ICAM/NCTS annual meeting: "Frontiers of Quantum Matter"

National Tsing-Hua University, Hsinchu, Taiwan, ROC 16 January 2019

Emergent metric of "flux attachment" in the fractional quantum Hall effect.

F. Duncan M. Haldane
Princeton University

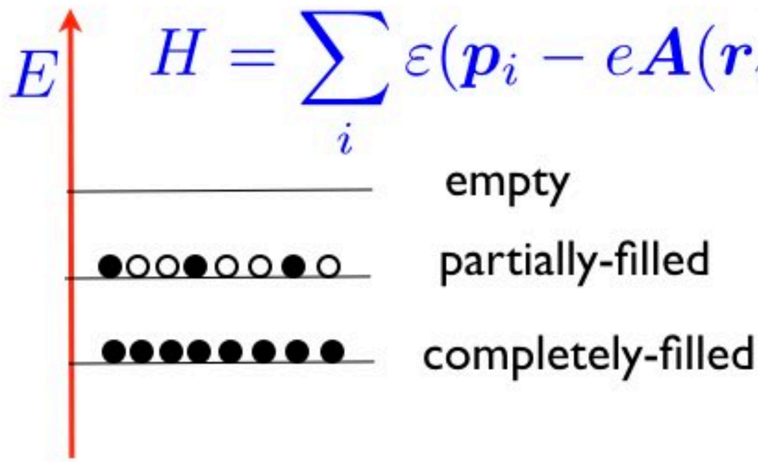
- flux attachment and the Laughlin state
- non-commutative geometry
- emergent geometry and metric of flux-attachment

- Topological states of matter have been a major theme in the recent developments in understanding novel quantum effects.
- key questions are: why do they occur, what features of materials favor such states, and how can we understand the energetics that drives their emergence.
- I will principally discuss the fractional quantum Hall effect, but this is a general question

- In 1982, the fractional quantum Hall effect was discovered
- Before then, most theorists believed that the powerful techniques of field theories, Feynman diagrams, second quantization, etc. could solve all problems.......
- they were wrong: the fractional quantum Hall effect was a new type of phenomenon that could not be adiabatically related to a weakly-interacting problem.

- the kinetic energy of electrons bound to a 2D surface through which a uniform magnetic flux passes undergoes Landau quantization into macroscopically-degenerate Landau levels

$$H = \sum_i \varepsilon(\boldsymbol{p}_i - e\boldsymbol{A}(\boldsymbol{r}_i))$$



one state in each level per London quantum of magnetic flux

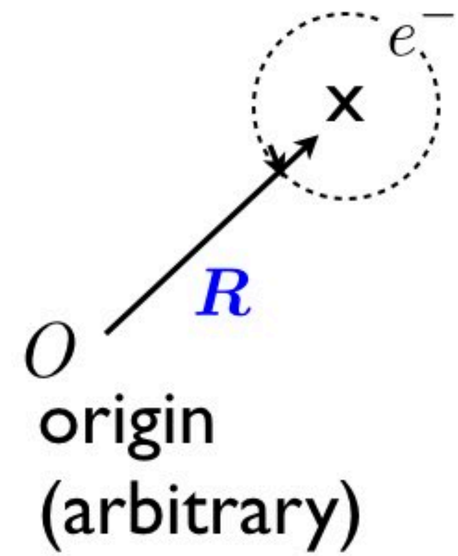
- In momentum space, the electrons move on closed contours of constant energy, similar to motion in phase space

$$[p_x, p_y] = i\hbar eB$$

$$\varepsilon(\boldsymbol{p}) = \frac{(p_x^2 + p_y^2)}{2m}$$

quantized Landau orbit around guiding center

- Their residual real-space degree of freedom is the center of their circular orbit, called the "guiding center" $\boldsymbol{R}$



$$\boldsymbol{R} = \boldsymbol{r} - (eB)^{-1}\hat{\boldsymbol{n}} \times (-i\hbar\boldsymbol{\nabla} - e\boldsymbol{A})$$

The origin ambiguity of $\boldsymbol{R}$ is also a gauge ambiguity

- Landau quantization of the orbital motion of the electrons leaves a residual problem of non-commutative geometry of the guiding centers of the orbits in a partially-filled Landau level.

$$H = \sum_{i<j} V(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

no kinetic energy!

$$[R^x, R^y] = -i\ell_B^2$$

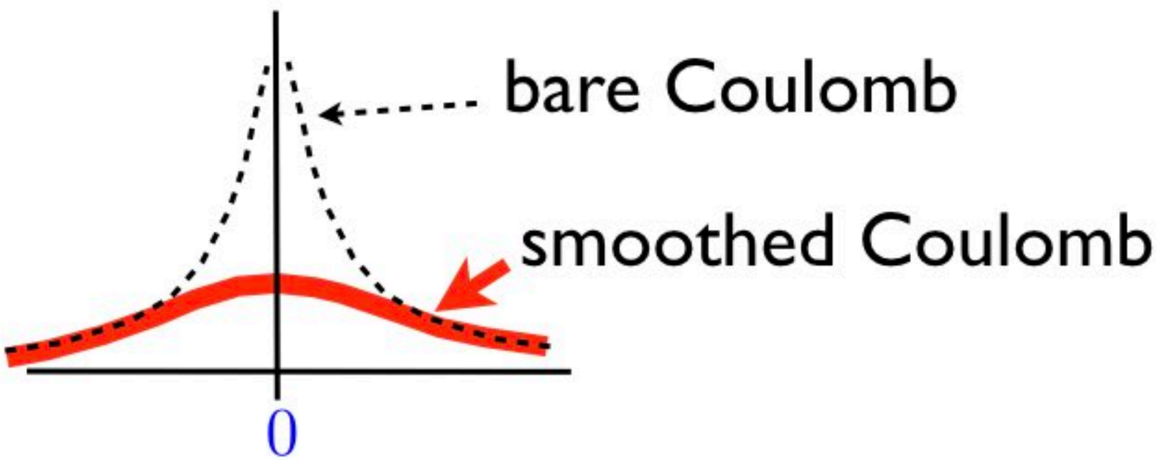
dynamics comes from non-commutative geometry!

$2\pi\ell_B^2 =$ **"quantum area"** through which one London quantum of magnetic flux passes

analogous to Planck area!

$$H = \sum_{i<j} V(\boldsymbol{R}_i - \boldsymbol{R}_j) \qquad [R^x, R^y] = -i\ell_B^2$$

- The interaction potential is very smooth because the short-distance singularity of the Coulomb interaction is smoothed out by the Landau orbit "form factor"

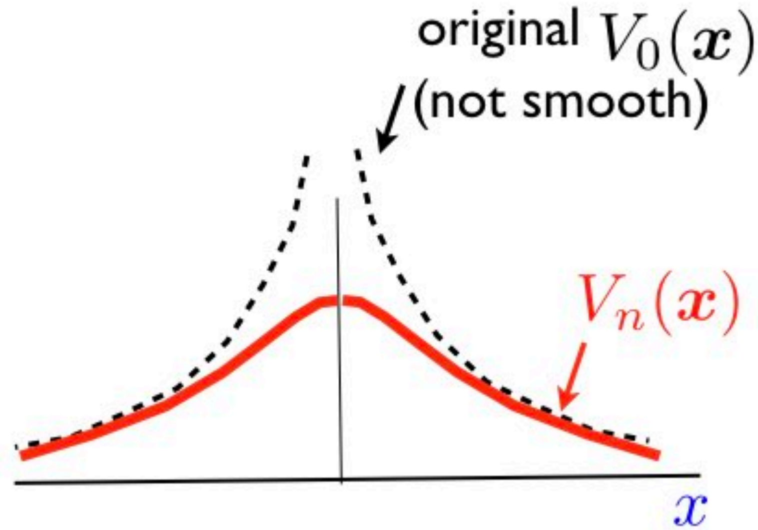


- The expansion of $V(r)$ about any point is absolutely convergent.
- This is needed for a function of non-commuting variables to "make sense"

$$[R^a, R^b] = -i\ell_B^2\epsilon^{ab}$$

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

Identical quantum particles
(fermions or boson)



We now have the final form of the problem:

- The potential $V_n(\boldsymbol{x})$ is a **very smooth** (in fact entire) function that depends on the form- factor of the partially-occupied Landau level
- The essential clean-limit symmetries are translation and inversion:

$$\boldsymbol{R}_i \mapsto \boldsymbol{a} \pm \boldsymbol{R}_i$$

$$H = \sum_{i<j} V(\boldsymbol{R}_i - \boldsymbol{R}_j) \qquad [R^x, R^y] = -i\ell_B^2$$

- This is a strongly interacting model with no free-particle limit! How can we study it?
- Numerical solution with a finite number N of particles has been the only quantitative source of information
- We can treat the rotationally-invariant case most easily $V(\boldsymbol{r}) = V(|\boldsymbol{r}|)$
- in 1983, Laughlin studied the N=3 case and was led to a remarkable model wavefunction

- The Laughlin state (Nobel 1998)

Landau level filling factor $\nu = 1/m$

$$\psi \propto \prod_{i<j} (z_i - z_j)^m \prod_i e^{-\frac{1}{4} z_i^* z_i / \ell_B^2}$$

- rotationally invariant, lowest Landau level, $z = x + iy$

- $m = 1$ is the uncorrelated Slater-determinant filled lowest Landau level, $m > 1$ is a highly correlated topologically-ordered state.

