Entanglement spectra of two-dimensional solvable models

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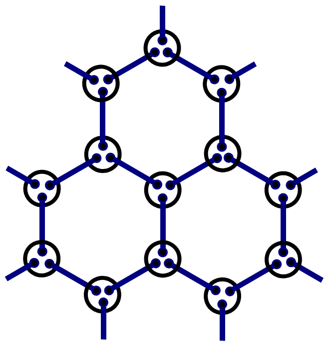
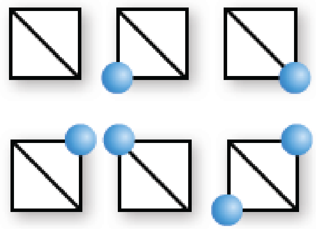




Outline

1. **Introduction**

- AKLT model & valence-bond-solid (VBS) state
- Schmidt decomposition & quantum entanglement

2. **Entanglement spectra in 2d AKLT model**

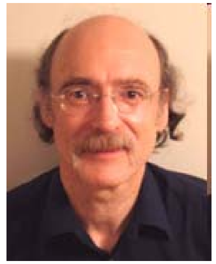
- Reflection symmetry & Gram matrix
- Entanglement entropy & spectrum
- Holographic spin chain (**VBS/CFT correspondence**)

3. **Entanglement spectra in quantum hard-square model**

- Tensor network states for interacting Rydberg atom systems
- Entanglement spectra
- Holographic minimal models (c<1)

A brief history of AKLT and valence-bond-solid (VBS) state

- **Haldane gap problem**



Haldane's conjecture (F. D. M. Haldane, *Phys. Rev. Lett*. **50** ('83).)

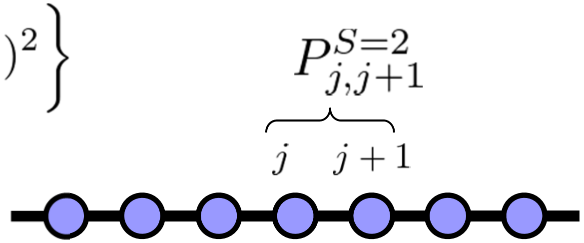
S=integer antiferromagnetic (AFM) Heisenberg chains are gapped.

$$H_{\mathrm{ex}} = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1}$$

Experiment in S=1 spin chain: $Ni(C_2H_8N_2)_2NO_2(ClO_4)$ (NENP) etc.

- **Affleck-Kennedy-Lieb-Tasaki (AKLT) model** (*PRL* **59** ('87), *CMP* **115** ('87).)

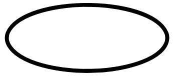
$$\begin{aligned} H_{\mathrm{AKLT}} &= J \sum_j \left\{ \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} (\vec{S}_j \cdot \vec{S}_{j+1})^2 \right\} \\ &= J \sum_j \left(2 P_{j,j+1}^{S=2} - \frac{2}{3} \right) \end{aligned}$$

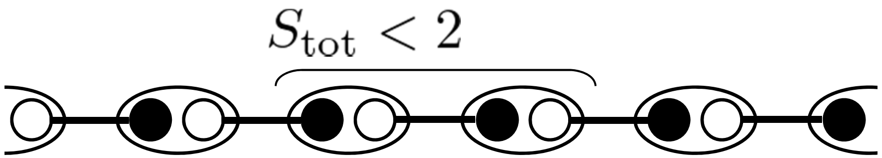


1. Exact unique ground state → valence-bond-solid (VBS) state
2. Rigorous proof of the 'Haldane' gap in this model
3. AFM correlation decays exponentially with distance. $\langle \vec{S}_0 \cdot \vec{S}_n \rangle = 4 \left(-\frac{1}{3} \right)^n$

- **Valence-bond-solid (VBS) state**


: spin singlet $\frac{1}{\sqrt{2}}(|\uparrow\rangle_\circ |\downarrow\rangle_\bullet - |\downarrow\rangle_\circ |\uparrow\rangle_\bullet)$

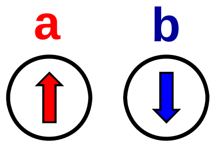

: symmetrization $P^{S=1}_{\bullet\circ}$



VBS state

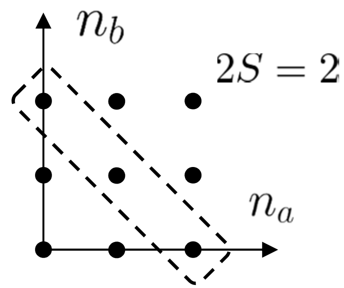
Scwinger boson(SB) rep. of spin operator

$$\begin{cases} S_j^+ = a_j^\dagger b_j, \ S_j^- = b_j^\dagger a_j \\ S^z = (a_j^\dagger a_j - b_j^\dagger b_j)/2 \end{cases}$$



$$|\uparrow\rangle = a^\dagger |{\rm vac}\rangle, \quad |\downarrow\rangle = b^\dagger |{\rm vac}\rangle$$

Constraint : $a_j^\dagger a_j + b_j^\dagger b_j = 2S$



S=1 state is spanned by $\{|0\rangle, |+\rangle, |-\rangle\}$.

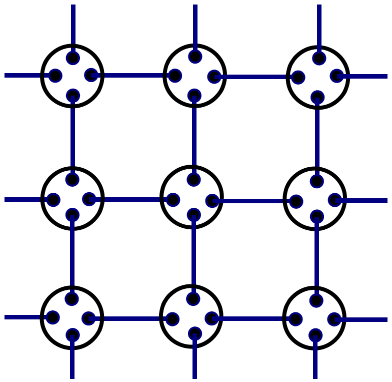
$$|0\rangle = a^\dagger b^\dagger |{\rm vac}\rangle, \quad |+\rangle = \frac{1}{\sqrt{2}}(a^\dagger)^2 |{\rm vac}\rangle, \quad |-\rangle = \frac{1}{\sqrt{2}}(b^\dagger)^2 |{\rm vac}\rangle$$

$$|{\rm VBS}\rangle = \prod_{j=0}^{L} (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger) |{\rm vac}\rangle$$

Matrix product state (MPS)
Fannes *et al*.,(1989), Kluemper *et al*.,(1991).

