Unbounded Growth of Entanglement in Models of Many-Body Localization

Frank Pollmann
Max-Planck-Institut für komplexer Systeme, Dresden, Germany

Together with: **Jens Bárðarson and Joel E. Moore**

J. H. Bardason, F. Pollmann, J. M. Moore, Phys. Rev. Lett. **109**, 017202 (2012)

Sep. 11 2012 Hsinchu

Unbounded Growth of Entanglement in Models of Many-Body Localization

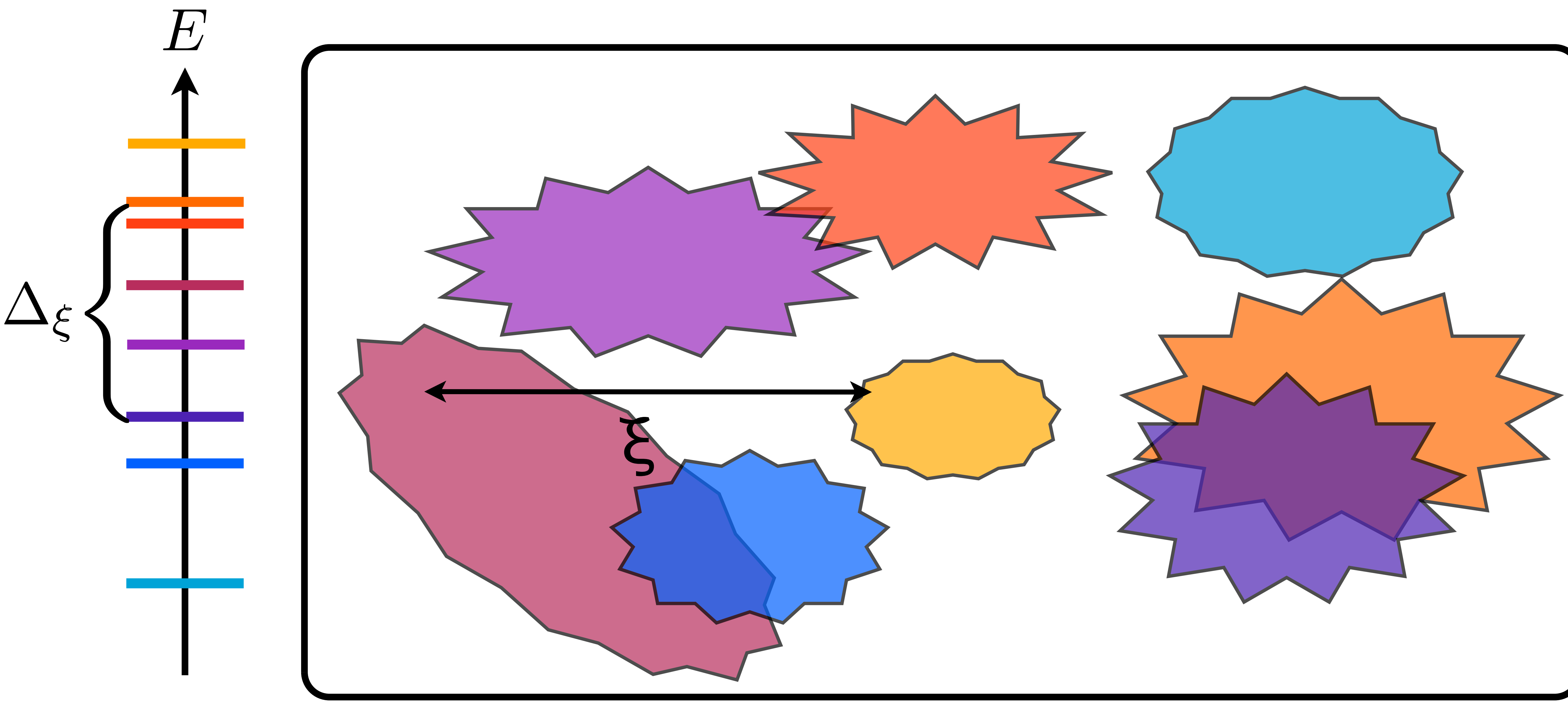
Overview

- Introduction many-body localization
- Dynamics after a quench in disordered systems: Entanglement and other physical quantities
- Entanglement in excited states

Introduction

- Many-body localization

What about interactions?

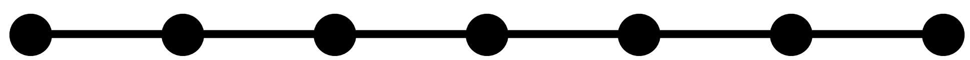


Single particle states localized with localization length ξ
States close in energy separated in space - no overlap
States close in space, separated in energy

Anderson (1958)

Introduction

- **Model system**: Anisotropic Heisenberg S=1/2 chain
- Equivalent to spinless fermions



$$H = J_\perp \sum_i \underbrace{(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)}_{\text{hopping}} + \underbrace{\sum_i h_i S_i^z}_{\text{random potential}} + \underbrace{J_z \sum_i S_i^z S_{i+1}^z}_{\text{interaction}}$$