- thirty years after its experimental discovery and theoretical description in terms of the Laughlin state, the fractional quantum Hall effect remains a rich source of new ideas in condensed matter physics.
- The key concept is "**flux attachment**" that forms "**composite particles**" and leads to topological order.
- Recently, it has been realized that flux attachment also has interesting **geometric** properties

- Laughlin thought of his state as a Lowest Landau Level (LLL) wavefunction, using the fact that in the symmetric gauge, and with rotational symmetry, a LLL one-particle wavefunction has the form

$$\psi(x,y) = f(z) e^{-\frac{1}{4} z^* z / \ell_B^2}$$

a holomorphic function

- It also has no obvious continuously-variable parameter (m must be an integer)

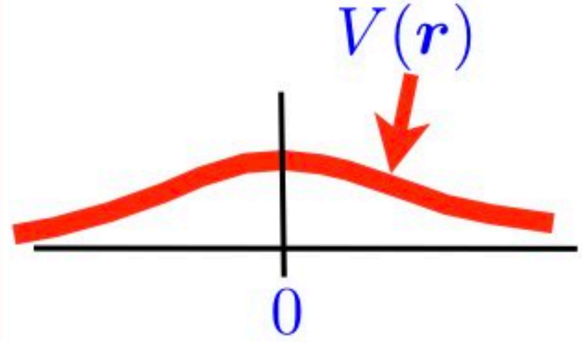
- **In fact, as we will see, both these commonly-held beliefs are incorrect!**

$$\psi \propto \prod_{i<j}(z_i - z_j)^m \prod_i e^{-\frac{1}{4}z_i^* z_i/\ell_B^2}$$

- The Laughlin state was soon shown to be the exact ground state of a "toy model" short-range interaction, and a very good approximation to the ground state of the Coulomb interaction
- It has been interpreted in terms of "flux attachment", (composite bosons and composite fermions) and its excitations identified as obeying (Abelian) "fractional statistics" and exhibiting "topological order"
- Its holomorphic part was recognized as a "conformal block" of an (Abelian) conformal field theory correlation function, leading even more interesting non-Abelian model states, which could be used for "topological quantum computing"

$$H = \sum_{i<j} V(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

$$[R^x, R^y] = -i\ell_B^2$$



- The only parameter in this model is the interaction, **no explicit mention of the Landau level (lowest or otherwise)**
- Non-commutative geometry is intrinsically "fuzzy", so has no valid "Schrödinger" real-space wavefunction formalism, just a Heisenberg Hilbert-space picture

$$\psi(\boldsymbol{x}) = \langle \boldsymbol{x} | \Psi \rangle$$

Schrödinger Heisenberg

$$\langle \boldsymbol{x} | \boldsymbol{x}' \rangle = 0$$
$$\boldsymbol{x} \neq \boldsymbol{x}'$$

needed for Schrödinger, but fails in non-commutative quantum geometry

$$H = \sum_{i<j} V(\boldsymbol{R}_i - \boldsymbol{R}_j) \qquad [R^x, R^y] = -i\ell_B^2$$

- so what is the actual meaning of the "Laughlin wavefunction"?

- choose **any** unit-determinant ("unimodular") metric g_{ab}

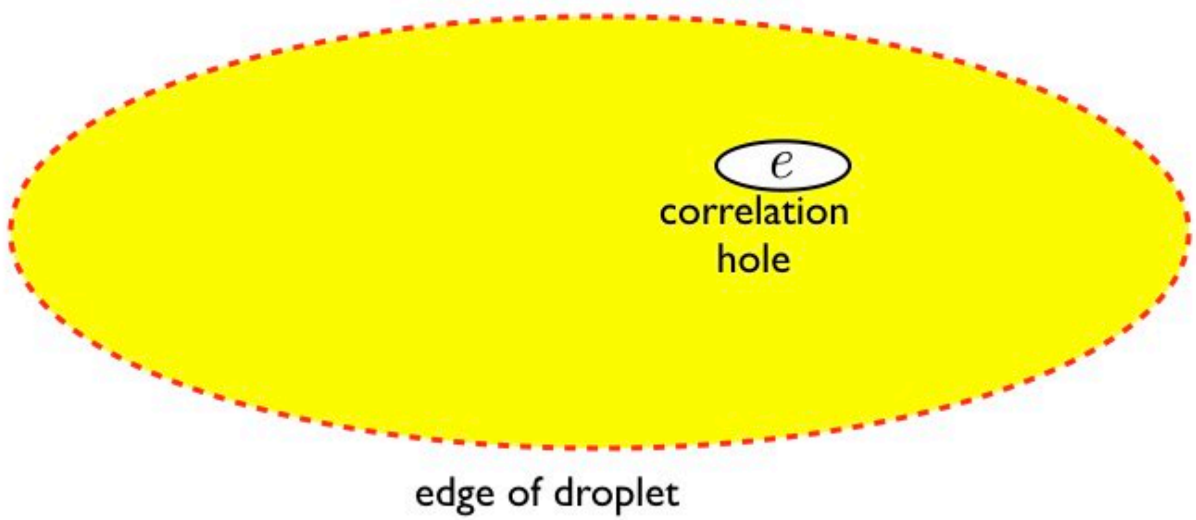
- diagonalize $L = \frac{g_{ab}}{2\ell_B^2} R^a R^b = \frac{1}{2}(a^\dagger a + a a^\dagger)$

$$|\Psi_L(g)\rangle = \prod_{i<j} (a_i^\dagger - a_j^\dagger)^m |0\rangle \qquad \begin{array}{c} a_i|0\rangle = 0 \\ [a_i, a_j^\dagger] = \delta_{ij} \end{array}$$

Heisenberg form of the (unnormalized) Laughlin state

- for $m > 1$ (but not $m = 1$), the metric is a **hidden parameter** of the state

- The original form of the Laughlin state is a finite-size droplet of N particles on the infinite plane.
- Somewhat confusingly, in this droplet state the metric parameter fixes both the shape of the droplet state **<u>and</u>** the shape of the correlation hole around each particle formed by "flux attachment":



- to remove the edge, compactify on the torus with N_Φ flux quanta:
- An unnormalized holomorphic single-particle state has the form

$$|\psi\rangle = \prod_{i=1}^{N_\Phi} \sigma(a^\dagger - w_i)|0\rangle, \qquad \sum_{i=1}^{N_\Phi} w_i = 0$$

generalized Weierstrass sigma function

$$\sigma(z) = e^{\frac{1}{2}C_2 z^2} z \prod_{L\neq 0} \left(1 - \frac{z}{L}\right) e^{\frac{z}{L} + \frac{1}{2}\left(\frac{z}{L}\right)^2}$$

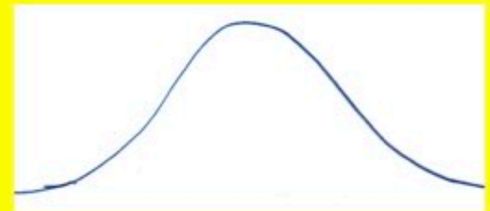
C_2 is an "almost holomorphic modular invariant"

Filled Landau level $N = N_\Phi$

$$|\Psi_{\text{filledLL}}\rangle = \sigma(\textstyle\sum_i a_i^\dagger) \displaystyle\prod_{i<j} \sigma(a_i^\dagger - a_j^\dagger)|0\rangle$$

independent of choice of metric, after normalization

This is the **entire** problem: nothing other than this matters!



$$H = \sum_{i<j} U(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

$$[R^x, R^y] = -i\ell_B^2$$

like phase-space, has Heisenberg uncertainty principle

- H has translation and <u>inversion</u> symmetry

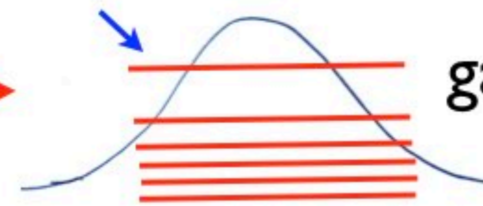
$$[(R_1^x + R_2^x), (R_1^y - R_2^y)] = 0$$

$$[H, \textstyle\sum_i \boldsymbol{R}_i] = 0$$

- generator of translations <u>and</u> electric dipole moment!

