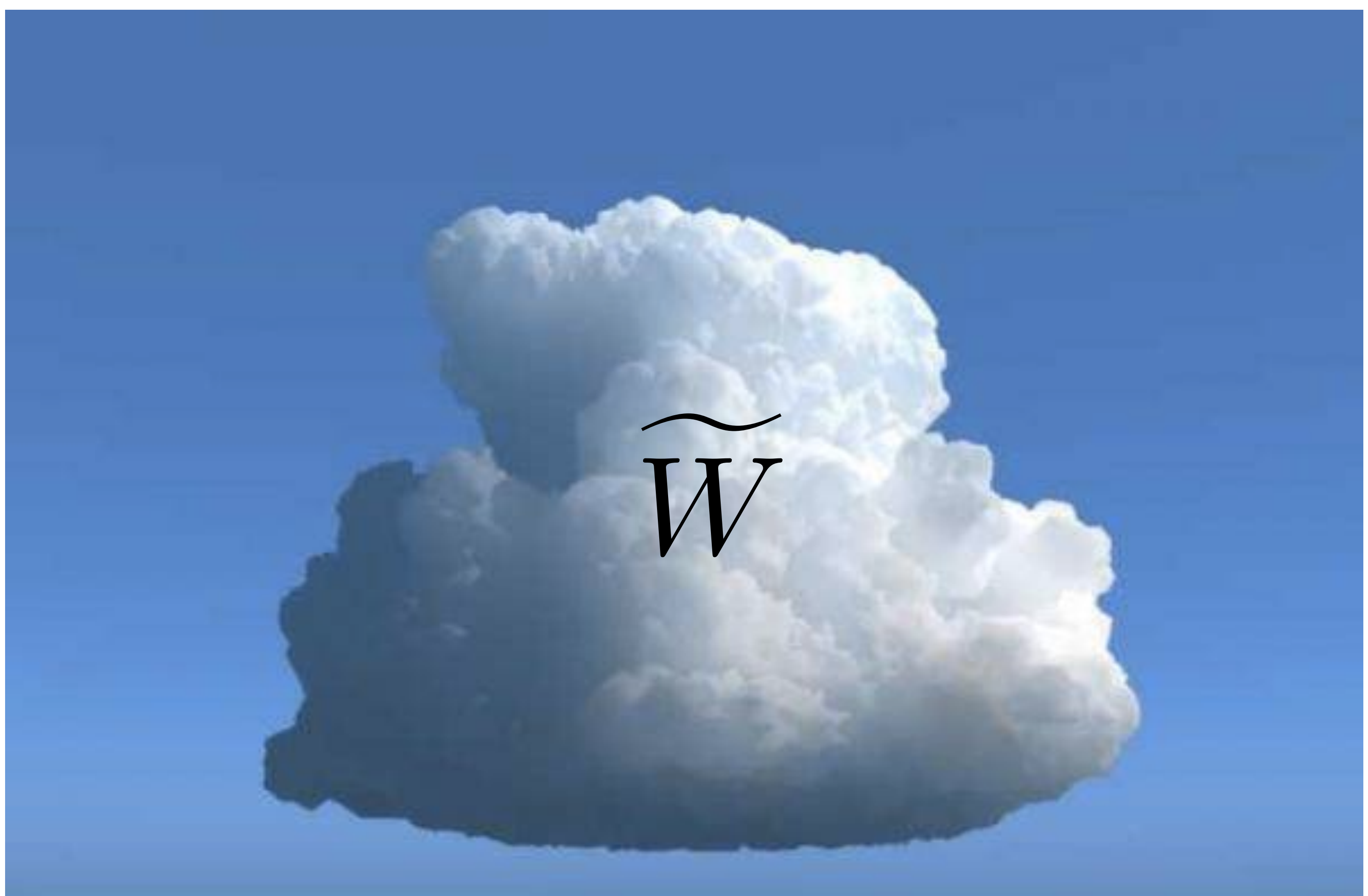
$$\rho_\alpha = (1-\alpha)|\phi^+\rangle\langle\phi^+| + \alpha\frac{1}{4}I \qquad\qquad \rho_\beta = (1-\beta)|\phi^-\rangle\langle\phi^-| + \beta\frac{1}{4}I$$