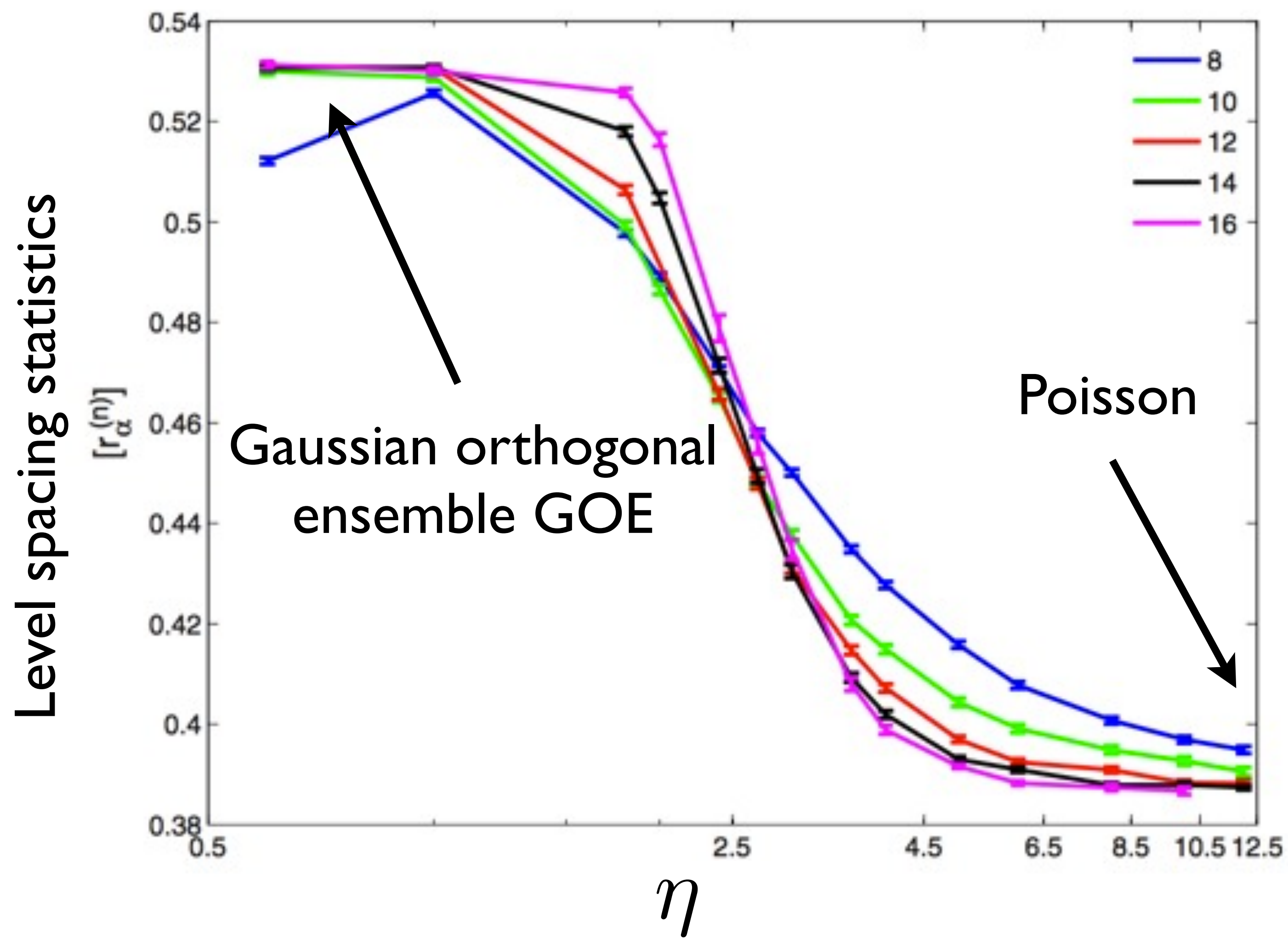
with $h_i \in [-\eta, \eta]$

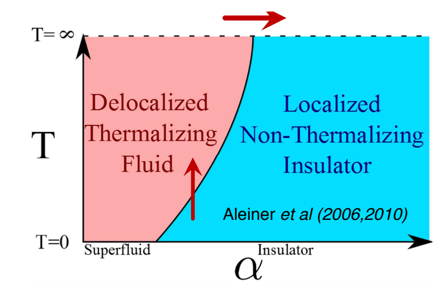
- **All single particle states localized** for $\eta \neq 0$

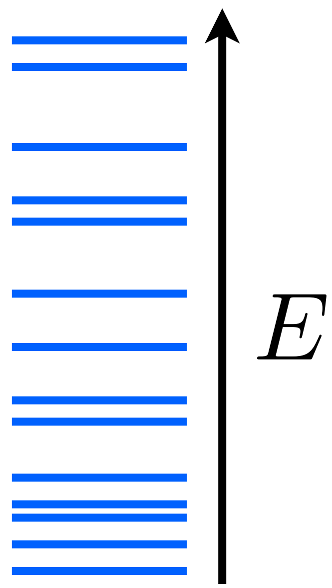
Anderson (1958)

Introduction

- Isotropic Heisenberg S=1/2 chain: Indications for a transition between localized and delocalized phase at $T = \infty$





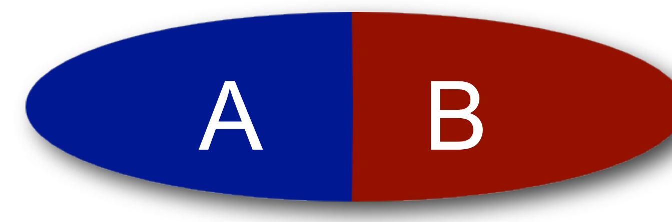


- **Properties of the localized Phase?** Dynamics?

Oganesyan and Huse (2007), Pal and Huse (2010)

Dynamics of entanglement entropy

Schmidt decomposition (SVD $C = UDV^\dagger$)



- Decompose a state $|\psi\rangle$ into a superposition of product states:

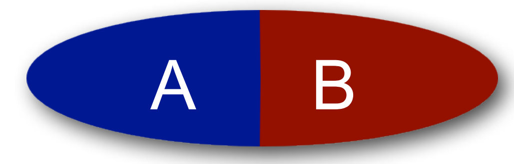
$$|\psi\rangle = \sum_{i=1}^{N_A}\sum_{j=1}^{N_B} C_{ij}|i\rangle_A|j\rangle_B = \sum_\gamma \lambda_\gamma |\phi_\gamma\rangle_A |\phi_\gamma\rangle_B$$

- **Schmidt states:** $|\phi_\gamma\rangle$, **Schmidt values:** λ_γ
- $|\phi_\gamma\rangle$ are eigenstates of the reduced density matrix

$$\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi| \quad \text{with} \quad \rho_A|\phi_\gamma\rangle_A = \lambda_\gamma^2|\phi_\gamma\rangle_A$$

Dynamics of entanglement entropy

- **“Entanglement”?**



  - product state (=non-entangled):

$$|\psi\rangle = \frac{1}{2}\Big(|\uparrow\rangle_A + |\downarrow\rangle_A\Big)\Big(|\uparrow\rangle_B + |\downarrow\rangle_B\Big) \quad \Rightarrow S = 0$$

  - entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B\Big) \quad \Rightarrow S = \log 2$$

- **Entanglement entropy** as a measure for the amount of entanglement

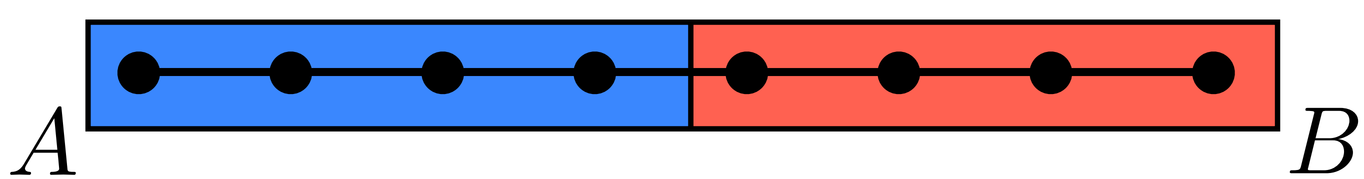
$$\begin{aligned} S &= -\mathrm{Tr}\rho_A \log \rho_A \\ &= -\sum \lambda_\gamma^2 \log \lambda_\gamma^2 \end{aligned}$$