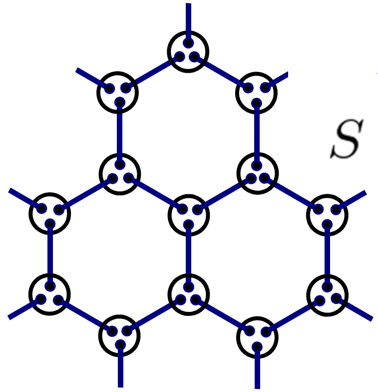
VBS on arbitrary graphs

Square lattice



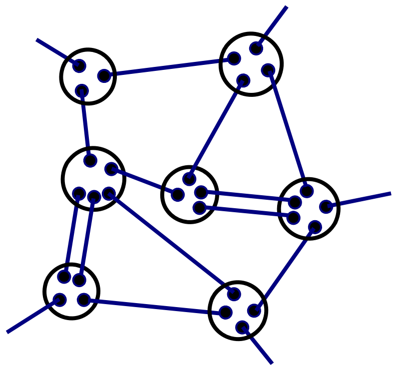
$$S = 2$$

Hexagonal lattice



$$S = 3/2$$

$$\frac{1}{90}(\vec{S}_k \cdot \vec{S}_l)^3 + \frac{29}{360}(\vec{S}_k \cdot \vec{S}_l)^2 + \frac{27}{160}(\vec{S}_k \cdot \vec{S}_l)$$



$$\prod_{\langle kl \rangle} (a_k^\dagger b_l^\dagger - b_k^\dagger a_l^\dagger)^{M_{kl}} |\mathrm{vac}\rangle$$

- **Projected pair entangled state (PEPS), Tensor network state (TNS)**
  Verstraete, Cirac (PRA **70** (R) ('04)), Gross, Eisert (PRL **98** ('07)).

- **Universal quantum computation using 2d VBS state**
  Wei, Affleck, Raussendorf (*PRL* **106** ('11)), Miyake (*Ann. Phys*. **326** ('11)).

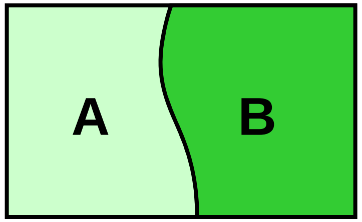
Very few results are known in 2d…

1. No (rigorous) proof of the gap, 2. Entanglement entropy and spectrum?

Kennedy-Lieb-Tasaki (J. Stat. Phys. **53** ('87))

Square & hexagonal VBS (exponential decays of correlations)

Schmidt decomposition & quantum entanglement



Schmidt decomposition:

$$|\Psi\rangle = \sum_\alpha \lambda_\alpha |\phi_\alpha^{[A]}\rangle \otimes |\varphi_\alpha^{[B]}\rangle$$

$\{|\phi_\alpha^{[A]}\rangle\},\ \{|\varphi_\alpha^{[B]}\rangle\}$: orthonormal basis

$|\Psi\rangle$ is normalized. $(\langle\Psi|\Psi\rangle = 1)$

Entanglement spectrum:

$$\lambda_\alpha = e^{-\xi_\alpha/2} \qquad (\alpha = 1, 2, \ldots)$$

Reduced density matrix:

$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| = \sum_\alpha \lambda_\alpha^2 |\phi_\alpha^{[A]}\rangle\langle\phi_\alpha^{[A]}|$$

Entanglement entropy (von Neumann entropy):

$$\mathcal{S} = -\sum_\alpha \lambda_\alpha^2 \ln \lambda_\alpha^2$$

Applications: see L. Amico *et al.*, *Rev. Mod. Phys.* **80**, 517 (2008).

Example (maximally entangled state):

$$|\Psi\rangle = \sum_{\alpha=1}^{D} \frac{1}{\sqrt{D}} |\phi_\alpha^{[A]}\rangle \otimes |\varphi_\alpha^{[B]}\rangle \quad \Rightarrow \quad \mathcal{S} = \ln D$$

$\mathcal{S} = 0$ when D=1 (Direct product state).

Entanglement Hamiltonian

- Reduced density matrix

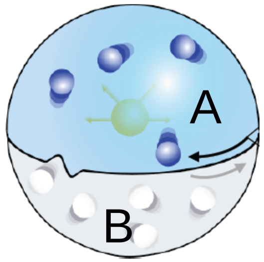
$$\rho_A = \sum_\alpha e^{-\xi_\alpha} |\phi_\alpha^{[A]}\rangle\langle\phi_\alpha^{[A]}|$$

- Entanglement Hamiltonian

$$\rho_A = e^{-H_{\mathrm{E}}} \quad (H_{\mathrm{E}} = -\log \rho_A)$$

- Li-Haldane conjecture

H. Li & F. D. M. Haldane, *PRL* **101**, 010504 (2008).
A. Chandran *et al*., arXiv:1102.2218 (2011).

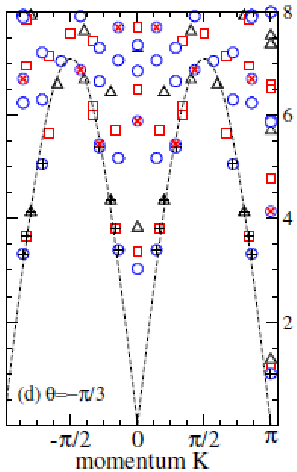


Edges of bulk FQHE states are described by CFT.
The low-energy entanglement spectrum is the same CFT!

A precise correspondence between the entanglement Hamiltonian and the physical Hamiltonian with open boundaries.

- **Examples of 2d topological systems**
  Topological insulators: Fidkowski, *PRL* **104** ('10).
  Turner, Zhang & Vishwanath, *PRB* **82** ('10).
  Honeycomb Kitaev model: Yao & Qi, *PRL* **105** ('10).
  General FQH states and BCFT: Qi, H.K. & Ludwig, PRL **108** ('12).

- **Other examples**
  Spin chain: Pollmann *et al*., *PRB* **81** ('10).
  Spin ladder: Poilblanc, *PRL* **105** ('10).
  General argument: Peschel & Chung, *Europhys. Lett*. **91** ('11).



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   - Entanglement spectra
   - Holographic minimal models ($c<1$)

Gram matrix & reflection symmetry

$$|\Psi\rangle = \sum_\alpha |\phi_\alpha^{[A]}\rangle \otimes |\varphi_\alpha^{[B]}\rangle$$

$|\Psi\rangle$ is not necessarily normalized.

$\{|\phi_\alpha^{[A]}\rangle\}$ and $\{|\varphi_\alpha^{[B]}\rangle\}$ may not be orthonormal.