$$[(R_1^x - R_2^x), (R_1^y - R_2^y)] = -2i\ell_B^2$$

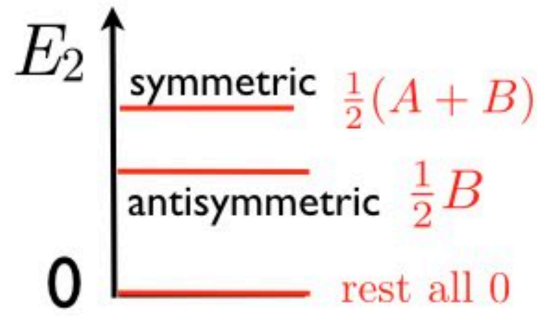
- relative coordinate of a pair of particles behaves like a single particle



two-particle energy levels

- Solvable model! ("short-range pseudopotential")

$$U(r_{12}) = \left(A + B\left(\frac{(r_{12})^2}{\ell_B^2}\right)\right) e^{-\frac{(r_{12})^2}{2\ell_B^2}}$$



- Laughlin state

$$|\Psi_L^m\rangle = \prod_{i<j}\left(a_i^\dagger - a_j^\dagger\right)^m |0\rangle$$

$$a_i|0\rangle = 0 \qquad a_i^\dagger = \frac{R^x + iR^y}{\sqrt{2}\ell_B}$$

$$E_L = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

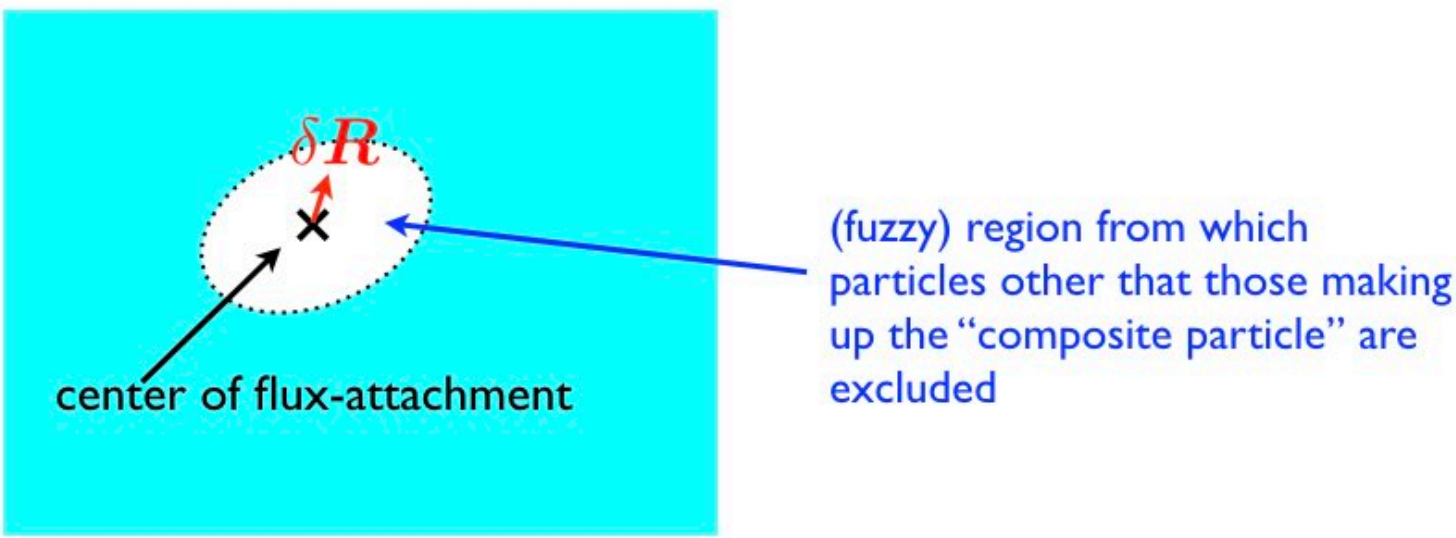
maximum density null state

- m=2: (bosons): all pairs avoid the symmetric state $E_2 = \frac{1}{2}(A+B)$
- m=3: (fermions): all pairs avoid the antisymmetric state $E_2 = \frac{1}{2}B$

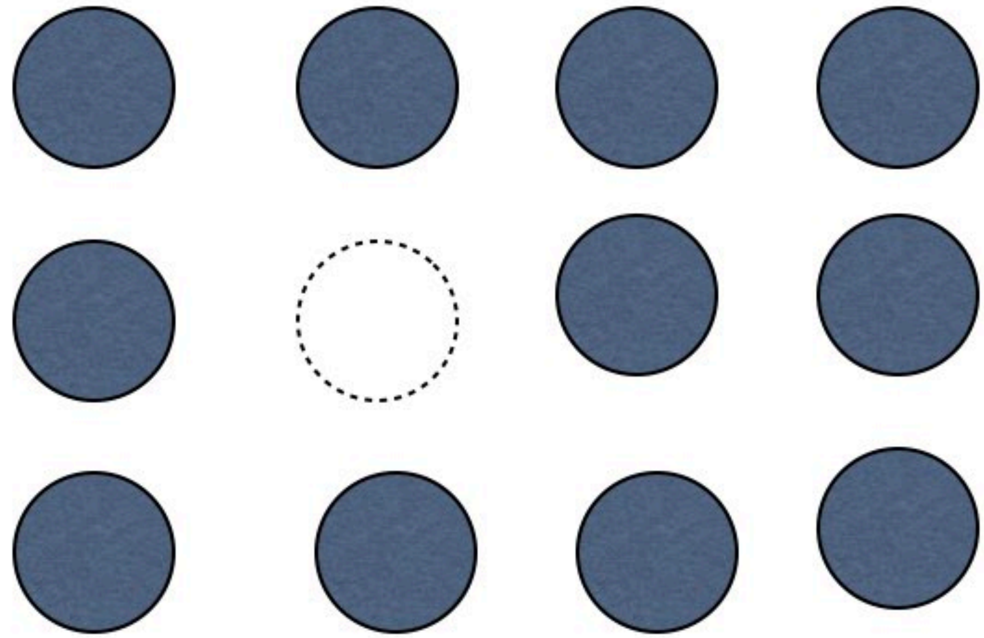
- the essential unit of the 1/3 Laughlin state is the electron bound to a correlation hole corresponding to three "units of flux", or three of the available single-particle states which are exclusively occupied by the particle to which they are "attached"
- In general, the elementary unit of the FQHE fluid is a "composite boson" of p particles with q "attached flux quanta"
- This is the analog of a unit cell in a solid....

- Flux attachment is a gauge condensation that removes the gauge ambiguity of the guiding centers, giving each one a "natural" origin, so they define a physical electric dipole moment of the "composite particle" in which they are bound by the "attached flux".

- This is analogous to how the "the vector potential becomes an observable" (in a hand-waving way) in the London equations for a superconductor.



- quantum solid
- unit cell is correlation hole
- defines geometry



- repulsion of other particles make an attractive

potential well strong enough to bind particle

solid melts if well is not strong enough to contain zero-point motion (Helium liquids)