Dynamics of entanglement entropy

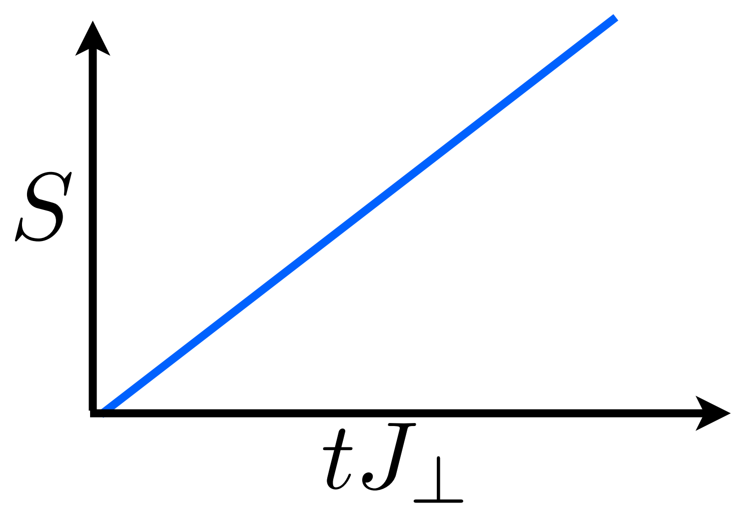
- Start from an unentangled product state ($S = 0$)

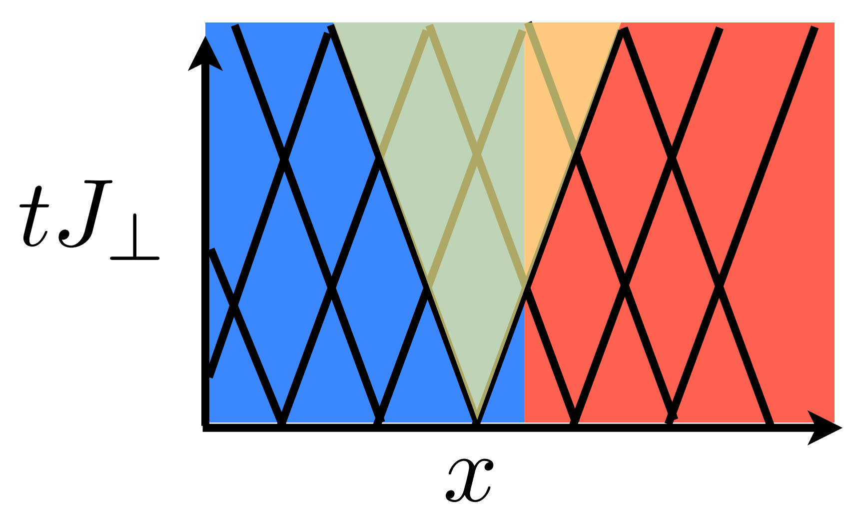
$$|\psi_0\rangle = |\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle$$

- Measure the entanglement after quench and the time evolution with $U(t) = e^{-itH}$



- Clean system:

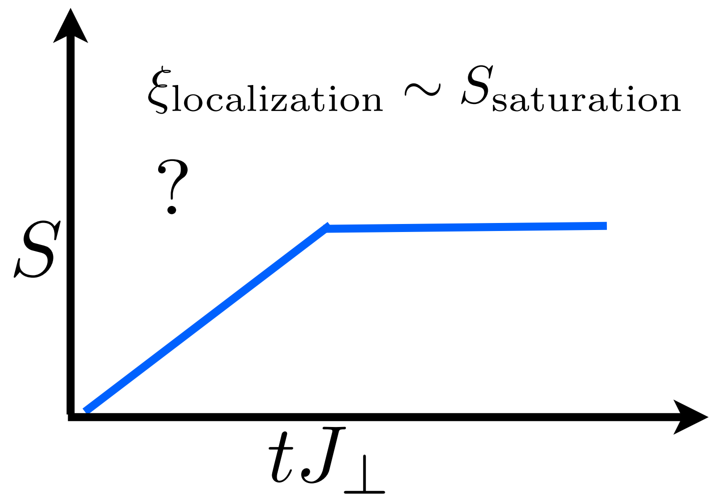


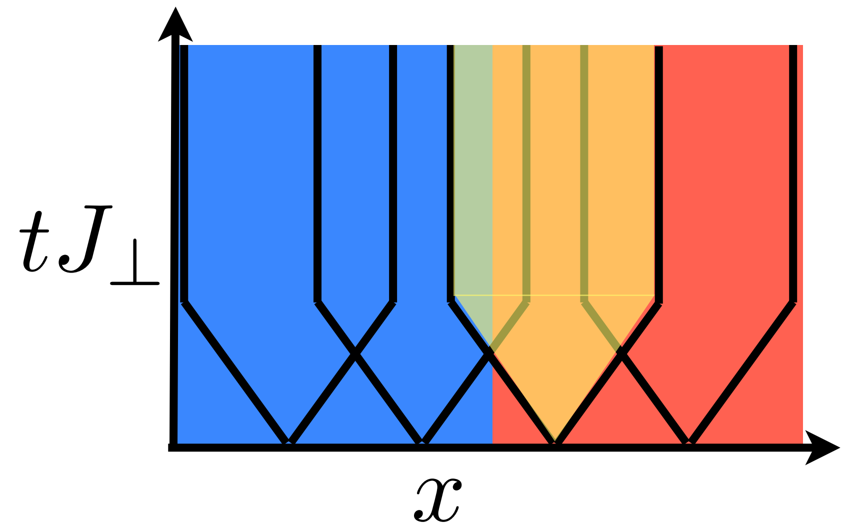


Lieb and Robinson (1972)
P. Calabrese and J. Cardy (2006)

Dynamics of entanglement entropy

- What about the evolution in the localized systems?