Gram matrices:

$$(M^{[A]})_{\alpha\beta} = \langle \phi_\alpha^{[A]} | \phi_\beta^{[A]} \rangle, \quad (M^{[B]})_{\alpha\beta} = \langle \varphi_\alpha^{[B]} | \varphi_\beta^{[B]} \rangle$$

Useful fact (see Katsura *et al*., *J. Phys. A* **43**, 255303 ('10))

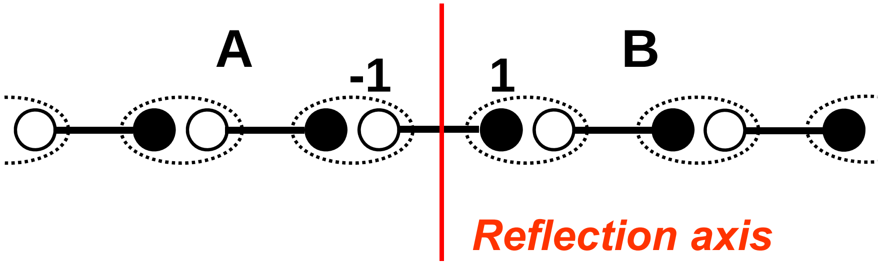
If $M^{[A]} = M^{[B]} = M$ and M is real symmetric matrix, then we have $|\Psi\rangle = \sum_\alpha d_\alpha |e_\alpha\rangle \otimes |f_\alpha\rangle$ where d_α are the eigenvalues of M and $\langle e_\alpha | e_\beta \rangle = \langle f_\alpha | f_\beta \rangle = \delta_{\alpha\beta}$. (Schmidt decomposition)

$$\rho_A = \frac{M^2}{\mathrm{Tr}[M^2]}$$ ⇒ Spectrum: $\xi_\alpha = -\ln \rho_{A,\alpha}$ Entropy: $\mathcal{S} = \sum_\alpha \xi_\alpha e^{-\xi_\alpha}$

Calculation of entanglement spectrum and entropy **=** ***Diagonalization of overlap matrix (M)***

This technique can be applied to VBS state on a ***reflection symmetric*** graph!

Application to 1d VBS states



$$|\mathrm{VBS}\rangle = a_{-1}^{\dagger}|\mathrm{VBS}^{[A]}\rangle \otimes b_1^{\dagger}|\mathrm{VBS}^{[B]}\rangle - b_{-1}^{\dagger}|\mathrm{VBS}^{[A]}\rangle \otimes a_1^{\dagger}|\mathrm{VBS}^{[B]}\rangle$$

$$|\phi_{\uparrow}^{[A]}\rangle = a_{-1}^{\dagger}|\mathrm{VBS}^{[A]}\rangle, \ |\phi_{\downarrow}^{[A]}\rangle = b_{-1}^{\dagger}|\mathrm{VBS}^{[A]}\rangle$$

Connected to each other by $S_{\mathrm{tot}}^{\pm}$.

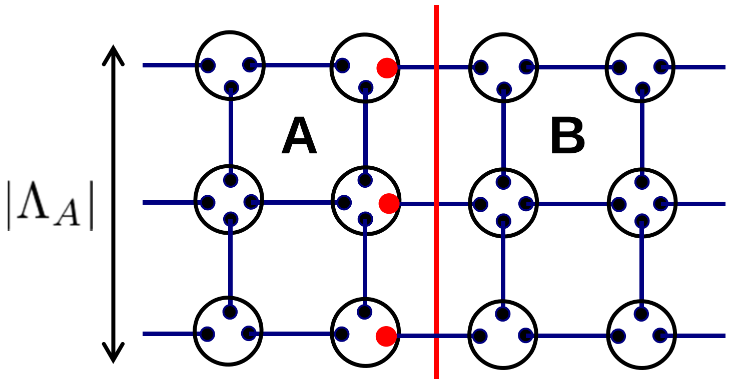
⇒

$$(M^{[A]})_{\alpha\beta} = (M^{[B]})_{\alpha\beta} = d\,\delta_{\alpha\beta}$$

$$\rho_{A,1} = \rho_{A,2} = \frac{1}{2}, \quad \mathcal{S} = \ln 2$$

1d results: Fan, Korepin, Roychowdhury, PRL **93** (‘04),
H.K., Hirano, Hatsugai, PRB **76** (‘07), Xu, H. K., Hirano, Korepin, J. Stat. Phys. **133** (‘08).

What about 2d VBS states?



$$|\phi_{\alpha}^{[A]}\rangle = \prod_{i\in\Lambda_A} (a_i^{\dagger})^{1/2+\alpha_i}(b_i^{\dagger})^{1/2-\alpha_i}|\mathrm{VBS}^{[A]}\rangle$$

$$|\mathrm{VBS}\rangle = \sum_{\alpha} |\phi_{\alpha}^{[A]}\rangle \otimes |\phi_{\alpha}^{[B]}\rangle$$

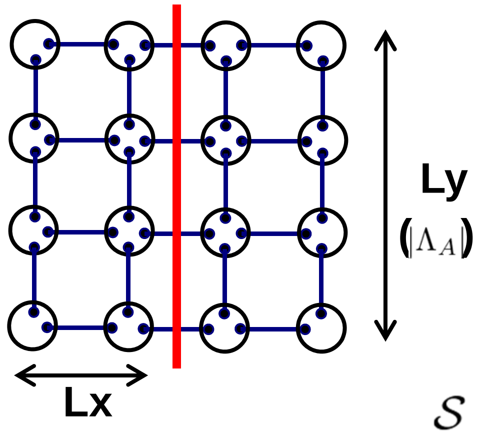
Symmetry is not large enough to determine the Gram matrix $M^{[A]}$***.***
Numerics are required.

Naïve guess (valence bond EE)

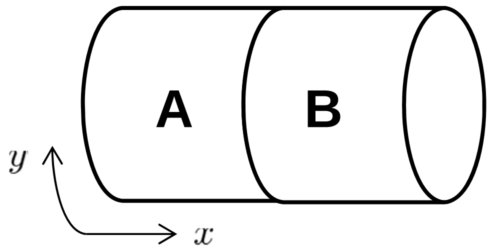
~~$\mathcal{S} = |\Lambda_A| \ln 2$~~

Numerical results of entanglement entropy

1. Matrix elements can be calculated by Monte Carlo method.
2. Exact diagonalization of M which is $2^{|\Lambda_A|}$ -dimensional matrix.