Connected by Partial Transpose

$$\widetilde{W} = \frac{1}{10}\left(\begin{array}{cc|cc} 2 & 0 & 0 & -2 \\ 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ -2 & 0 & 0 & 2 \end{array}\right), \qquad \frac{3}{20} \leq \mathrm{tr}[\sigma_{sep}\widetilde{W}] \leq \frac{7}{20} \quad \forall \sigma_{sep}$$

$$p^*/d^2 \leq \mathrm{tr}[\widetilde{W}\sigma_{\mathrm{sep}}] \leq q^*/d^2$$

**Result 2 : POVM Cloud = EWs**



How can I detect entangled states?

mathoverflow

Questions Tags Users

## positive not completely positive maps

14

In extension to this question Positive but not completely positive? I'd like to know, fo
examples of $k$-positive linear maps of a matrix algebra into itself that are not $k+1$-p
know a single one.) By M.D. Choi's theorem the size of the matrices involved must g
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How can I detect entangled states?

$$p^*/d^2 \leq \mathrm{tr}[\widetilde{W}\sigma_{\mathrm{sep}}] \leq q^*/d^2$$


$$L = \min_{\sigma_{\mathrm{sep}}}[\widetilde{W}\sigma_{\mathrm{sep}}] \leq \widetilde{W} \leq U = \max_{\sigma_{\mathrm{sep}}}[\widetilde{W}\sigma_{\mathrm{sep}}]$$

# General Picture of Entanglement Detection

**Unknown Quantum States** → (quantum state identification) → **Quantum State Tomography**

**I don't know the state** but,

want to know if it's entangled

**I know the state** and, (though NP-Hard)

can apply Positive Maps, SDP

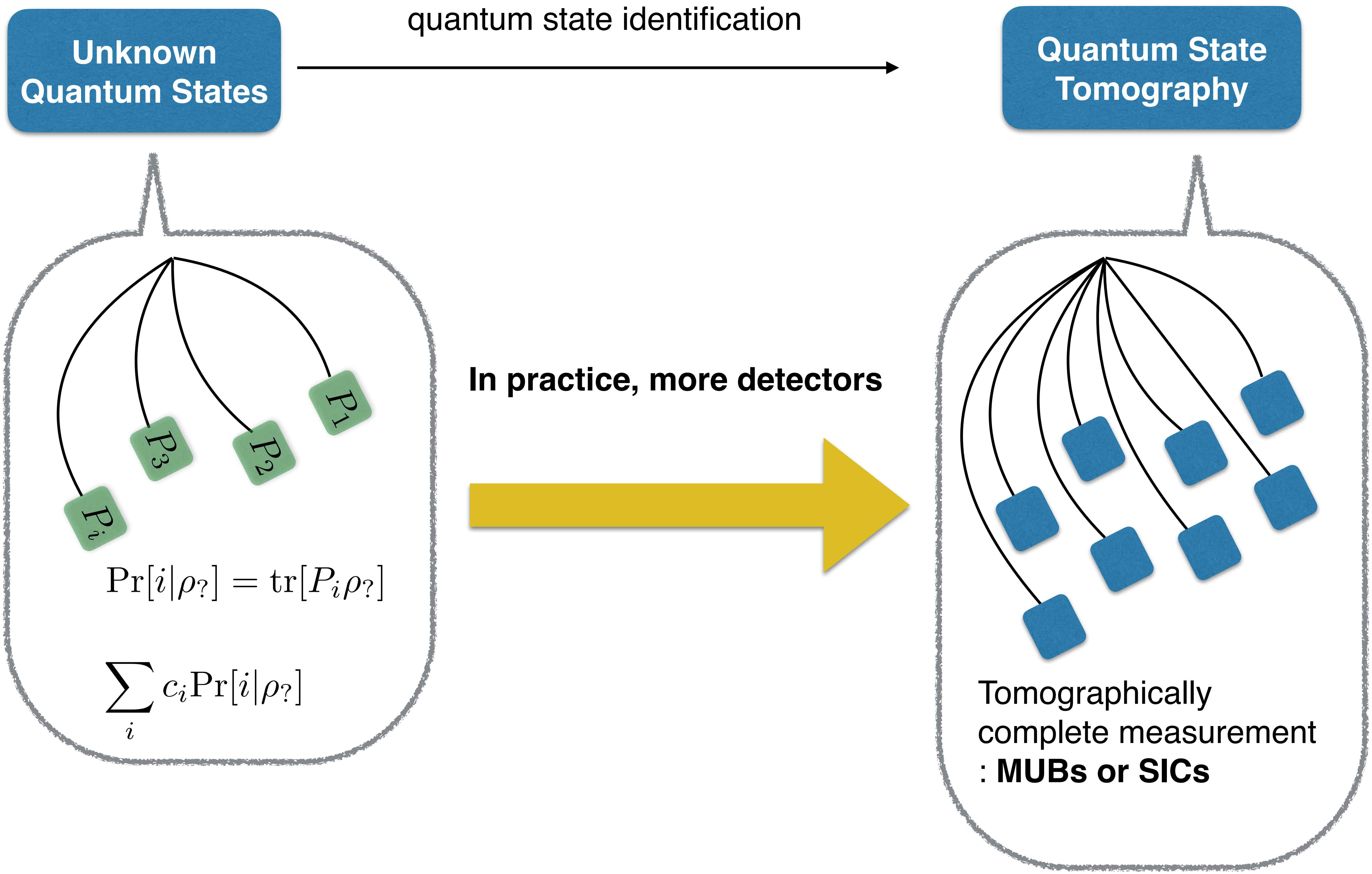
$$\mathrm{tr}[W\rho_{\mathrm{ent}}] < 0$$