- First consider the **non-interacting system**

- Entanglement entropy for $J_z = 0$ can be simply calculated using a mapping to chain of free fermions

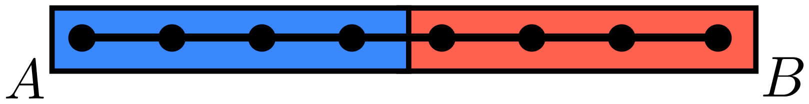
$$\rho_A \sim \exp\left[\sum_{i,j\in A} h_{ij} c_i^\dagger c_j\right]$$

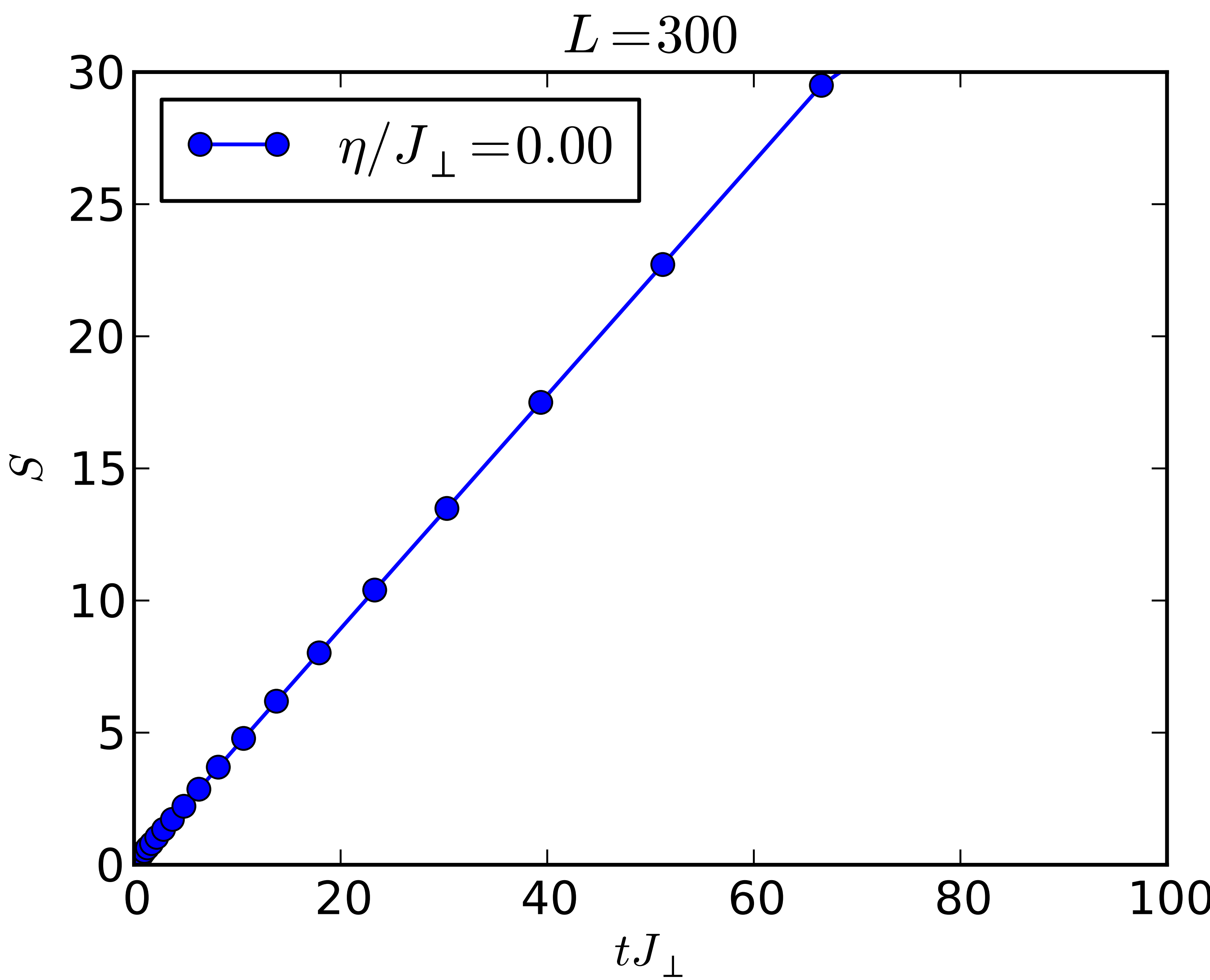
with $h = \ln\left[(1-C)/C\right]$

M.-C. Chung and I. Peschel (2001)

Dynamics of entanglement entropy

- Evolution of the non-interacting system





- Entanglement grows linearly as function of time in clean system
- Saturation of the entanglement for any amount of disorder

$$e^{-itH} | \ldots \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow \ldots \rangle$$

Dynamics of entanglement entropy

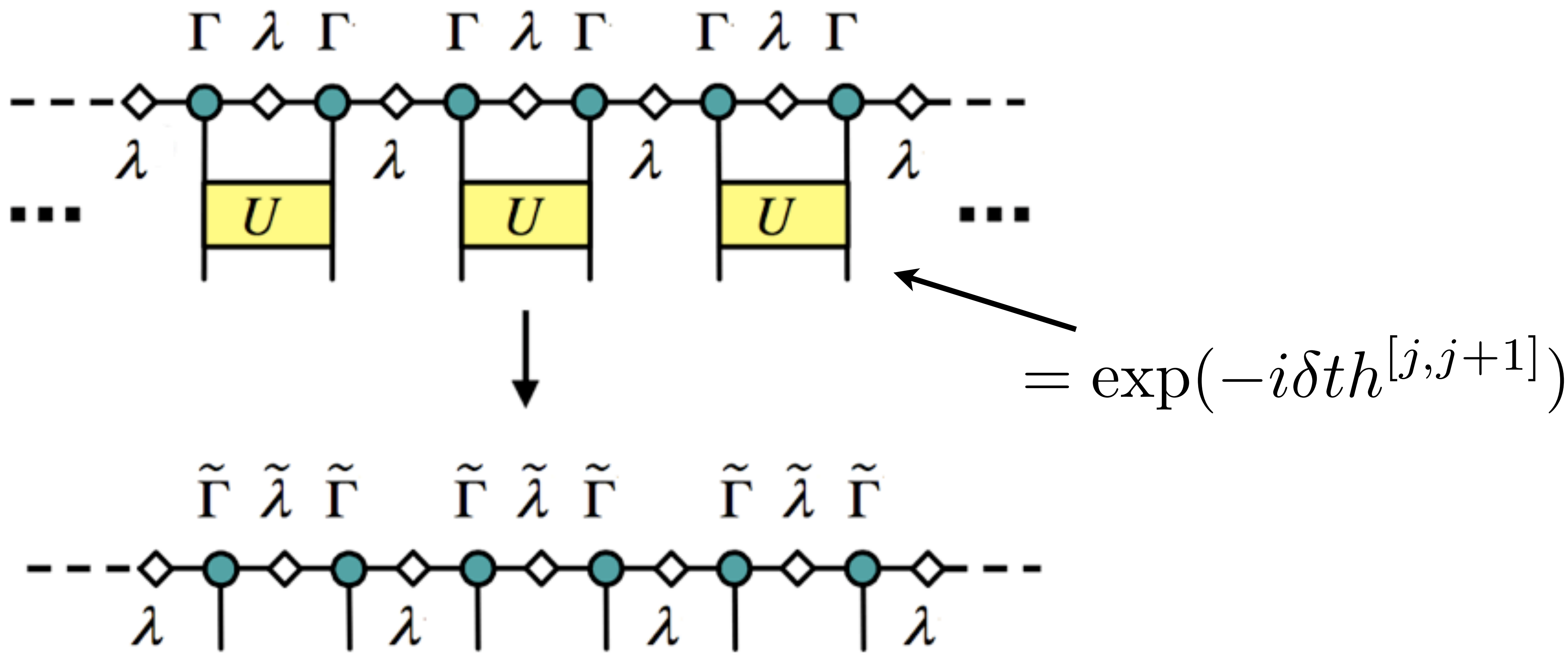
- The effect of interactions: **Properties of the many-body localized phase**
- Use **matrix-product state based methods:**