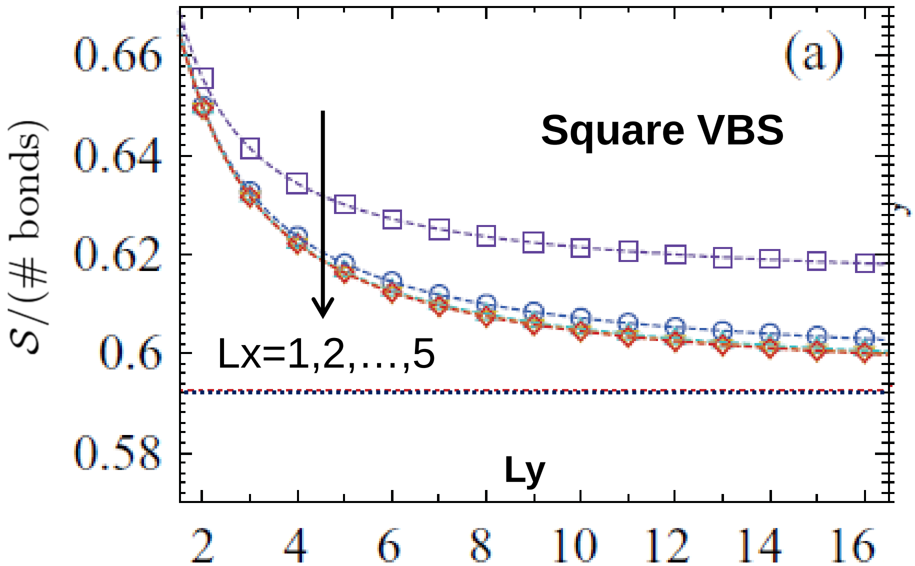


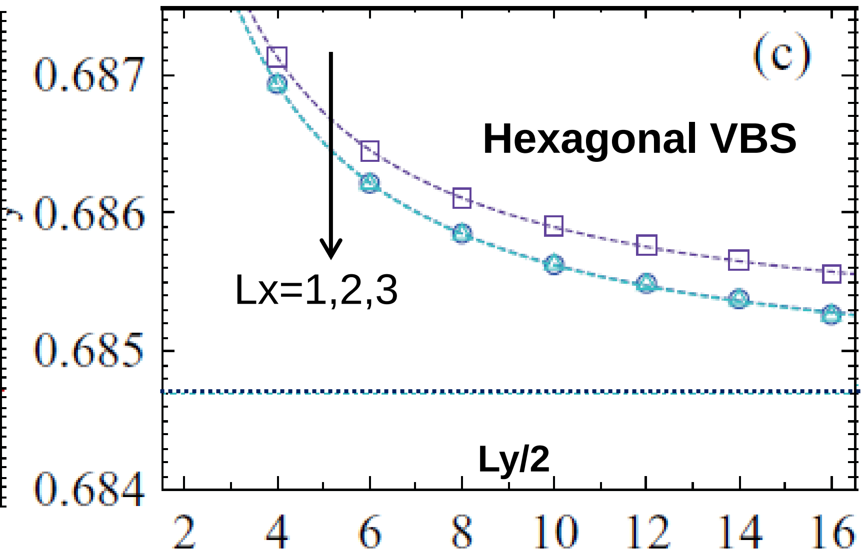
Cylindrical geometry



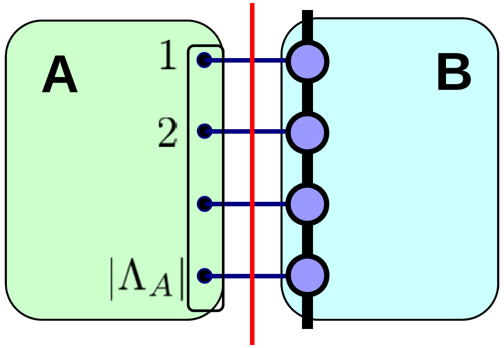
$$\mathcal{S} = \alpha L_y + \mathcal{S}_0$$

Prefactor α is less than ln2=0.6931.





Analytical results - Loop model approach -

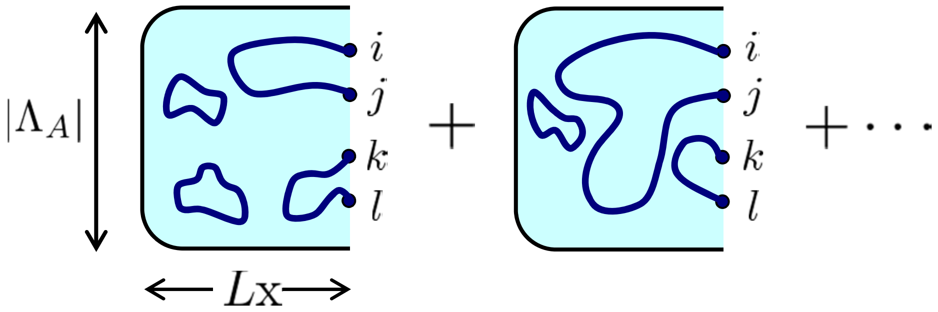


Gram matrix
= holographic spin chain (polynomial in $(\vec{\sigma}_i \cdot \vec{\sigma}_j)$ ***)***

$$M = \int \left(\prod_{i \in A} \frac{(2S_i+1)!}{4\pi} d\hat{\Omega}_i \right) \prod_{k \in \Lambda_A} \left(\frac{1+\hat{\Omega}_k \cdot \vec{\sigma}_k}{2} \right) \prod_{(i,j) \in \mathcal{B}_A} \left(\frac{1-\hat{\Omega}_i \cdot \hat{\Omega}_j}{2} \right)$$

Useful formula: $\int \frac{d\hat{\Omega}}{4\pi} (\hat{\Omega}_1 \cdot \hat{\Omega})(\hat{\Omega} \cdot \hat{\Omega}_2) = \frac{1}{3} (\hat{\Omega}_1 \cdot \hat{\Omega}_2)$

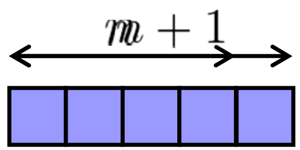
Ex) Graphical rep. for $A_{ijkl}\,(\vec{\sigma}_i \cdot \vec{\sigma}_j)(\vec{\sigma}_k \cdot \vec{\sigma}_l)$



Bond weight: **1/3**

Loop weight: **3**

Non-critical loop model…



- **Transfer matrix approach for strip systems**

Recursion relation between $Lx = n$ and $Lx = n + 1$.

Perron-Frobenius vector of the transfer matrix → M in the limit of large Lx

We can obtain $M\ (Lx \to \infty)$ ***without using MC method!***

Comparison of 'analytical' and numerical results

Square lattice		Lx = 1	Lx = 2	Lx = 3	Lx = 4	Lx = 5
Ly = 2	Exact	0.6553433	0.6498531	0.6494635	0.6494368	0.6494349
	MC	0.6553431	0.6498533	0.6494621	0.6494342	0.6494373
Ly = 3	Exact	0.6413153	0.6325619	0.6316999	0.6316095	0.6315995
	MC	0.6413145	0.6325626	0.6316999	0.6316080	0.6315866

Hexagonal lattice		Lx = 1	Lx = 2	Lx = 3	Lx = 4	Lx = 5
Ly = 4	Exact	0.6891577	0.6890932	0.6890927	0.6890927	0.6890927
	MC	0.6891575	0.6890924	0.6890929	0.6890925	0.6890840
Ly = 6	Exact	0.6878024	0.6876554	0.6876523	0.6876522	0.6876522
	MC	0.6878027	0.6876558	0.6876513	0.6876537	0.6875899
Ly = 8	Exact	0.6871254	0.6869344	0.6869295	0.6869293	0.6869293
	MC	0.6871243	0.6869385	0.6869363	0.6868834	0.6867750