$$\mathrm{tr}[W\sigma_{\mathrm{sep}}] \geq 0 \quad \forall \sigma_{\mathrm{sep}}$$

→ (sufficiently many EWs) → Entangled or Separable?

**Advantage of EWs: Entanglement of unknown states can be detected !**

# General Picture of Entanglement Detection



$$p^*/d^2 \leq \mathrm{tr}[\widetilde{W}\sigma_{\mathrm{sep}}] \leq q^*/d^2$$

**Result 3 : POVM Cloud = MUBs and SICs (for tomography)**

$$W = (I \otimes T)(|\phi^+\rangle\langle\phi^+|)$$

**pSPA** $\Longrightarrow$ $\widetilde{W} = (1-p^*)W + p^*\dfrac{I \otimes I}{d^2}$

**Quantum 2-design** $\Longleftrightarrow$ $\widetilde{W} = \mathrm{Sym}_{\mathcal{H}\otimes\mathcal{H}}$

$$= \mathrm{supp}\{\mathrm{MUBs\ or\ SICs}\}$$

# POVM cloud with quantum 2-designs

$$P_k^{(\mathrm{MUB})}(i,i) = \mathrm{tr}[|b_i^k\rangle\langle b_i^k|^{\otimes 2}\rho]$$

$$P^{(\mathrm{SIC})}(j,j) = \mathrm{tr}[|s_j\rangle\langle s_j|^{\otimes 2}\rho]$$