$$|\Psi\rangle = \sum_{j_1,\ldots,j_L} \underbrace{B^T A_{j_1} \ldots A_{j_L} B}_{\psi_{j_1,\ldots,j_L}} |j_1, \ldots, j_L\rangle$$

- **Very efficient if states are only slightly entangled**: suitable for one-dimensional systems (complexity growth exponentially with S)
- Efficient time evolution using the **Time Evolving Block Decimation method** G.Vidal (2007)

Dynamics of entanglement entropy

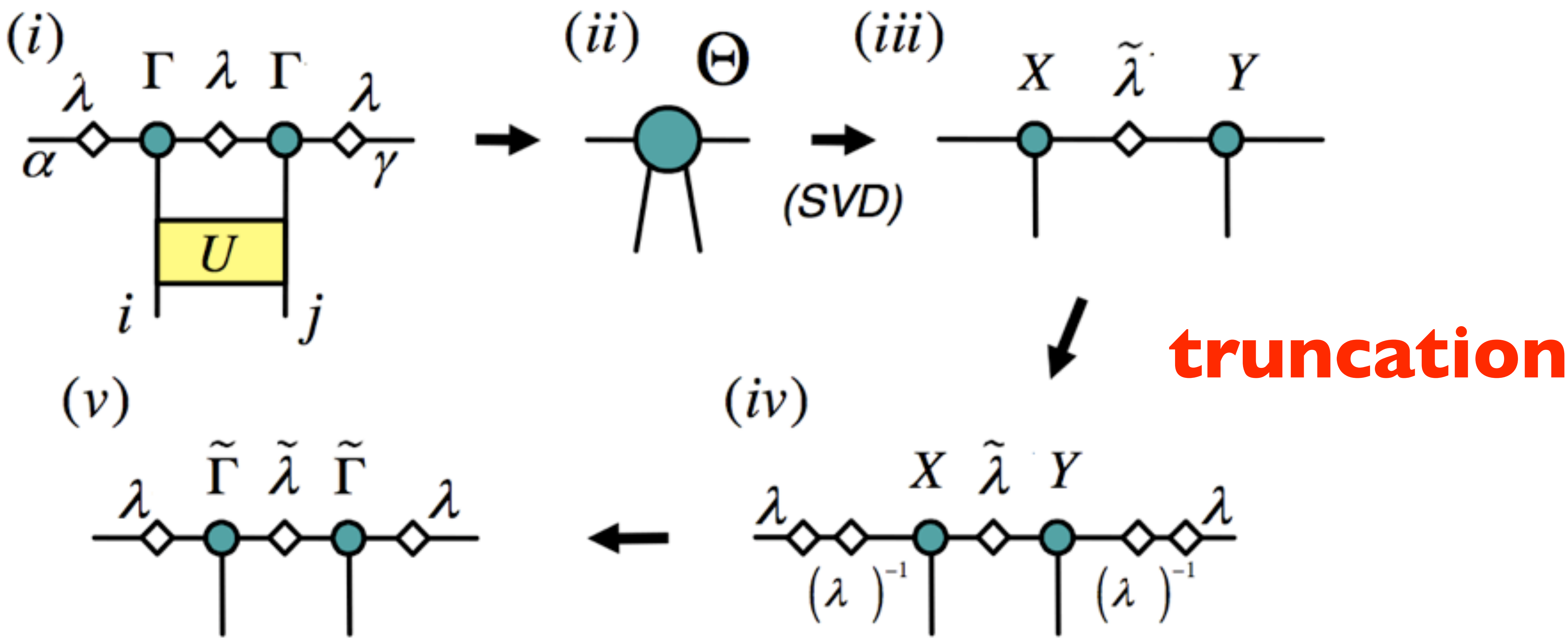
- Trotter-Suzuki decomposition of the time evolution operator ($B^j_{\alpha\beta} = \Gamma^j_{\alpha\beta}\lambda_\beta$)



G.Vidal (2007)