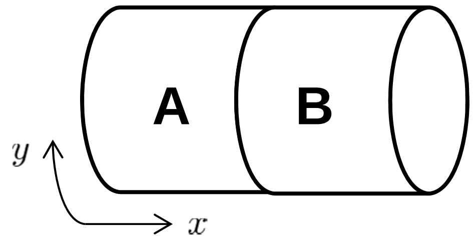
Entanglement spectrum

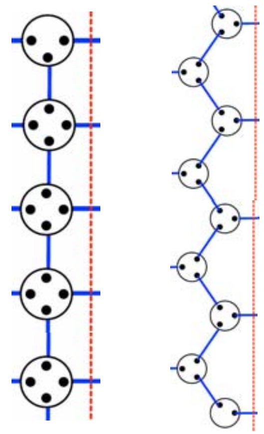
- Entanglement Hamiltonian (reminder)

$$\rho_A = e^{-H_\mathrm{E}}$$

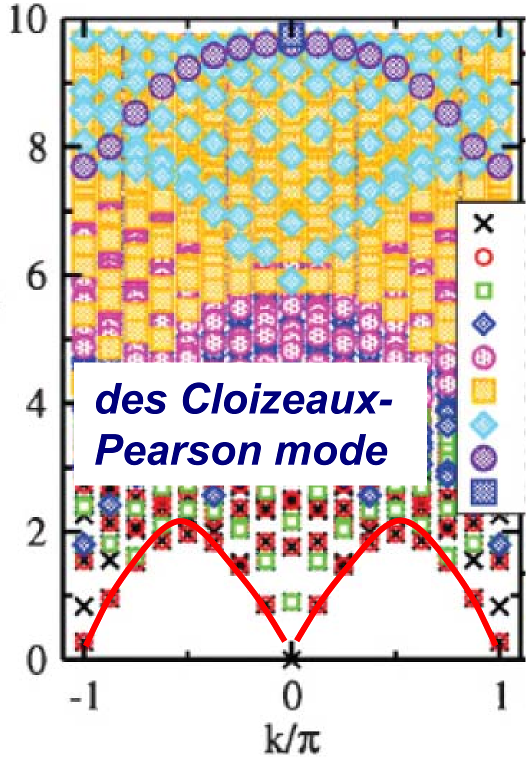
Momentum in *y*-direction is a good quantum number.
→ Momentum resolved spectrum!

Cylindrical geometry

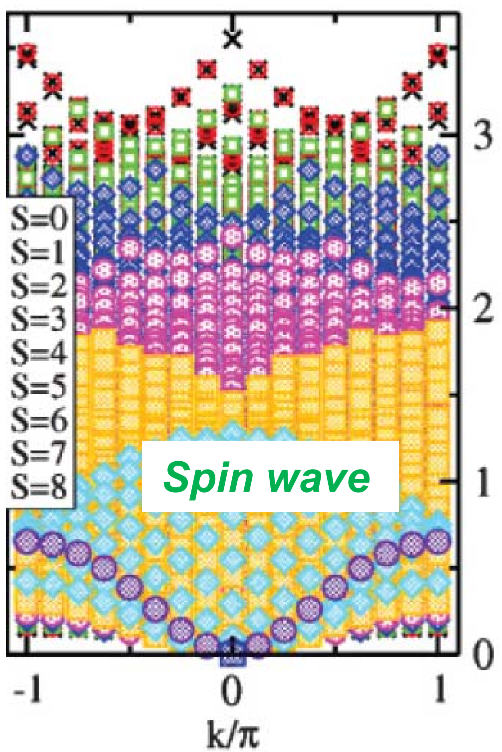




Square (Lx=5, Ly=16)

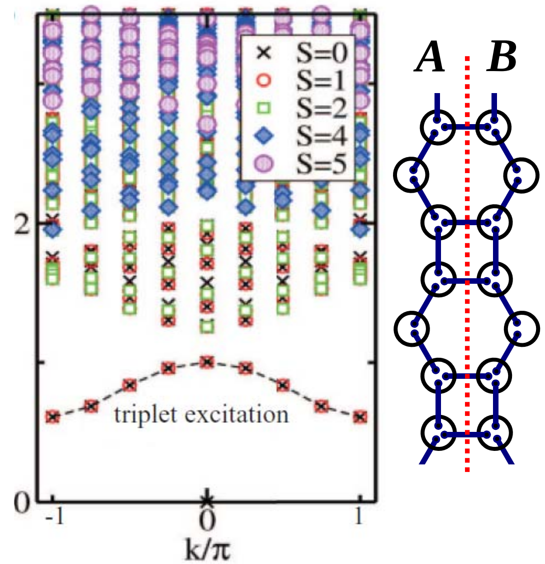


Hexagonal (Lx=5, Ly=32)



Gapped example

Mixed lattice



Correspondence between 2d VBS and 1d spin chain

Entanglement Hamiltonian	physical Hamiltonian
Square lattice VBS	AFM spin chain
Hexagonal lattice VBS	FM spin chain
Mixed lattice VBS	J1-J2 spin chain

Reduced density matrix

$$\rho_A \sim \frac{1}{Z}\exp(-\beta H_{\mathrm{Heis}})$$

Lattice geometry is very important!

Nested entanglement entropy (square VBS)

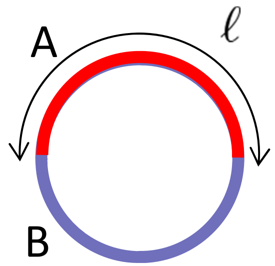
- "entanglement" ground state (g.s.) := g.s. of H_{E}

Maximum eigenvalue

$$H_{\mathrm{E}}|\Psi_0\rangle = E_{\mathrm{gs}}|\Psi_0\rangle \quad \Longleftrightarrow \quad (\rho_A|\Psi_0\rangle = \rho_0|\Psi_0\rangle)$$

- Nested reduced density matrix $\rho(\ell) := \mathrm{Tr}_{\ell+1,\ldots,L}[|\Psi_0\rangle\langle\Psi_0|]$
- Nested entanglement entropy (new concept)

$$\mathcal{S}(\ell, L) = -\mathrm{Tr}_{1,2,\ldots,\ell}[\rho(\ell)\ln\rho(\ell)]$$



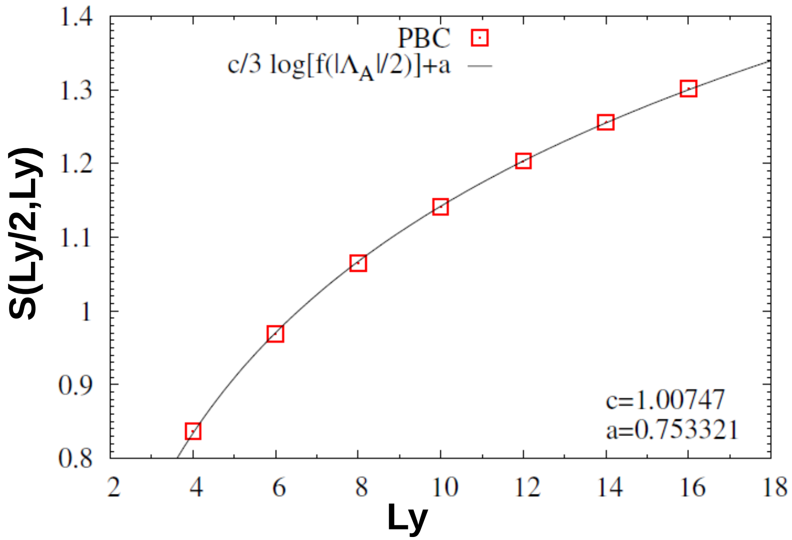
CFT predictions (Calabrese-Cardy, Affleck, …):