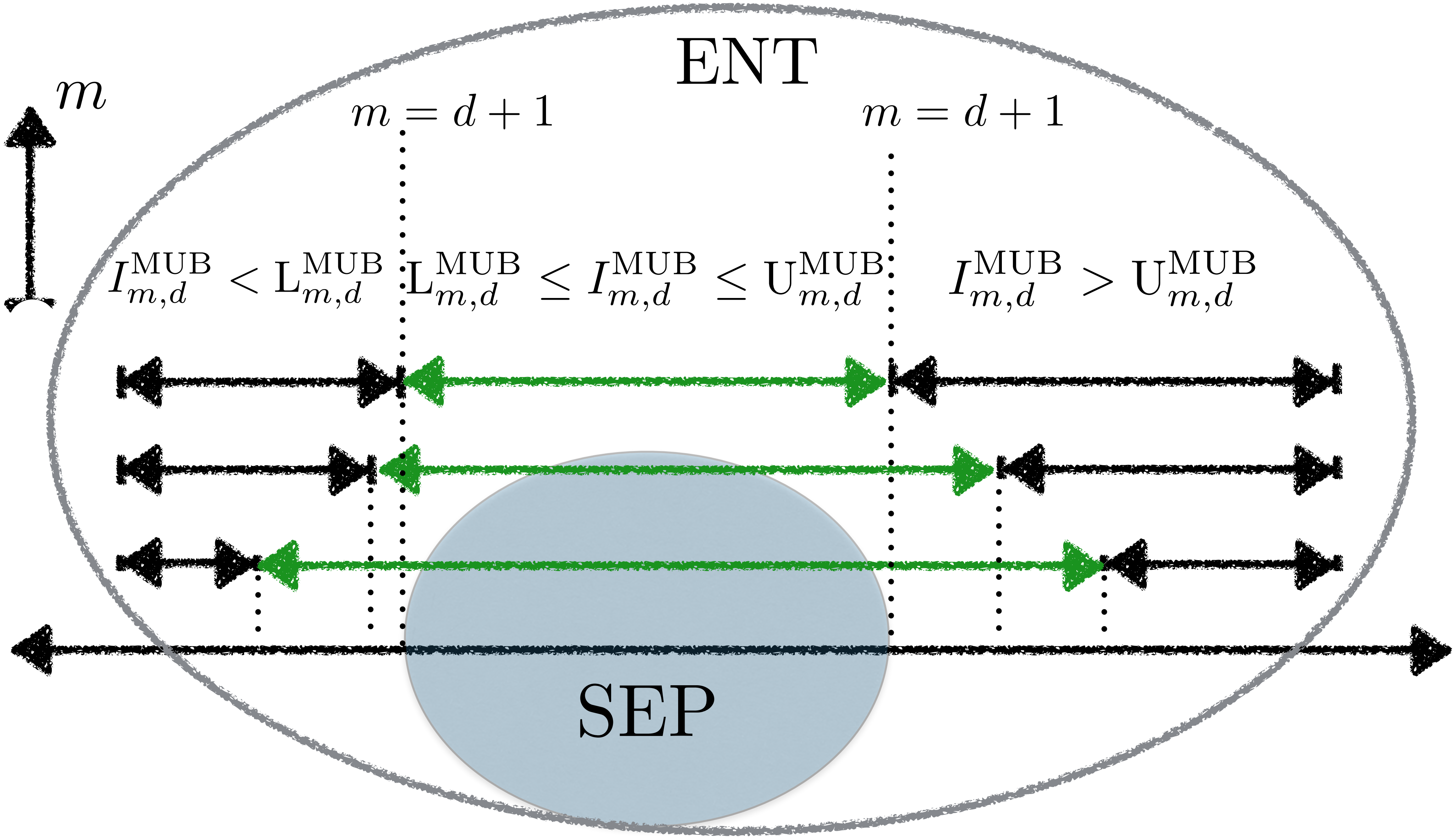
$$\Rightarrow \quad I_m^{(\mathrm{MUB})}(\rho) = \sum_{k=1}^{m}\sum_{i=1}^{d} P_k^{(\mathrm{MUB})}(i,i)$$

$$I_m^{(\mathrm{SIC})}(\rho) = \sum_{j=1}^{m} P^{(\mathrm{SIC})}(j,j)$$

$$\Rightarrow \quad L_m^{(\mathrm{MUB})} \leq I_m^{(\mathrm{MUB})}(\sigma_{\mathrm{sep}}) \leq U_m^{(\mathrm{MUB})}$$

$$L_m^{(\mathrm{SIC})} \leq I_m^{(\mathrm{SIC})}(\sigma_{\mathrm{sep}}) \leq U_m^{(\mathrm{SIC})}$$

## Our Scheme of Detecting Entanglement: *TWICE*




**Upper bounds detect entangled isotropic states**

**Lower bounds detect entangled Werner states**

**Throughout technical parts, main results: a number of inequalities**

**d=2**

$$0.5 \leq I_2^{(\mathrm{MUB})}(\sigma_{\mathrm{sep}}) \leq 1.5$$

$$1 \leq I_3^{(\mathrm{MUB})}(\sigma_{\mathrm{sep}}) \leq 2$$ **QST can be applied**

**d=3**

$$0.211 \leq I_2^{(\mathrm{MUB})}(\sigma_{\mathrm{sep}}) \leq 1.333$$

$$0.5 \leq I_3^{(\mathrm{MUB})}(\sigma_{\mathrm{sep}}) \leq 1.666$$

$$1 \leq I_4^{(\mathrm{MUB})}(\sigma_{\mathrm{sep}}) \leq 2$$ **QST can be applied**

**Throughout technical parts, main results: a number of inequalities**

**d=4**

$$0 \leq I_2^{(\mathrm{MUB})}(\sigma_{\mathrm{sep}}) \leq 1.25$$

$$0.5 \leq I_3^{(\mathrm{MUB})}(\sigma_{\mathrm{sep}}) \leq 1.5$$

$$0.5 \leq I_4^{(\mathrm{MUB})}(\sigma_{\mathrm{sep}}) \leq 1.75$$

$$1 \leq I_5^{(\mathrm{MUB})}(\sigma_{\mathrm{sep}}) \leq 2$$

**QST can be applied**

**Remarks. MUBs vs. Capability of Entanglement Detection**

**Entanglement vs. properties of MUBs**

**Throughout technical parts, main results: a number of inequalities**

**d=2**

$$
\begin{aligned}
0 \leq \quad & I_{2,2}^{(\mathrm{SIC})} \quad \leq \frac{(\sqrt{3}+1)^2}{6} \\
\frac{4}{15} \leq \quad & I_{3,2}^{(\mathrm{SIC})} \quad \leq \frac{4}{3} \\
\frac{2}{3} \leq \quad & I_{4,2}^{(\mathrm{SIC})} \quad \leq \frac{4}{3}.
\end{aligned}
$$