Dynamics of entanglement entropy

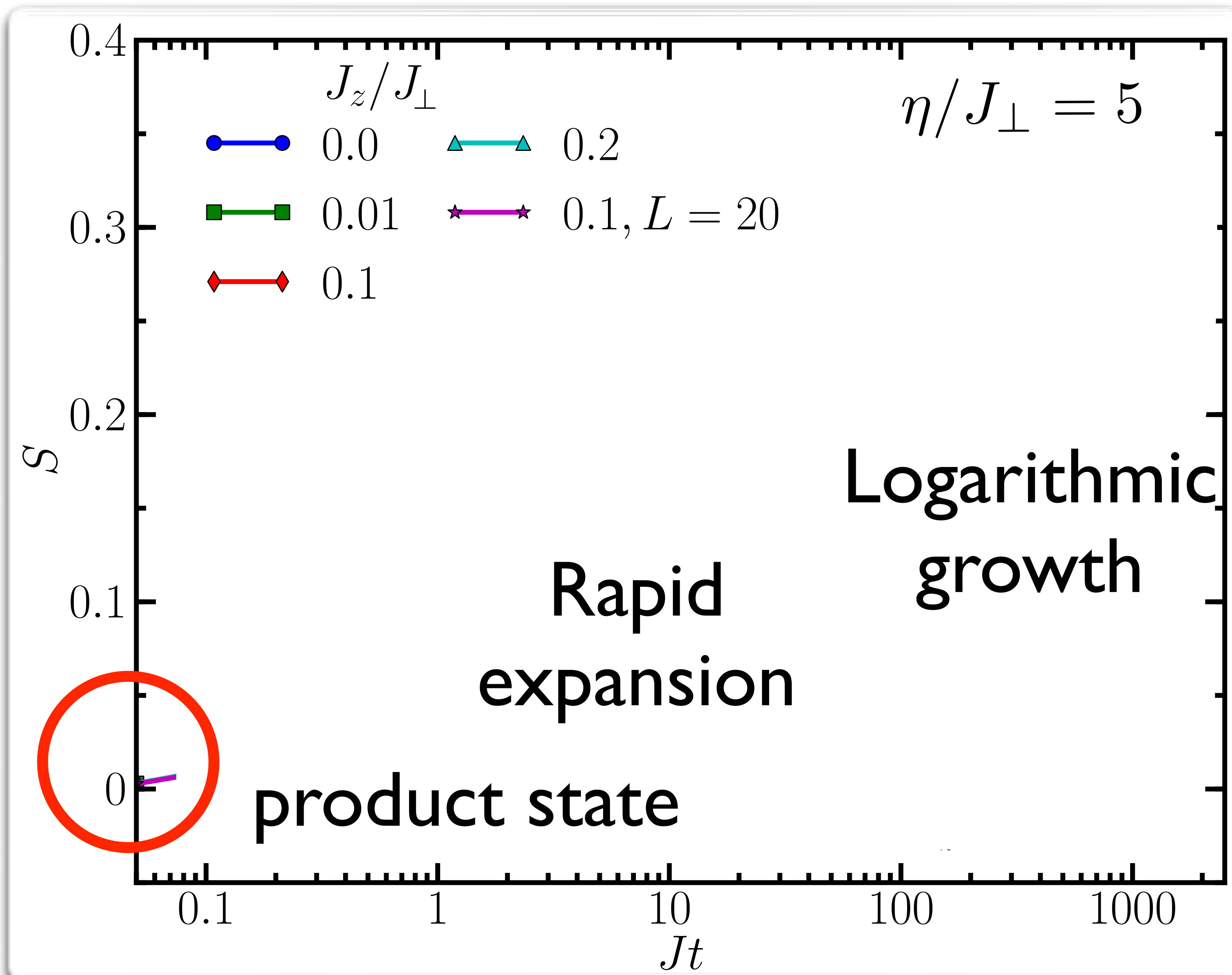
- Time Evolving Block Decimation algorithm [Vidal 03]