- In Maxwell's equations, the momentum density is

$$\pi_i = \epsilon_{ijk} D^j B_k \quad D^i = \epsilon_0 \delta^{ij} E_j + P^i$$

- The momentum of the condensed matter is

$$\boldsymbol{p} = \boldsymbol{d} \times \boldsymbol{B}$$

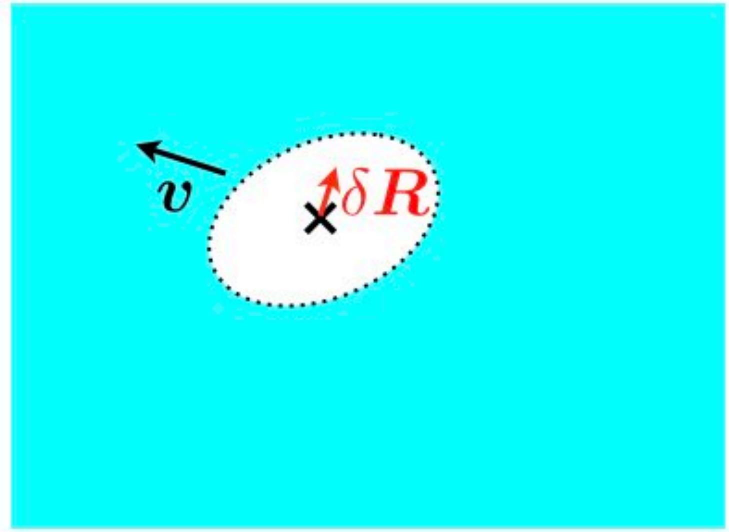
↖ electric dipole moment

- in 2D the guiding-center momentum then is

$$p_a = eB\epsilon_{ab}\delta R^b$$

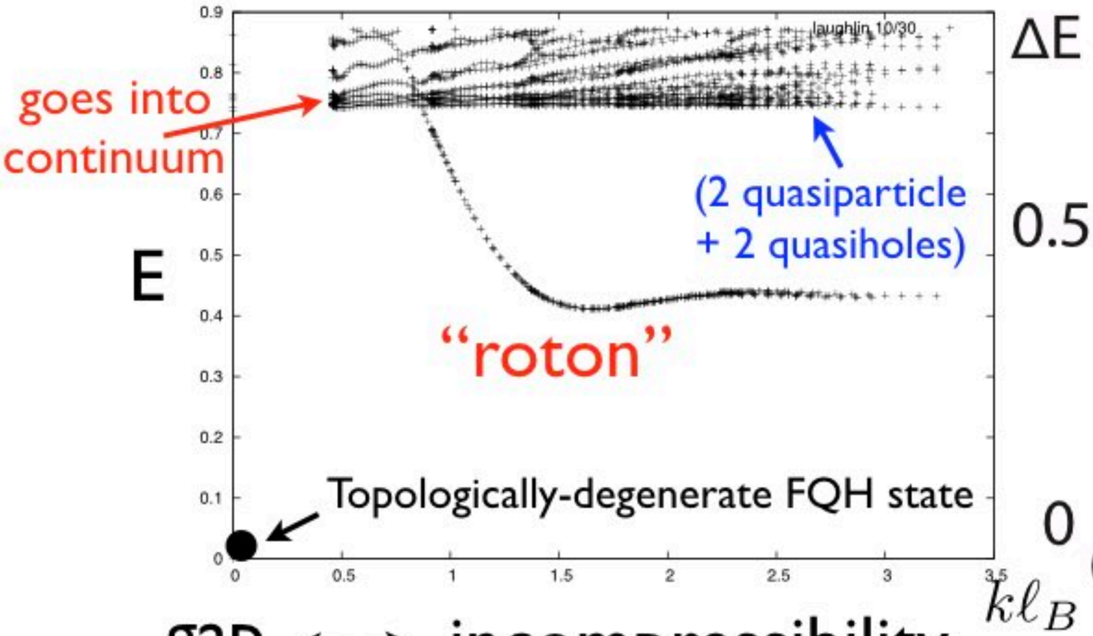
- The electrical polarization energy of the dielectric composite particle then gives its energy-momentum dispersion relation, with no involvement of any "Newtonian inertia" involving an effective mass

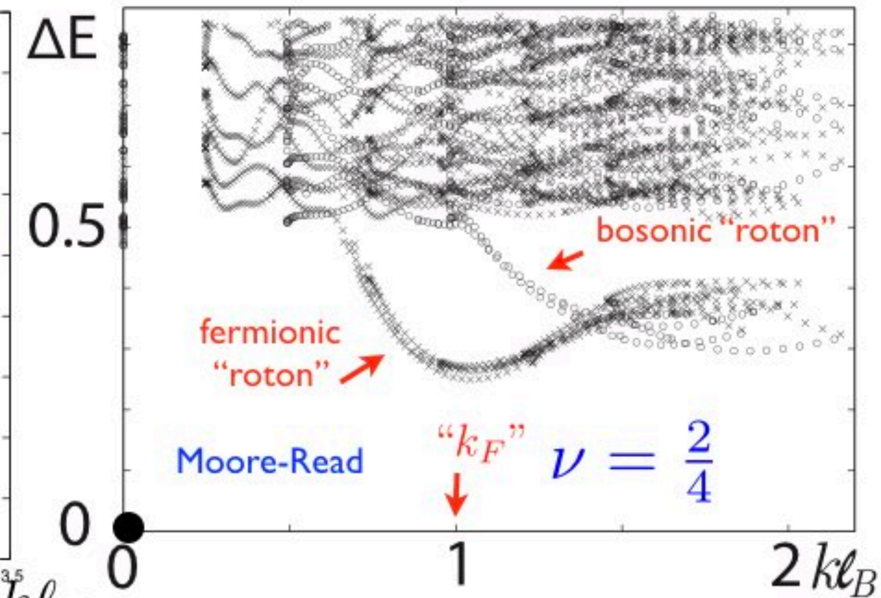
- The Berry phase generated by motion of the "other particles" that "get out of the way" as the vortex-like "flux-attachment" moves with the particle(s) it encloses can be formally-described as a Chern-Simons gauge field that cancels the Bohm-Aharonov phase, so that the composite object propagates like a neutral particle.



- If the composite particle is a boson, it condenses into the zero-momentum **(zero electric dipole-moment)** inversion-symmetric state, giving an incompressible-fluid Fractional Quantum Hall state, with an energy gap for excitations that carry momentum or electric dipole moment ("quantum incompressibility", no transmission of pressure through the bulk).

- All FQH states have an elementary unit (analogous to the unit cell of a crystal) that is a composite boson under exchange.
- It may be sometimes be useful to describe this boson as a a bound state of composite fermions (with their own preexisting flux attachment) bound by extra flux (Jain's picture)





gap ⟷ incompressibility

Collective mode with short-range V_1 pseudopotential, 1/3 filling (Laughlin state is exact ground state in that case)

Collective mode with short-range three-body pseudopotential, 1/2 filling (Moore-Read state is exact ground state in that case)

- momentum $\hbar k$ of a quasiparticle-quasihole pair is proportional to its **electric dipole moment $\mathbf{p}_e$**

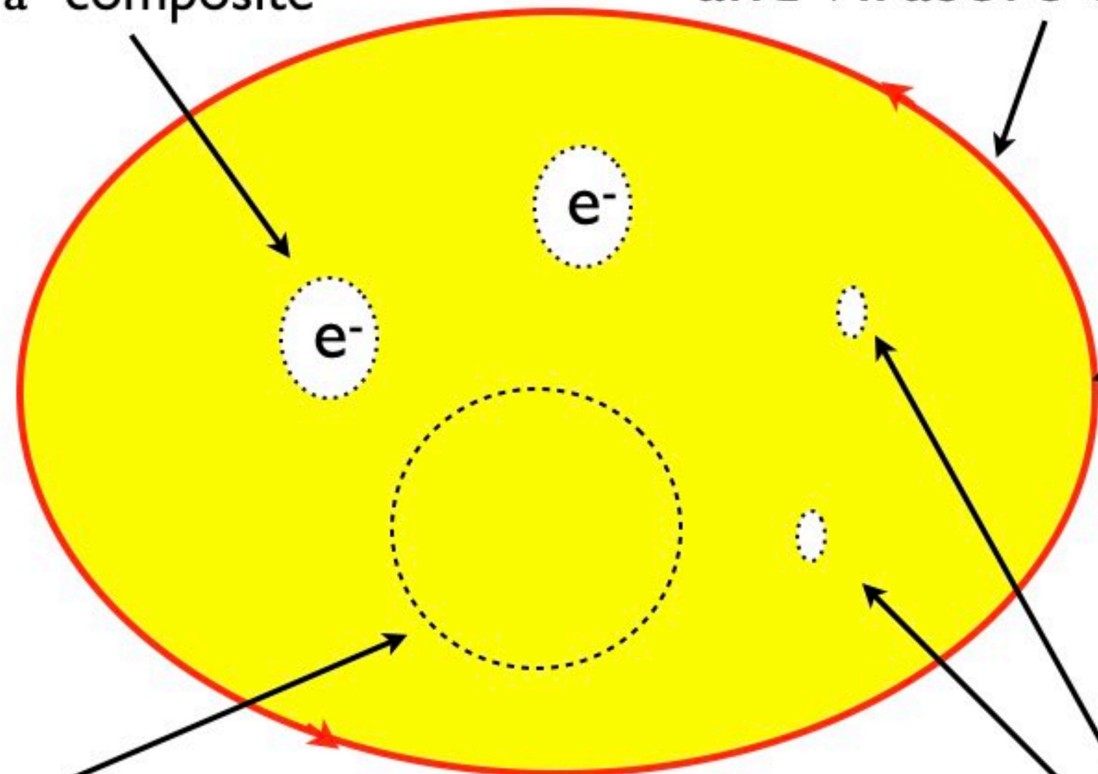
$$\hbar k_a = \epsilon_{ab} B p_e^b$$

gap for electric dipole excitations is a MUCH stronger condition than charge gap: fluid **does not transmit pressure through bulk!**

• Anatomy of Laughlin state

electron with "flux attachment" to form a "composite boson"

Chiral edge mode with chiral anomaly and Virasoro anomaly



geometric edge dipole moment determined by Hall viscosity

(Wen-Zee term)

Topological and geometric bulk properties revealed by entanglement spectrum of cut

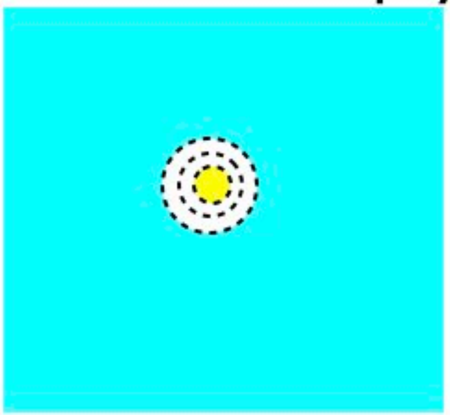
fractionally-charged e/3 quasiholes obeying (Abelian) fractional statistics

- the essential unit of the 1/3 Laughlin state is the electron bound to a correlation hole corresponding to "units of flux", or three of the available single-particle states which are exclusively occupied by the particle to which they are "attached"

- In general, the elementary unit of the FQHE fluid is a "composite boson" of p particles with q "attached flux quanta"

- This is the analog of a unit cell in a solid....