**QST can be applied**

## Throughout technical parts, main results: a number of inequalities

**d=3**

$$0 \leq I_{3,3}^{(\mathrm{SIC})} \leq \frac{9}{8}$$

$$0 \leq I_{4,3}^{(\mathrm{SIC})} \leq 1.25414$$

$$0 \leq I_{5,3}^{(\mathrm{SIC})} \leq 1.39952$$

$$0.1123 \leq I_{6,3}^{(\mathrm{SIC})} \leq 1.48175$$

$$\frac{3}{20} \leq I_{7,3}^{(\mathrm{SIC})} \leq \frac{3}{2}$$

$$\frac{3}{8} \leq I_{8,3}^{(\mathrm{SIC})} \leq \frac{3}{2}.$$

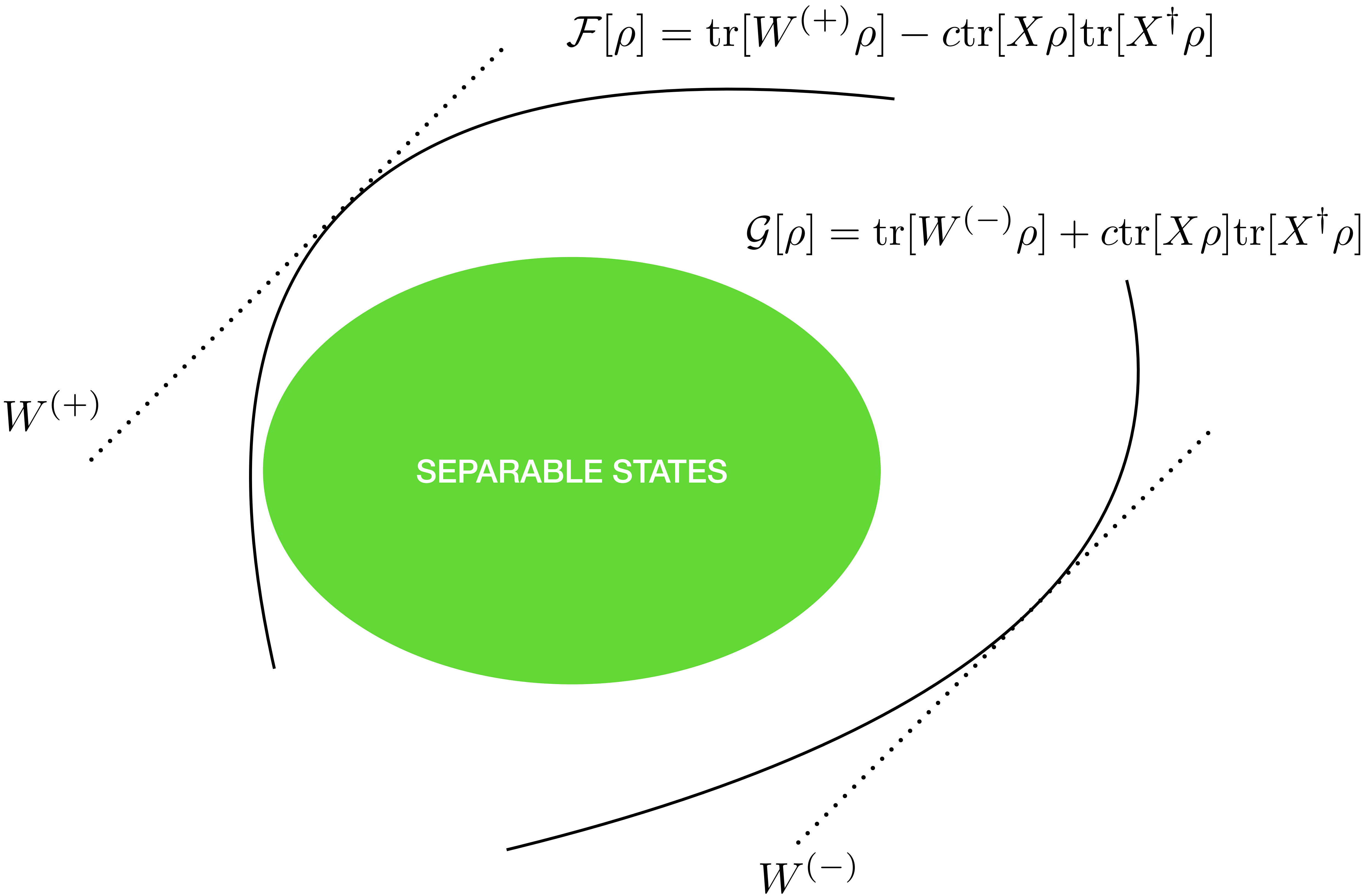
$$\frac{3}{4} \leq I_{9,3}^{(\mathrm{SIC})} \leq \frac{3}{2}.$$