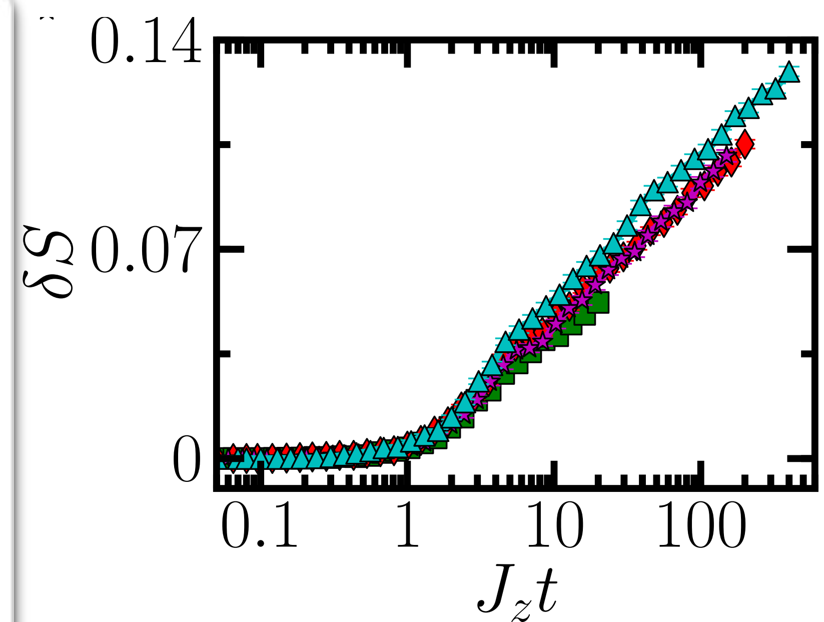


- Computationally very easy: **But entanglement growth quickly with time!**

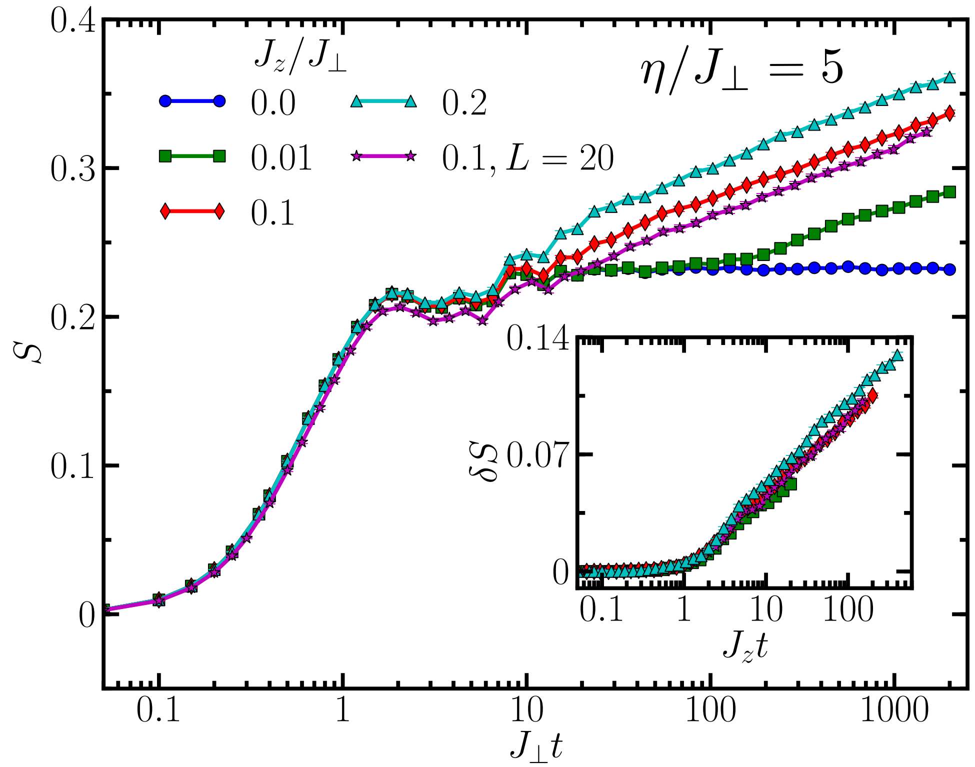
Dynamics of entanglement entropy

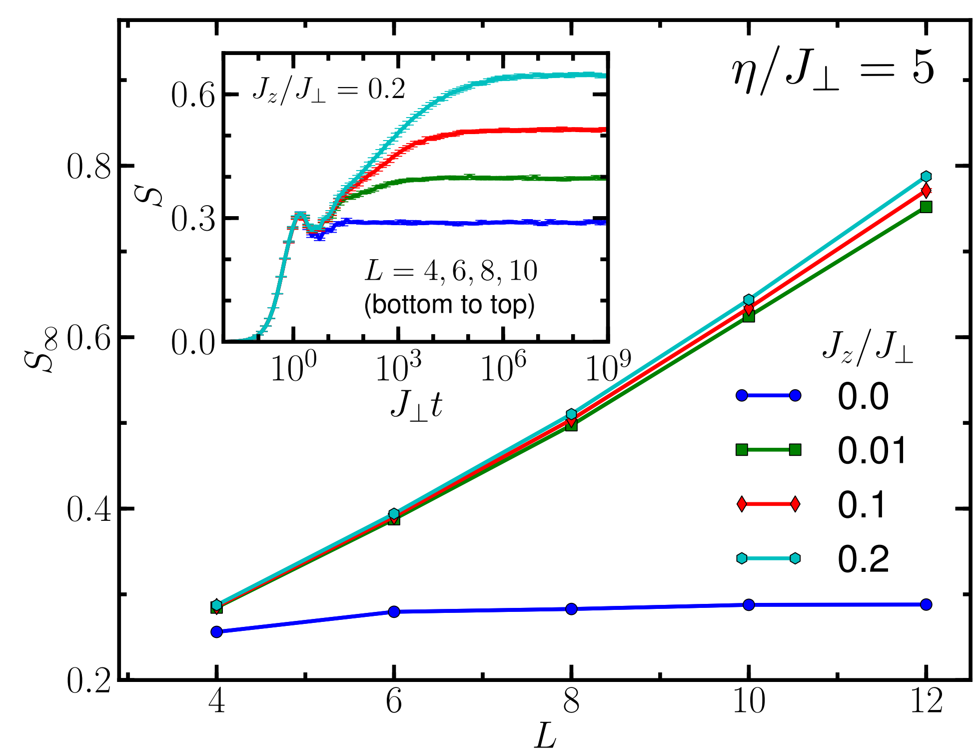
- Time evolution of S in the interacting system





Dynamics of entanglement entropy



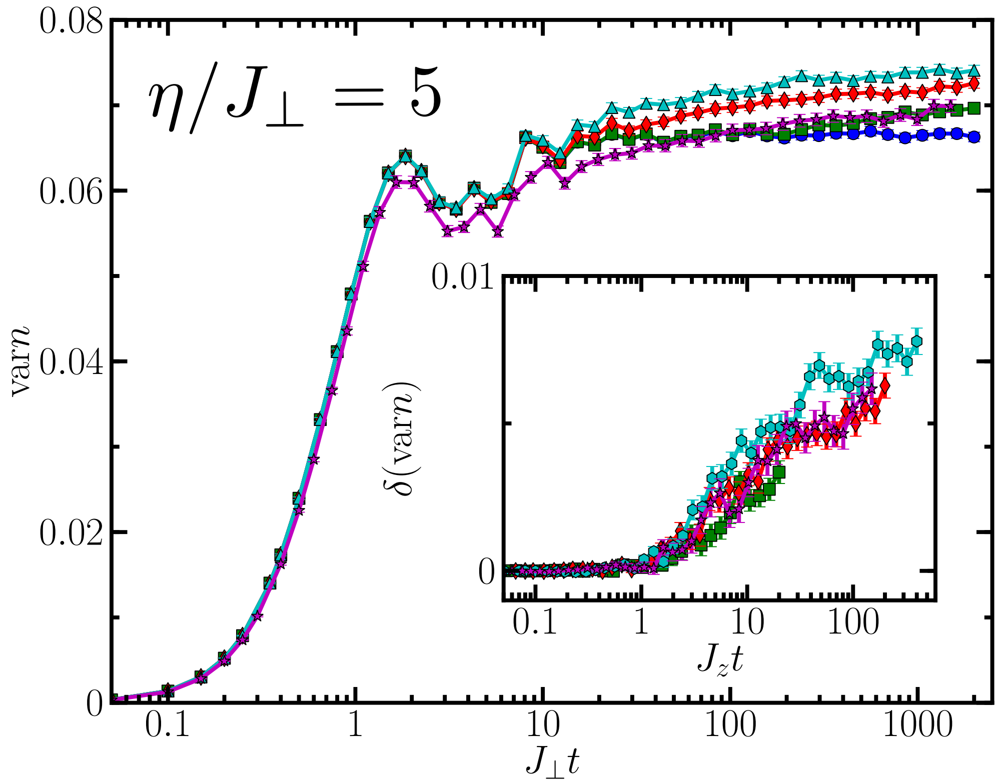


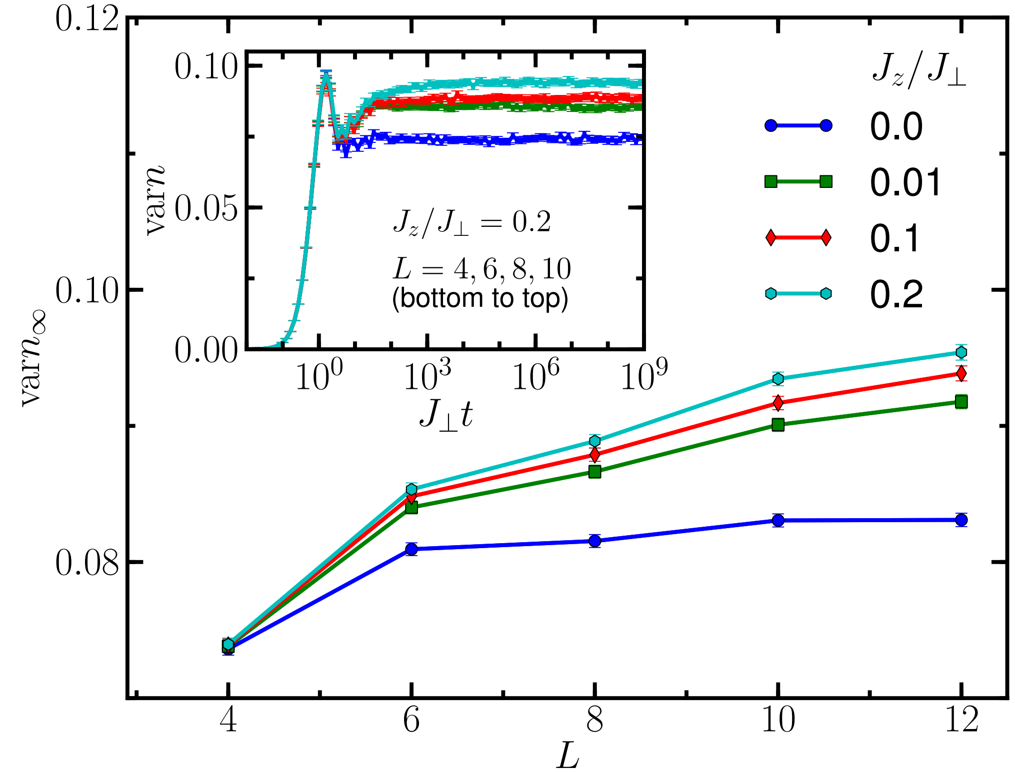
- Time evolution of S in the interacting system: Any non-zero interaction J_z gives rise to a **logarithmic growth of entanglement**