- The Laughlin state is parametrized by a unimodular metric: what is its physical meaning?



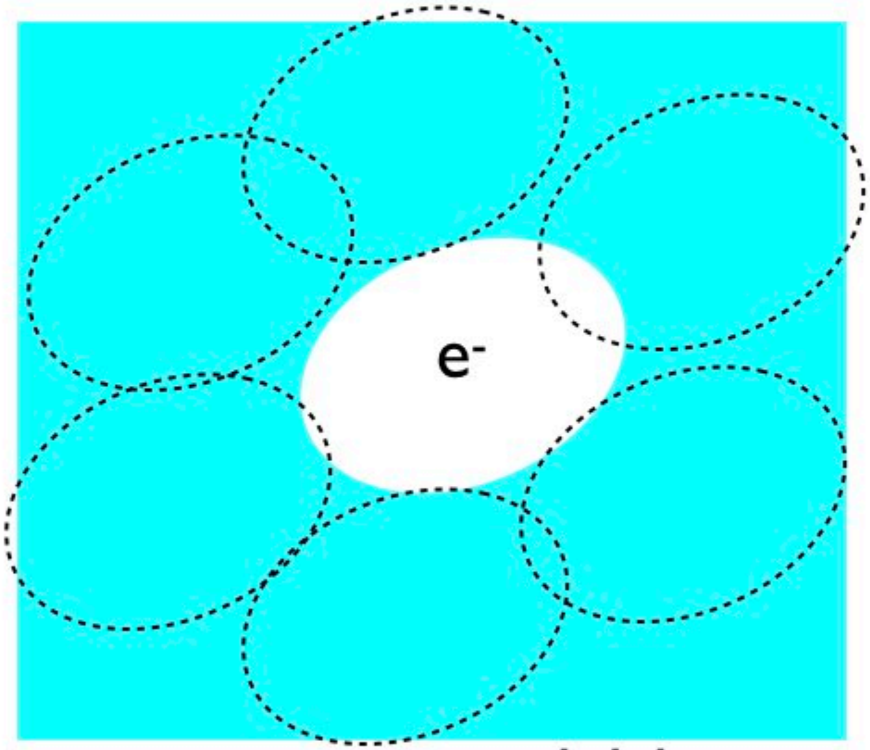
correlation holes in two states with different metrics

- In the $\nu = 1/3$ Laughlin state, each electron sits in a correlation hole with an area containing 3 flux quanta. The metric controls the *shape* of the correlation hole.

- In the $\nu = 1$ filled LL Slater-determinant state, there is <u>no</u> correlation hole (just an exchange hole), and this state does <u>**not**</u> depend on a metric

but no broken symmetry

- similar story in FQHE:



- “flux attachment” creates correlation hole
- defines an emergent geometry
- potential well must be strong enough to bind electron
- continuum model, but similar physics to Hubbard model
- new physics: Hall viscosity, geometry............

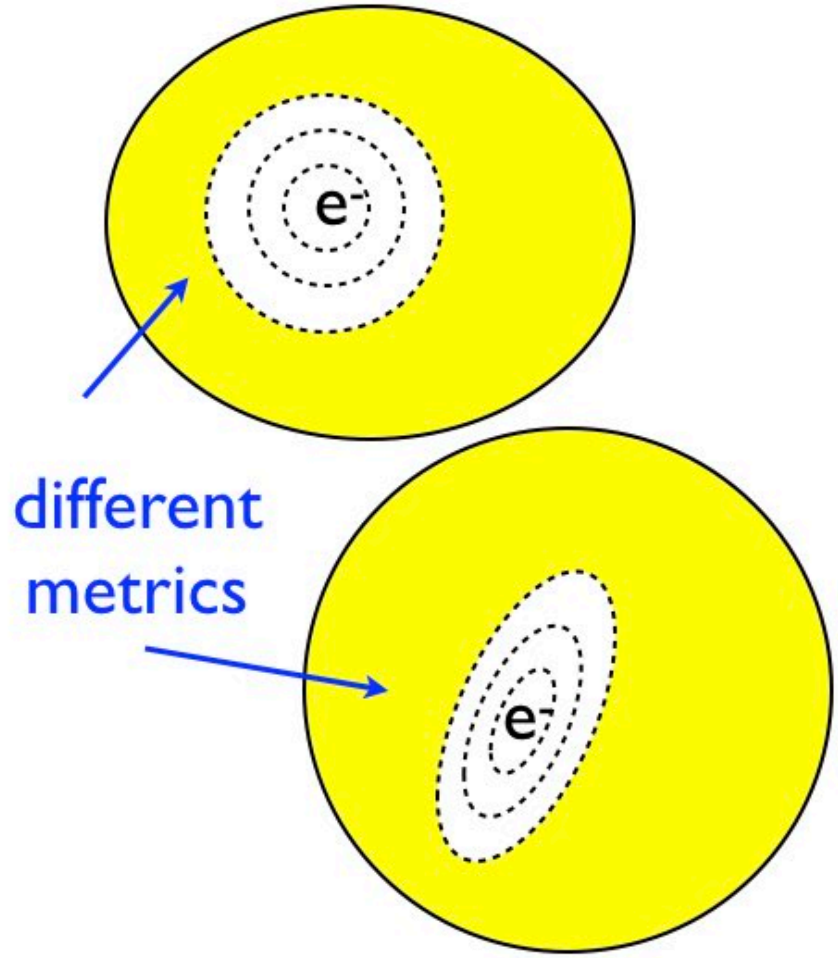
- composite boson: if the central orbital of a basis of eigenstates of L(g) is filled, the next two are empty

$$|\Psi_L^3\rangle = \prod_{i<j}\left(a_i^\dagger - a_j^\dagger\right)^3 |0\rangle$$

$$L(g)|\psi_m\rangle = (m + \tfrac{1}{2})|\psi_m\rangle$$

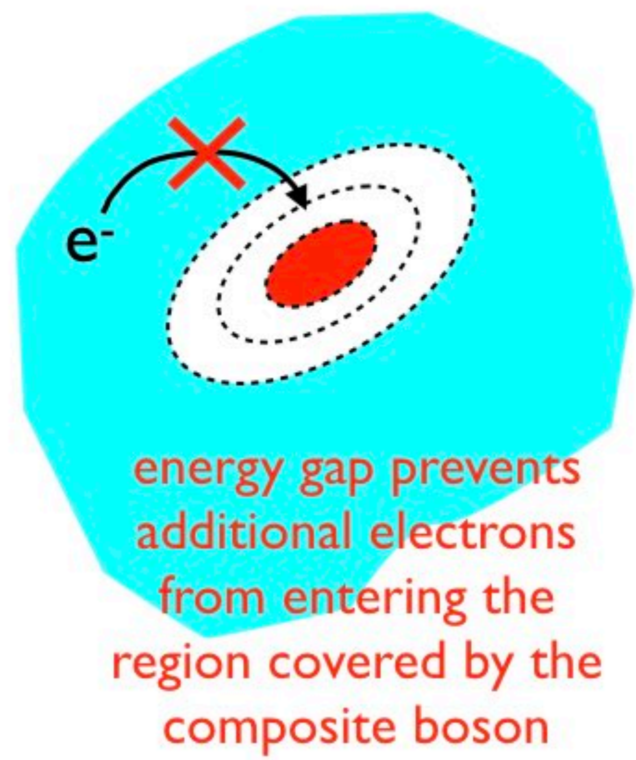
- this correlation hole is equivalent to "attachment of three flux quanta" or vortices that travel with the particle, generating a Berry phase that cancels the Bohm-Aharonov phase and transmutes Fermi to Bose exchange statistics.