For PBC (cylinder),

$$\mathcal{S}^{\mathrm{PBC}}(\ell, L) = \frac{c}{3}\ln\left[\frac{L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)\right] + s_1$$

(s_1: non-universal constant)

- Central charge (square VBS, Lx=1)



Central charge=1.00747…

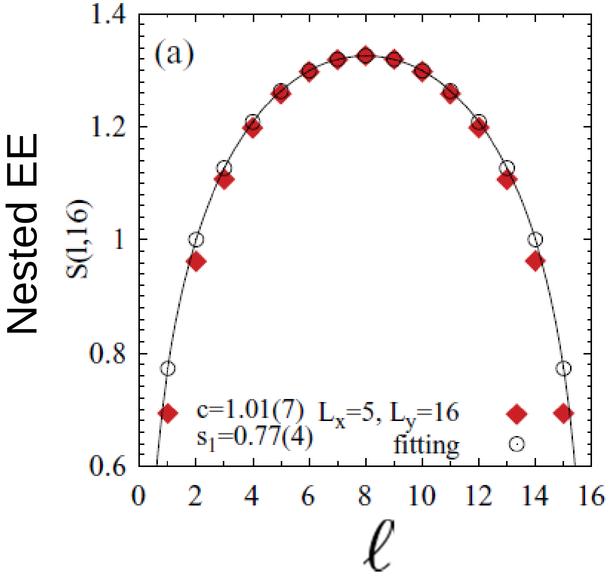
$L_x = 1$	$L_x = 2$	$L_x = 3$	$L_x = 4$	$L_x = 5$
1.007(4)	1.042(4)	1.055(4)	1.056(2)	1.059(2)

Low-energy effective field theory is *c=1* conformal field theory as in the case of the *S*=1/2 AFM Heisenberg chain!!

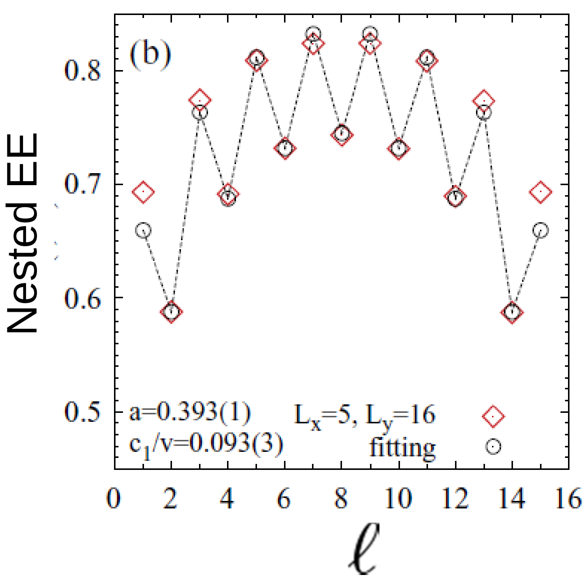
VBS/CFT correspondence

- Nested entanglement entropy (Lx=5, Ly=16)

Cylinder



Strip



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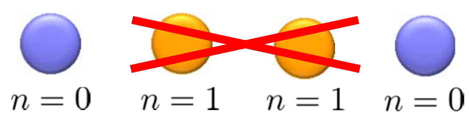
Quantum hard-square lattice gas

- **Rydberg lattice gas**

$$\mathcal{H}_{\text{Rydberg}} = \Omega \sum_{i\in\Lambda} \sigma_i^x + \Delta \sum_{i\in\Lambda} n_i + V \sum_{i,j\in\Lambda} \frac{n_i n_j}{|\mathbf{r}_j - \mathbf{r}_i|^\gamma}, \quad n_i = \frac{\sigma_i^z + 1}{2} \quad \gamma = 6$$

Strongly interacting regime: $|V| \gg |\Omega|, |\Delta|$

Variational ansatz (Tensor network state)



S. Ji et al., *PRL* **107**, 060406 (2011); I. Lesanovsky, *PRL* **108**, 105301 (2012).

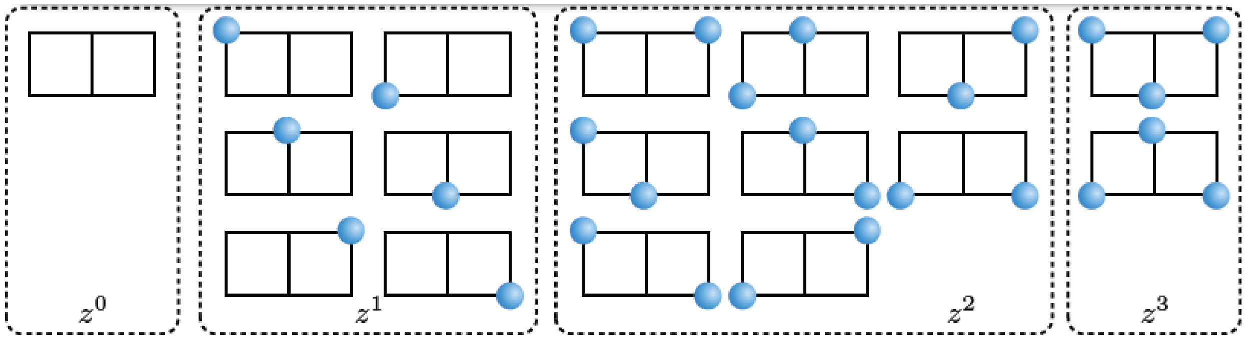
$$|z\rangle = \frac{1}{\sqrt{\Xi(z)}} \prod_{i\in\Lambda} \exp(\sqrt{z}\sigma_i^+ \mathcal{P}_{\langle i\rangle}) |\downarrow\downarrow\cdots\downarrow\rangle$$

$$\mathcal{P}_{\langle i\rangle} := \prod_{j\in G_i} (1-n_j)$$

Never have adjacent excited states.

$$|z\rangle \propto \sum_{\mathcal{C}\in\mathcal{S}} z^{n_{\mathcal{C}}/2} |\mathcal{C}\rangle$$

