QCD sum rules as an inverse problem

Hsiang-nan Li
Academia Sinica

Presented at NCTS

Dec. 09, 2020

QCD sum rules

- QCD sum rules broadly applied to extraction of nonperturbative observables from OPE at short distance (quark side)
- Based on analyticity, derived from dispersion relation

Shifman, Vainshtein, Zakharov
1979, 5000+ citations

- Quark-hadron duality for spectral density at low energy ~ 1 GeV (hadron side) usually assumed, causing large uncertainty
- Will propose alternative way to extraction

Quark side

- Correlator

$$\Pi_{\mu\nu}(q^2) = i\int d^4x e^{iq\cdot x}\langle 0|T[J_\mu(x)J_\nu(0)]|0\rangle$$
$$= (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2)$$

- Two ways to calculate correlator at large q^2
- Operator product expansion on quark side

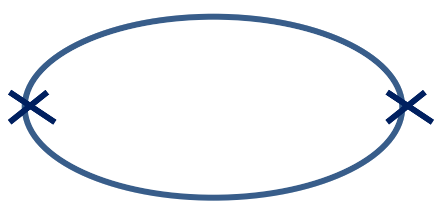
higher order

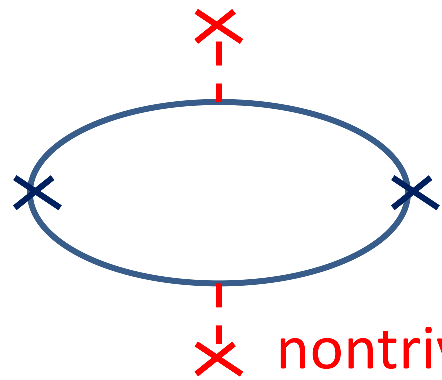
higher powers

$$\Pi^{\rm pert}(q^2) = \frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{\mu^2}{-q^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q\bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

perturbative QCD

4-quark condensate under factorization assumption





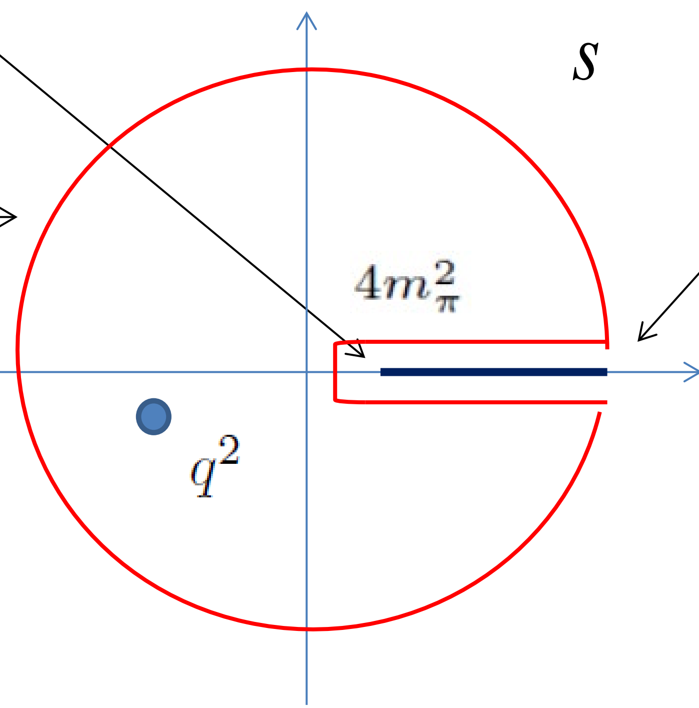
nontrivial vacuum

Dispersion relation

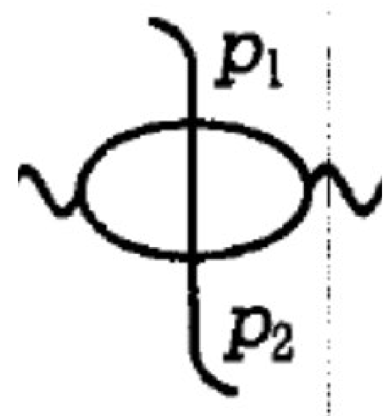
- Second way: dispersion relation

$$\Pi(q^2) = \frac{1}{\pi}\int_{s_i}^{\infty} ds \frac{\mathrm{Im}\Pi(s)}{s - q^2 - i\epsilon}$$

contribution from large radius assumed negligible



branch cut caused by physical intermediate states due to time-like $q^2 > 0$



$$\mathrm{Im}\Pi^{\mathrm{pert}}(q^2) = \frac{1}{4\pi}\left(1 + \frac{\alpha_s}{\pi}\right) \equiv a\pi.$$

Hadron side

- Insert intermediate states

phase space ↙

$$2\mathrm{Im}\Pi_{\mu\nu}(q^2) = \sum_n \langle 0|J_\mu|n\rangle\langle n|J_\nu|0\rangle d\Phi_n (2\pi)^4 \delta(q-p_n)$$