- this shape of the corelation hole - and hence its correlation energy - varies with the metric g_{ab}

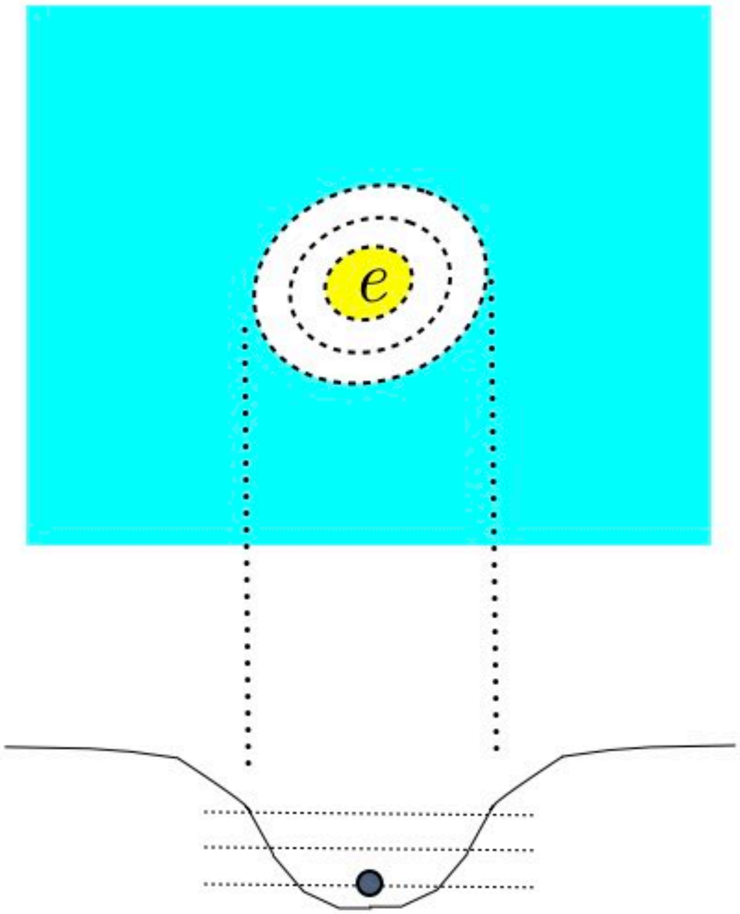


- Origin of FQHE incompressibility is analogous to origin of **Mott-Hubbard gap** in lattice systems.
- There is an energy gap for putting an **extra particle** in a quantized region that is **already occupied**

- **On the lattice** the "quantized region" is an atomic orbital with a fixed shape
- **In the FQHE** only the **area** of the "quantized region" is fixed. The **shape** must adjust to minimize the correlation energy.



$1/3$ Laughlin state



If the central orbital is filled, the next two are empty

The composite boson has inversion symmetry about its center

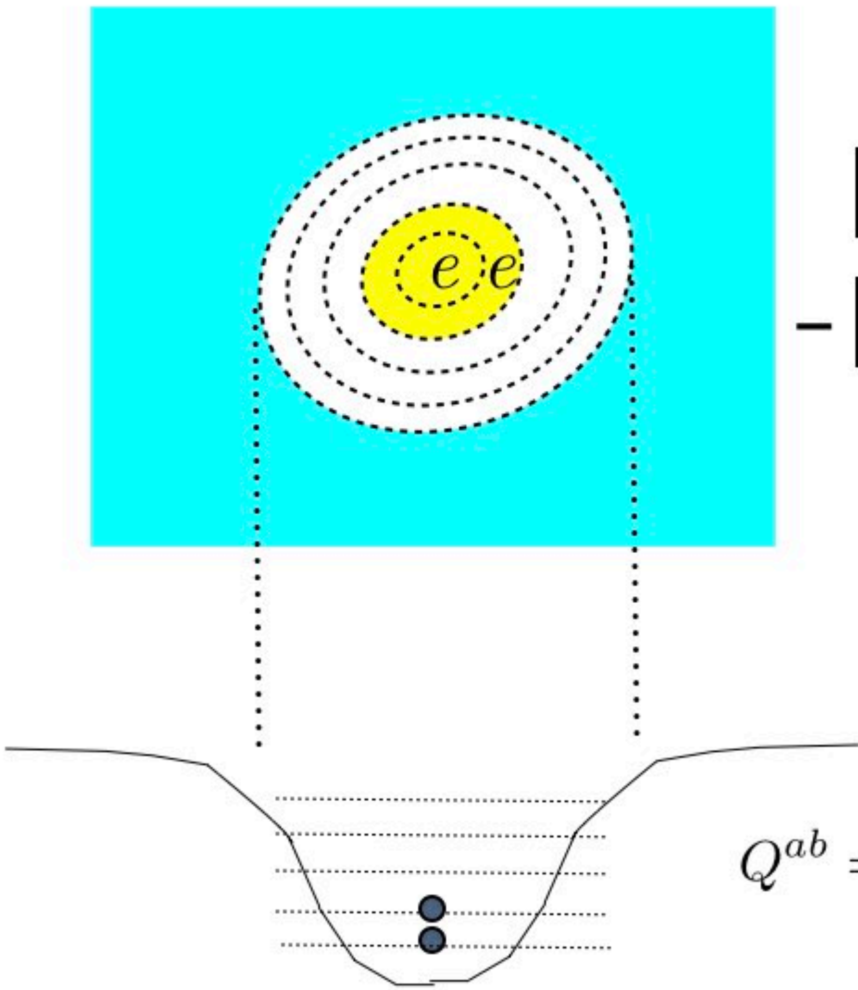
It has a "spin"

$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$		
1	0	0	.....	$L = \frac{1}{2}$
$-\ \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.....	$-\ L = \frac{3}{2}$

$$s = -1$$

the electron excludes other particles from a region containing 3 flux quanta, creating a potential well in which it is bound

$2/5$ state



$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$			
1	1	0	0	0	.....

$-$

$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	.....