Nearest neighbor exclusion



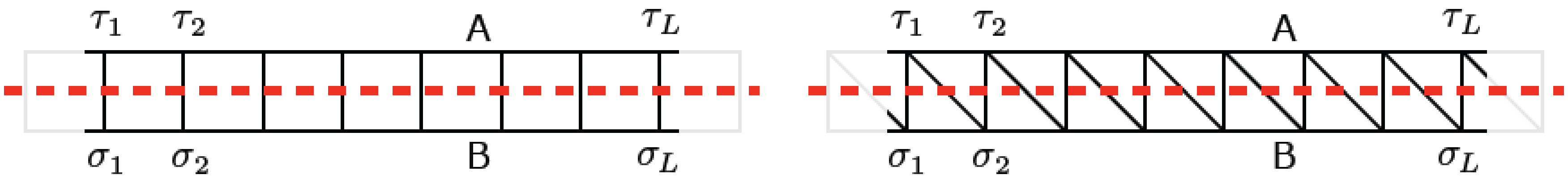
Parent Hamiltonian (RK construction):

$$\mathcal{H}_{\text{sol}} = \sum_{i\in\Lambda} h_i^\dagger(z) h_i(z), \quad h_i(z) := [\sigma_i^- - \sqrt{z}(1-n_i)]\mathcal{P}_{\langle i\rangle}$$

|z> is the zero-energy ground state.

Ground states on 2-leg ladders

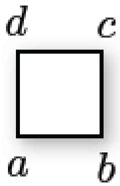
- **Setup**



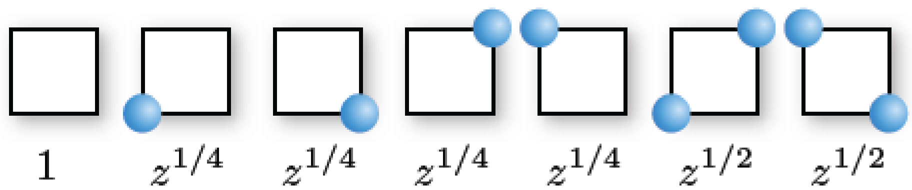
- **Ground state (un-normalized)**

$$|\Psi(z)\rangle = \sum_{\tau}\sum_{\sigma}[T(z)]_{\tau,\sigma}|\tau\rangle \otimes |\sigma\rangle, \qquad [T(z)]_{\tau,\sigma} := \prod_{i=1}^{L} w(\sigma_i, \sigma_{i+1}, \tau_{i+1}, \tau_i)$$

Face Boltzmann weight:



$= w(a,b,c,d)$

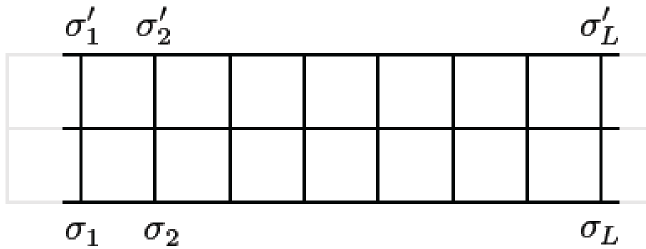


- **Gram matrix**

$$M = \frac{1}{\Xi(z)}\underline{\underline{[T(z)]^{\mathrm{T}}T(z)}} \quad \Rightarrow \quad M =: \exp(-\mathcal{H}_{\mathrm{E}})$$

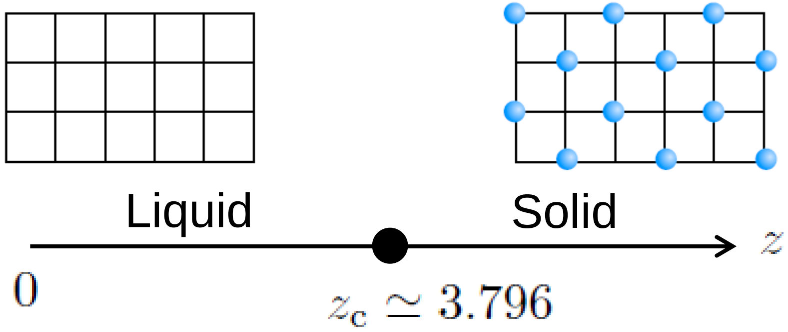
Entanglement Hamiltonian

Double-row transfer matrix (2d classical stat. mech.)

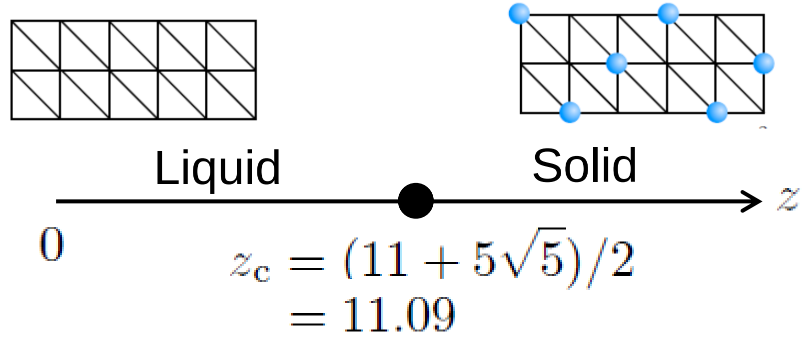


- **Classical hard-square & hard-hexagon models**

Hard square (Gaunt & Fisher, *JPC* **43** ('65))



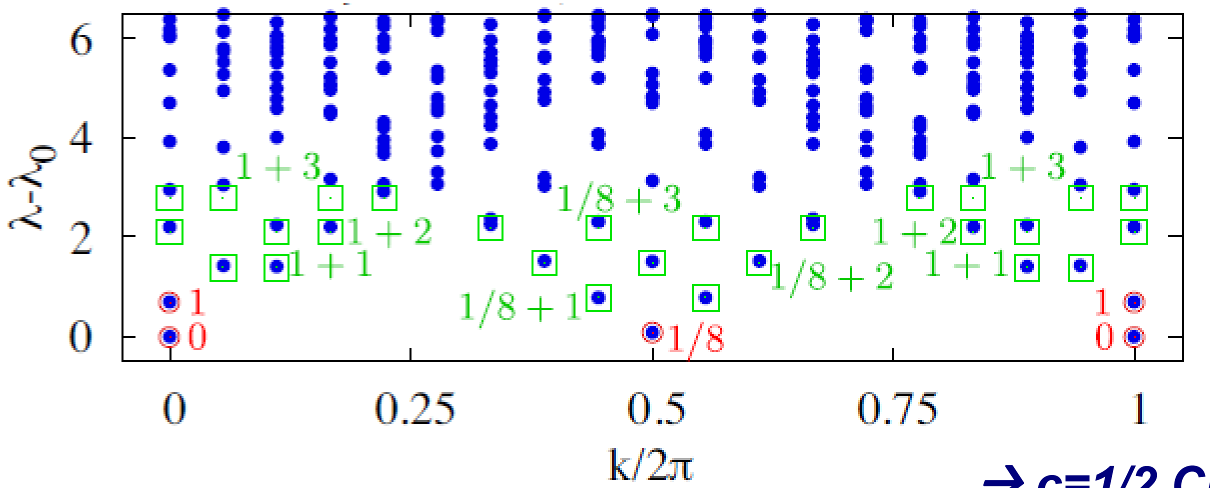
Hard hexagon (Baxter, *JPA* **13** ('80))



Entanglement spectrum should be critical at $z=z_c$.