**QST can be applied**

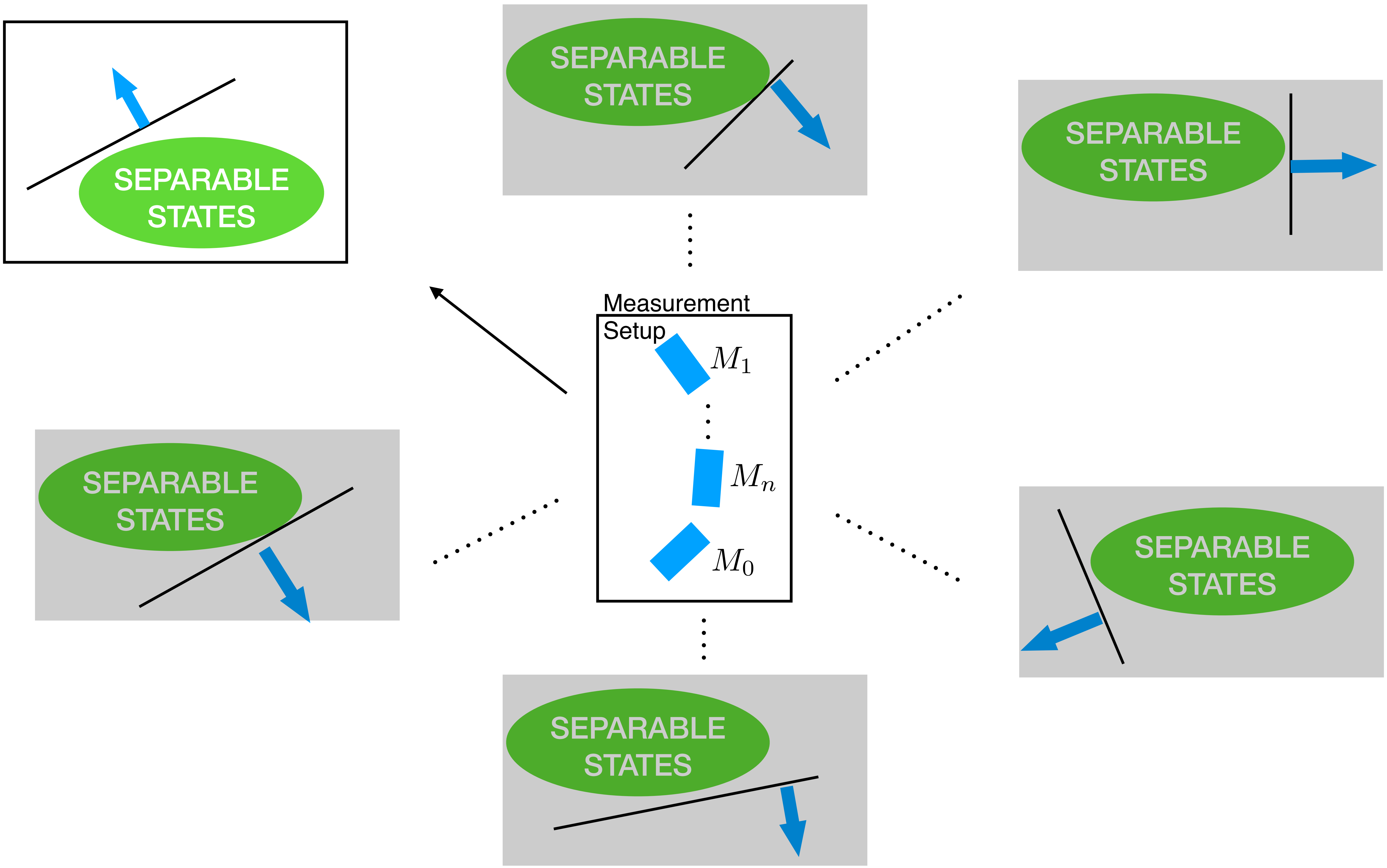
**Remarks. SICs vs. Capability of Entanglement Detection**