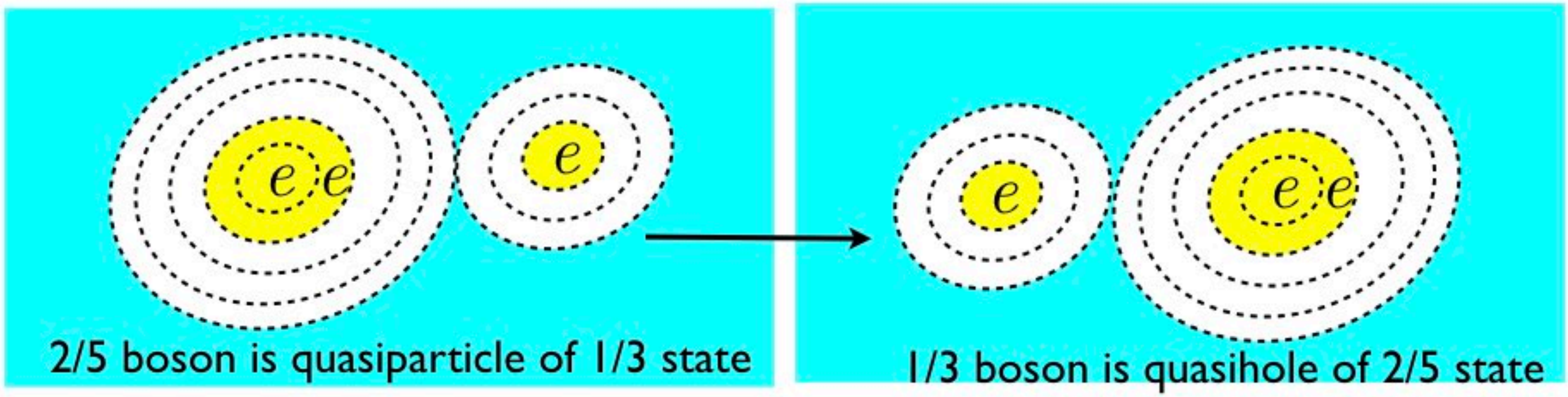
$$L = 2$$
$$-\,L = 5$$
$$s = -3$$

$$L = \frac{g_{ab}}{2\ell_B^2} \sum_i R_i^a R_i^b$$

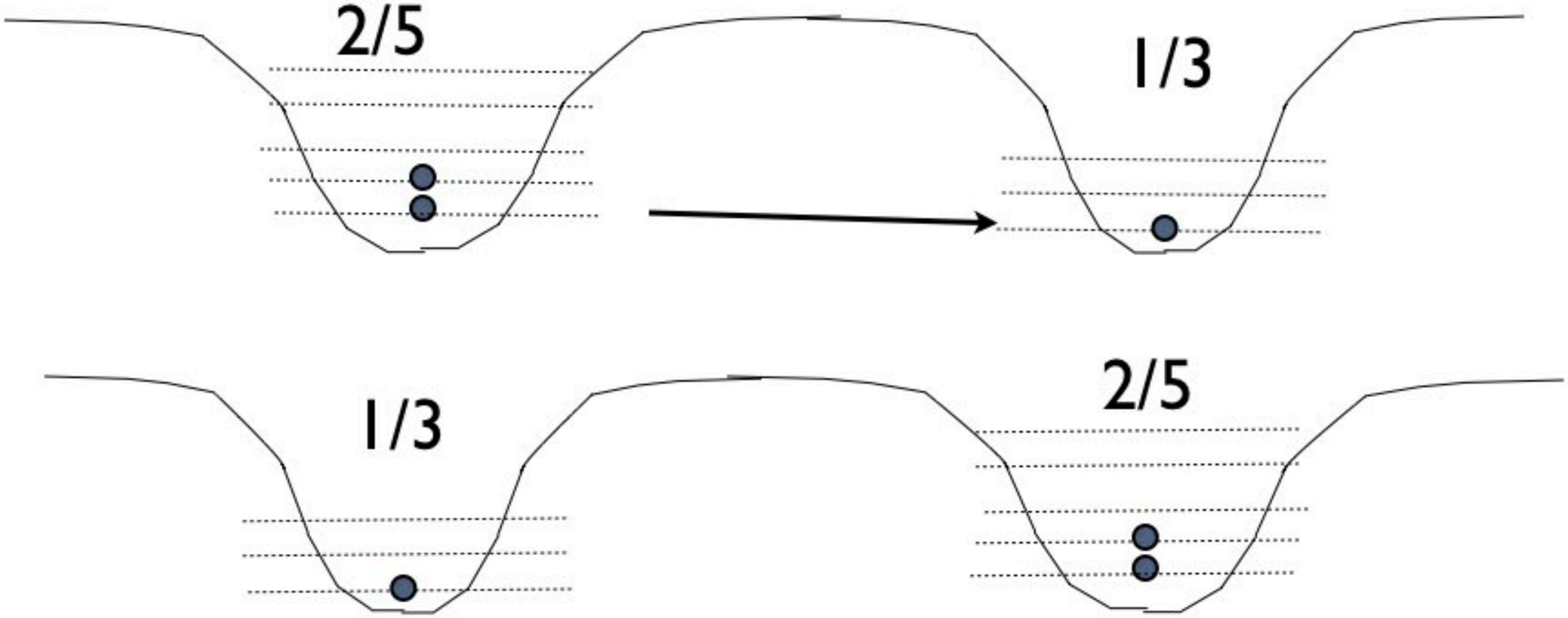
$$Q^{ab} = \int d^2 r \, r^a r^b \delta\rho(r) = s\ell_B^2 g^{ab}$$

second moment of neutral
composite boson
charge distribution

hopping of a "composite fermion" (electron + 2 flux quanta)



Jain's "pseudo Landau levels"

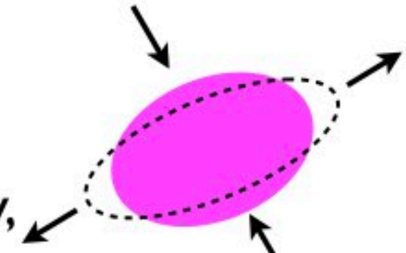


- The composite boson behaves as a neutral particle because the Berry phase (from the disturbance of the the other particles as its "exclusion zone" moves with it) cancels the Bohm-Aharonov phase

- It behaves as a boson provided its statistical spin cancels the particle exchange factor when two composite bosons are exchanged

p particles	$(-1)^{pq} = (-1)^p$	fermions
q orbitals	$(-1)^{pq} = 1$	bosons

- The metric (shape of the composite boson) has a preferred shape that minimizes the correlation energy, but fluctuates around that shape



- The zero-point fluctuations of the metric are seen as the O(q^4) behavior of the "guiding-center structure factor" (Girvin et al, (GMP), 1985)
- long-wavelength limit of GMP collective mode is fluctuations of (spatial) metric (analog of "graviton")

FDMH, Phys. Rev. Lett. **107**, 116801 (2011)

- Furthermore, the local electric charge density of the fluid with $\nu = p/q$ is determined by a combination of the magnetic flux density and the Gaussian curvature of the intrinsic metric

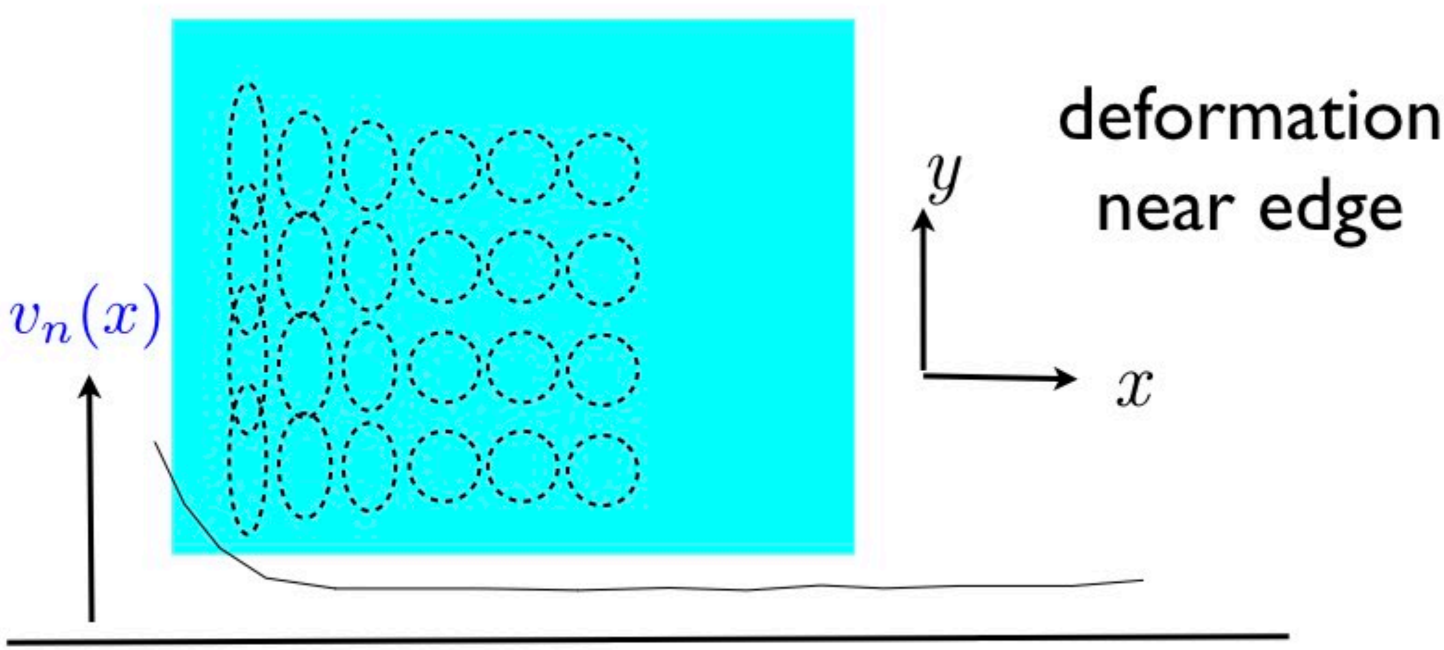
$$J_e^0(\boldsymbol{x}) = \frac{e}{2\pi q}\left(\frac{peB}{\hbar} - sK_g(\boldsymbol{x})\right)$$

Topologically quantized "guiding center spin"