- Saturation follows **volume law** but only **small fraction of phase space involved** [$\approx 1/5$ of the total volume]: No thermalization

Consistent with: Chiara et al J. Stat, Mech 2006 and Žnidarič et al PRB 2008

Dynamics of entanglement entropy





- Increase of particle number fluctuations much slower than entanglement
- Particle transport not ergodic, consistent with a many-body localized phase

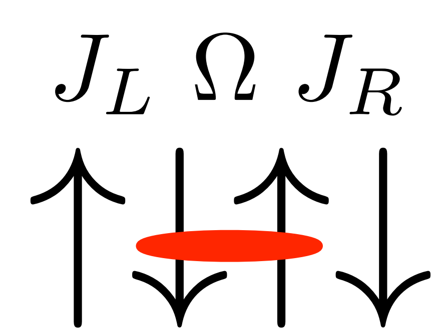
Dynamics of entanglement entropy

- Getting insight from a related model

$$H = \sum_i J_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right)$$

- Real space RG for the dynamics $e^{-itH} | \ldots \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow \ldots \rangle$

1. Short times described by rapid oscillations Ω performed by pairs of spins coupled by strongest bonds: Nothing else for $t \approx \Omega^{-1}$
2. Eliminating frequencies of order Ω (effective dynamics)
3. Iterate to obtain flow of the (distribution of) coupling constants



R. Vosk and E. Altmann (2012)
Das gupta & Ma (1979), D. S. Fisher (1992)

Dynamics of entanglement entropy

- Getting insight from a related model

$$H = \sum_i J_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right)$$

- Real space RG for the dynamics $e^{-itH} |\ldots \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow \ldots\rangle$
- Entanglement by counting decimated bonds that cut the interface

$$S(t) \sim \log(t/t_{\text{delay}}) \qquad t_{\text{delay}} < t \ll t_*$$

$$S(t) \sim \log(t/t_{\text{delay}})^{2/\phi} \qquad t_* \ll t$$

- Particle number Fluctuations

$$\text{var}_n(t) \sim \log\log(\Omega_0 t)$$

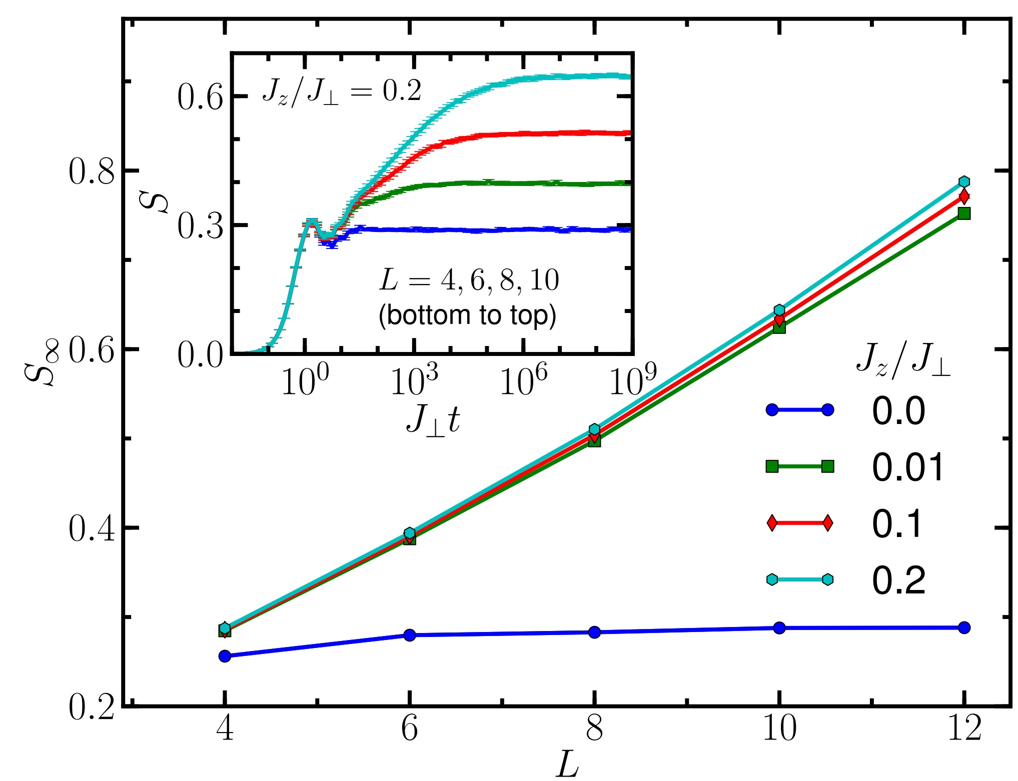
R. Vosk and E. Altmann (2012)

Dynamics of entanglement entropy

- Getting insight from a related model

$$H = \sum_i J_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right)$$

- Real space RG for the dynamics $e^{-itH}|\ldots\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\ldots\rangle$
- Non thermal steady state understood as Generalized Gibbs ensemble with the asymptotic conserved quantities: $(S_1^z S_2^z)_{\text{pair}}$
- How to generalize RG Scheme to include Zeemann term?



R. Vosk and E. Altmann (2012)

Conclusions

- Logarithmic growth of the entanglement entropy in the many-body localized phase
- Other observable (particle number fluctuations etc) appear to be localized
- Now: "Finite time" scaling could give us the localization transition
- Analytic understanding of the log-growth?
- Understand the entanglement of excited states better