**Entanglement vs. properties of SICs**

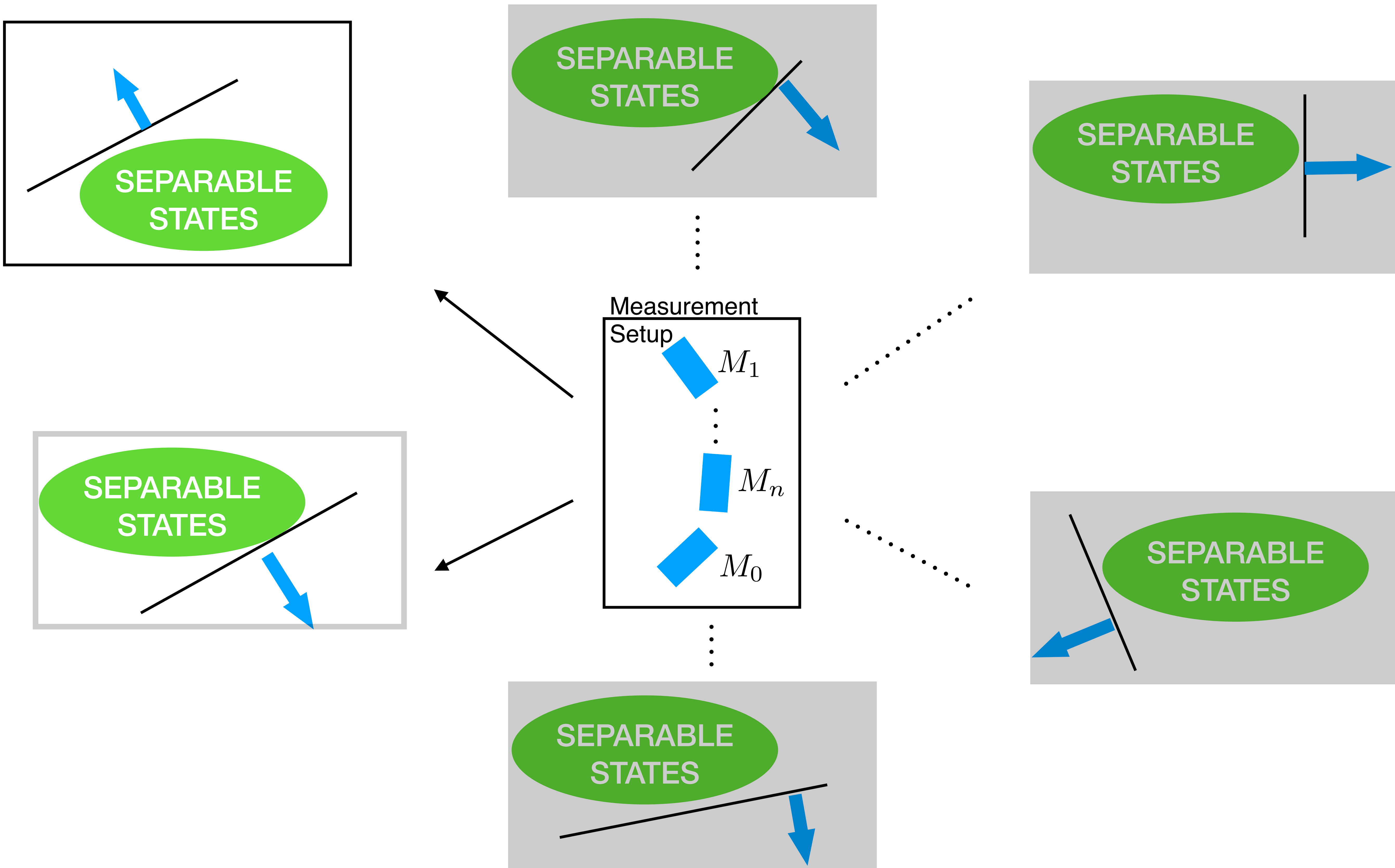
Non linear EW



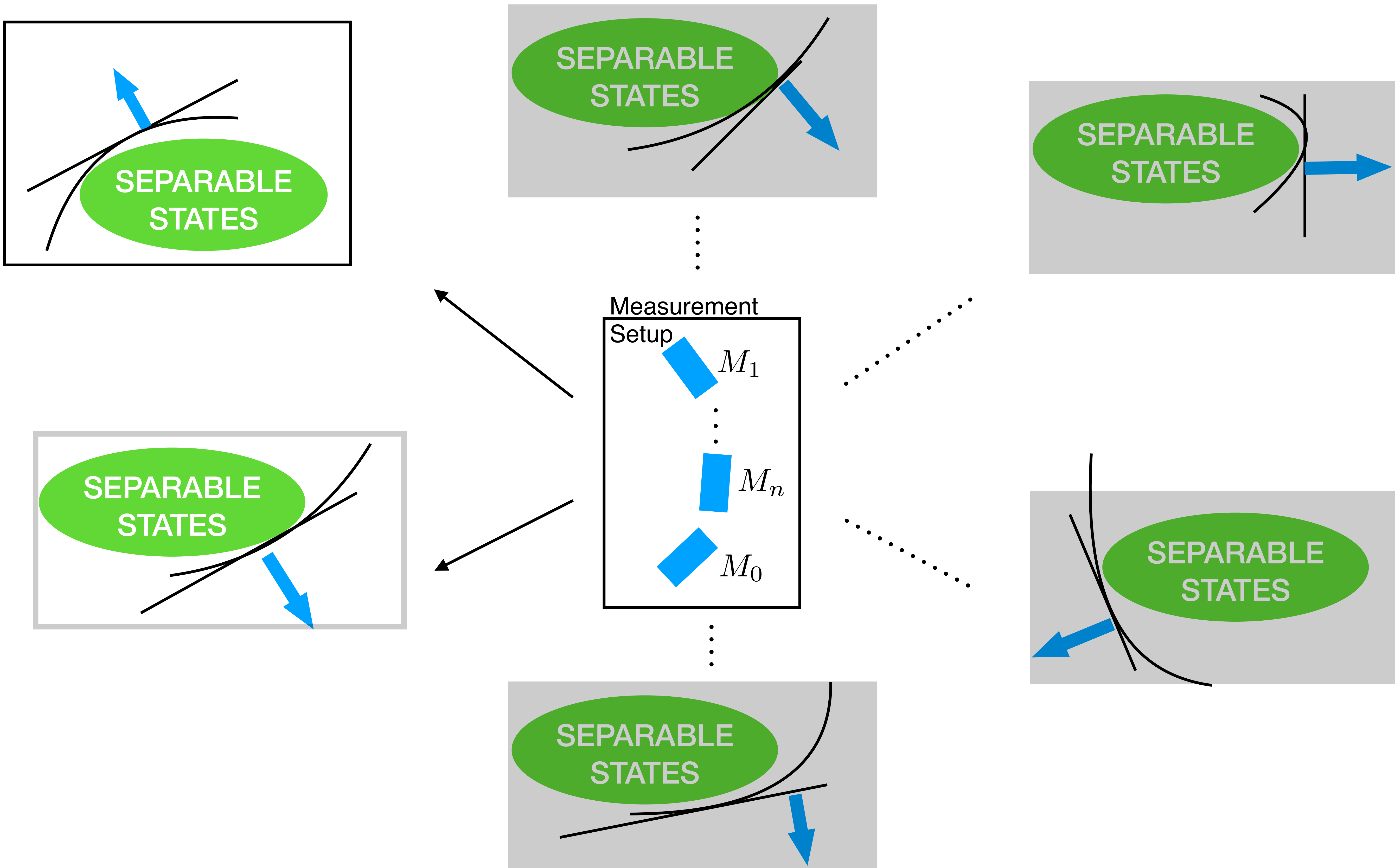
# Conclusion & Future Direction



# Conclusion & Future Direction



# Conclusion & Future Direction



**Introduction: Entanglement Witnesses (EWs)**

**Main Question: Entanglement Detection vs. Quantum State Tomography**

**Our contribution : EWs are more useful than we thought**

Theoretical parts: Many hyperplanes can be generated

Experimental proposal: Entanglement detection can be tested many times

**Discussions: I no longer need Positive Maps to construct EWs.**

**Applications : MUBs, the conjecture in d=6, etc.**

**On-going directions and questions**

I learned that **EWs are more useful than I thought**