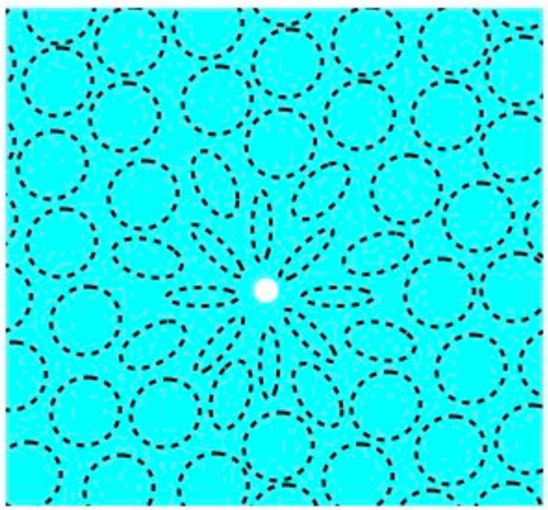
Gaussian curvature of the metric

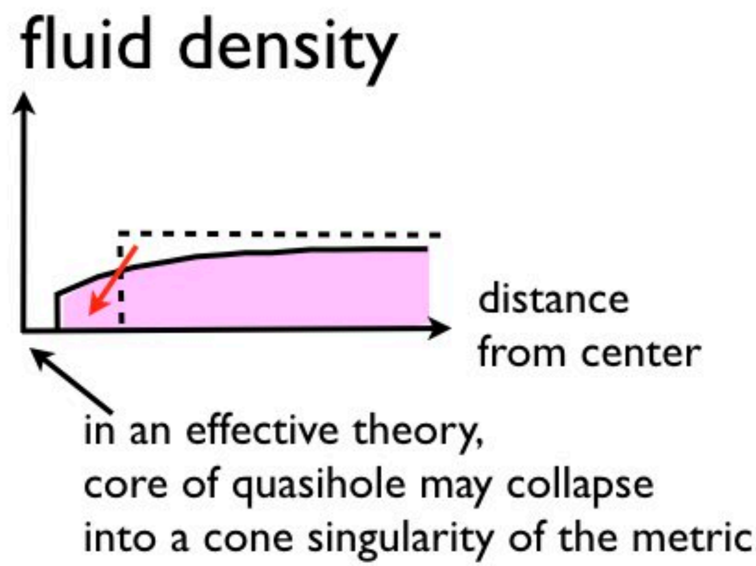
- In fact, it is locally determined, if there is an inhomogeneous slowly-varying substrate potential

$$H = \sum_i v_n(\boldsymbol{R}_i) + \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

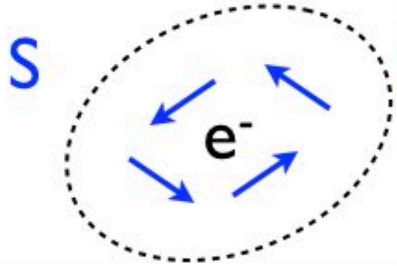


- “skyrmion”-like “cone”-like structure moves charge away from quasihole by introducing negative Gaussian curvature



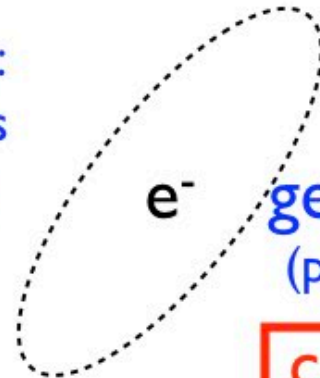


- shape of correlation hole (flux attachment) fluctuates, adapts to environment (electric field gradients)



new property: "spin" couples to curvature

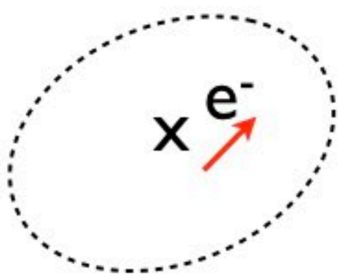
shape=metric



geometric distortion
(preserving inversion symmetry)

creates "curvature" of metric

- polarizable, B x electric dipole = momentum, origin of "inertial mass"



electric polarizability

- In the standard incompressible FQH states, the bulk interior of the fluid is described by a gapped topological field theory (TQFT).
- The gapless edge degrees of freedom are a direct sum of unitary representations of the Virasoro algebra.
- Can there be continuous second order transitions between FQH states at which the bulk gap collapses?

- The (fermion) “Gaffnian” model (Steve Simon et al)
- This is a model 2/5 state that (a) is an exact zero-energy state of a (three-body) interaction (b) has a non-unitary representation of the Virasoro algebra on its edge and (c) as a consequence is believed to have bulk gapless neutral excitations (Read).

- It is a Jack polynomial with a “root configuration exclusion statistics rule” of “not more than two particles in five consecutive orbitals”

- The "Gaffnian" interaction penalizes three-body states

$(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$ 11100

$(z_1 - z_2)(z_2 - z_3)(z_3 - z_1) \times \left((z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2)\right)$ 11001

$$H = V_0 P_{111} + V_2 P_{11001}$$

- On the torus, the 2/5 Gaffnian zero-energy states has a 10-fold degeneracy corresponding to the two sets of 5 "motifs"

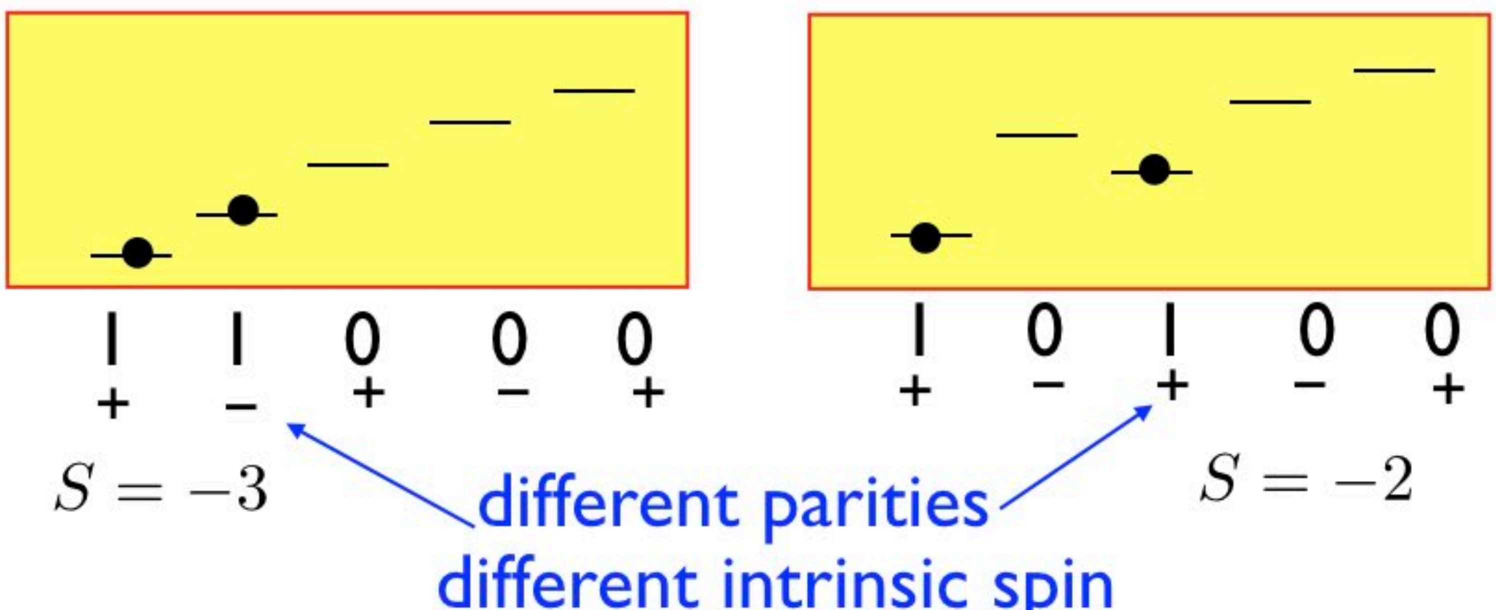
11000 01100 00110 00011 10001

10100 01010 00101 10010 01001

↑ lowest weight (most to left)

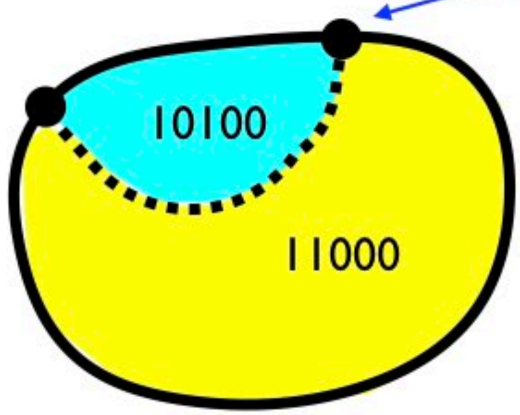
- A degeneracy beween two internal states of the 2/5 "composite boson" with different parity.

- In higher Landau levels the "10100" pattern may replace 11000 as the stable 2/5 pattern because of competition between the "vacancy potential" that favors putting the second particle in the second orbital, and repulsion from the first particle, which pushes it outwards



- Domain wall between states with different Wen-
  Zee term carries momentum density (electric
  dipole moment) but no chiral modes (no U(1) c
  Virasoro anomaly)

negative weight primary field
of non-unitary CFT ???



- gapless bulk if domain wall energy is zero or negative

<u>transmits pressure!</u>

- sliding of domain wall attachment point removes momentum from edge (non-unitary virasoro on edge)

- Many open questions about the gapless critical state (e.g. what is the dynamical critical exponent z (1or 2?))
- does charge gap exist for all ratios of the two parameters?
- develop a Full interpretation of the non-unitary Virasoro representation.