Entanglement spectra

- **Square ladder**



$$\lambda_\alpha - \lambda_0 = \frac{2\pi v}{L}(h_{L,\alpha} + h_{R,\alpha})$$

v : velocity

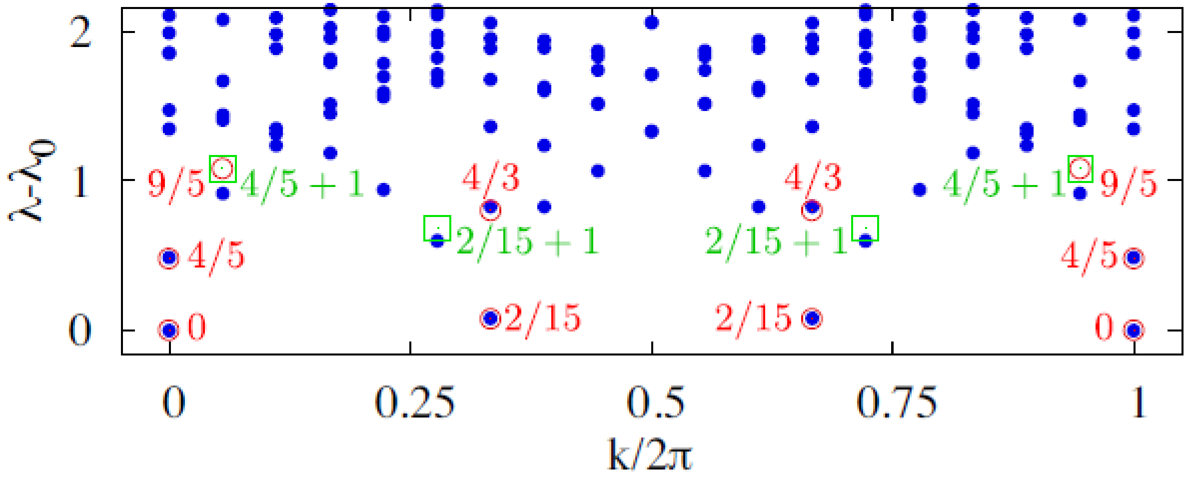
$h_{L,\alpha} + h_{R,\alpha}$ scaling dimension

primary field

descendant field

→ ***c=1/2 CFT (Ising criticality)***

• **Triangular ladder**



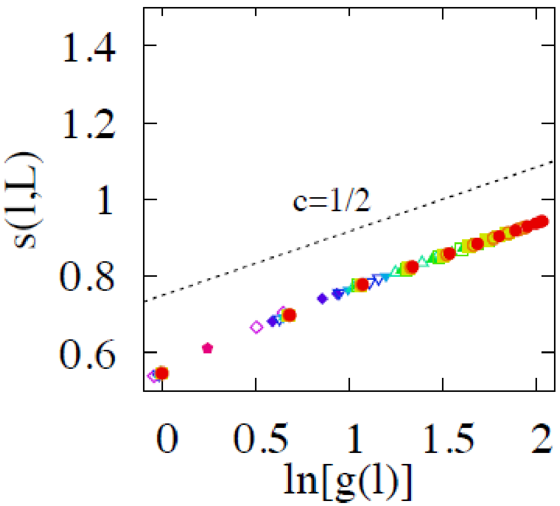
Transfer matrix → Baxter's hard-hexagon model

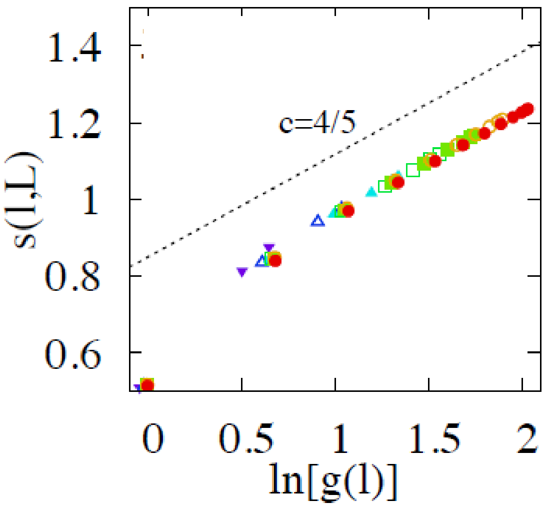
Exact scaling dimensions: Kluemper & Pearce, *J. Stat. Phys.* **64** ('91).

→ ***c=4/5 CFT (3-state Potts criticality)***

Entanglement Hamiltonian is integrable (although the original model is not).

Nested entanglement entropy





Calabrese-Cardy formula

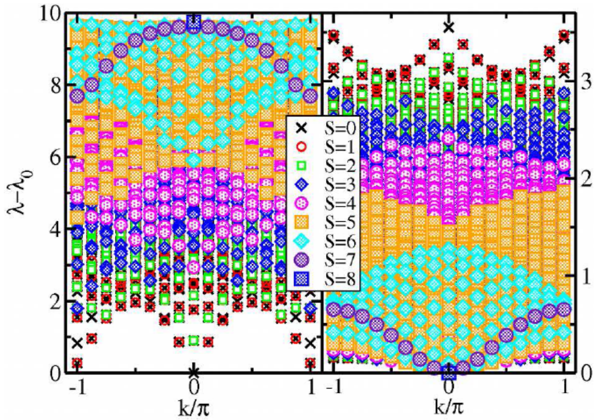
$$s(\ell, L) = \frac{c}{3}\ln[g(\ell)] + s_1$$

Square	2d Ising
Triangular	3-state Potts

Summary

1. 2d AKLT model

- RDM of 2d VBS ⇔ thermal DM of 1d spin chain



Square VBS	AFM spin chain
Hexagonal VBS	FM spin chain
Mixed VBS	J1-J2 spin chain

- Nested EE shows that the entanglement hamiltonian for square VBS is well described by c=1 CFT. [**VBS/CFT correspondence**]

➢ H.K., *et al*., *JPA* **43**, 255303 (2010).

➢ J. Lou, S. Tanaka, H.K. & N. Kawashima, *PRB* **84**, 245128 (2011).

2. Quantum hard-square model

- RDM of 2-leg ladder g.s. ⇔ transfer matrix of 2d classical stat. mech.

Square	2d Ising (c=1/2)
Triangular	3-state Potts (c=4/5)

Holographic minimal model CFTs (c<1)

- For the triangular ladder, entanglement Hamiltonian is integrable.

➢ S. Tanaka, R. Tamura, & H.K., arXiv:1207.6752 (to appear in *PRA*).