- Isolate ground state

$$\langle 0|J_\mu|V^\lambda\rangle = f_V m_V \epsilon_\mu^\lambda$$

$$\mathrm{Im}\Pi(q^2) = \pi f_V^2 \delta(q^2 - m_V^2) + \pi \rho^h(q^2)\theta(q^2 - s_h)$$

unknown vector decay constant to be calculated

- $s_h < 1\ \mathrm{GeV}^2$: threshold for continuum
- $\rho^h(q^2)$: spectral density fn from continuum
- Step function is assumption, not natural

Quark-hadron duality

- Need to estimate spectral density
- Local duality

  for large enough $s_0 > s_h$
  approximate $\rho^h(q^2)$ by $\mathrm{Im}\Pi^{\mathrm{pert}}(q^2)$

$$\mathrm{Im}\Pi(q^2) = \pi f_V^2 \delta(q^2 - m_V^2) + \mathrm{Im}\Pi^{\mathrm{pert}}(q^2)\theta(q^2 - s_0)$$

but $s_0 \sim 1\ \mathrm{GeV}^2$ large enough?

- Global duality

$$\int_{s_h}^{\Lambda} ds \frac{\rho^h(s)}{s - q^2} = \frac{1}{\pi} \int_{s_0}^{\Lambda} ds \frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s - q^2}$$

- Major assumption, theoretical uncertainty difficult to control

QCD sum rules (QSR)

- Hadron side = quark side leads to QCD sum rules

$$\frac{f_V^2}{(m_V^2 - q^2)} = \frac{1}{\pi}\int_{s_i}^{s_0} ds \frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s - q^2} + \frac{1}{12\pi}\frac{\langle \alpha_s G^2 \rangle}{(q^2)^2} + 2\frac{\langle m_q \bar{q} q \rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa \alpha_s \langle \bar{q} q \rangle^2}{(q^2)^3}$$

- In practical application, Borel transformation is applied to suppress uncertain continuum contribution on hadron side, and to improve perturbative expansion on quark side
- Less crucial for theoretical formulation

Stability?

- Obviously, f_V depends on q^2 from QSR, but it should not
- Tune the threshold s_0 to minimize q^2 dependence---stability
- Stability may not exist, and ambiguity appears
- Common criticism on QSR : you have to know answer first; you can get whatever you want; 10% error here, 10% there, soon you are talking about real numbers…

Leinweber 9510051

Our proposal

- Do not assume step function for continuum

$$\mathrm{Im}\Pi(q^2) = \pi f_V^2 \delta(q^2 - m_V^2) + \pi \rho^h(q^2)$$

smooth function in whole range

- Do not assume quark-hadron duality at low q^2
- Reasonable to approximate $\rho^h(q^2)$ by $\mathrm{Im}\Pi^{\mathrm{pert}}(q^2)$ for sufficiently large q^2

unknowns: f_V^2, m_V^2, $\rho^h(s)$; input: $\omega(q^2)$

$$\frac{f_V^2}{m_V^2 - q^2} + \int_0^\Lambda ds \frac{\rho^h(s)}{s - q^2} = \Pi(q^2) - \frac{1}{\pi}\int_\Lambda^\infty ds \frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s - q^2} \equiv \omega(q^2)$$

$$\omega(q^2) = a \ln\frac{q^2 - \Lambda}{q^2} + \frac{1}{12\pi}\frac{\langle \alpha_s G^2 \rangle}{(q^2)^2} + 2\frac{\langle m_q \bar{q} q \rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa \alpha_s \langle \bar{q} q \rangle^2}{(q^2)^3}$$

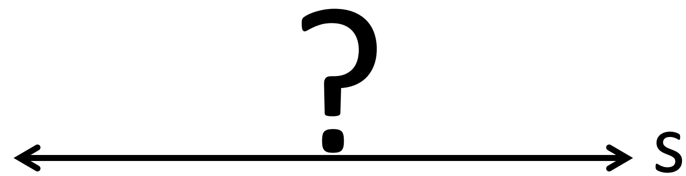
QSR turned into inverse problem

No need of stability

- Forward problem

known charge distribution at low s

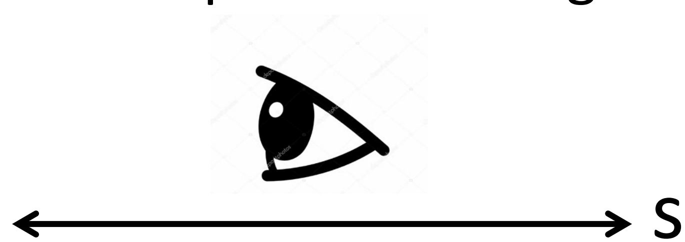
compute potential at high s



- Inverse problem

extract charge distribution at low s

known potential at high s



Spectral density

- Expand spectral density in series of Legendre polynomials $\rho^h(y) \equiv \rho^h(s = y\Lambda)$

$$\rho^h(y) = b_0 P_0(2y-1) + b_1 P_1(2y-1) + b_2 P_2(2y-1) + b_3 P_3(2y-1) + \cdots ,$$

$$P_0(y) = 1, \quad P_1(y) = y, \quad P_2(y) = \frac{1}{2}(3y^2 - 1), \quad P_3(y) = \frac{1}{2}(5y^3 - 3y)$$

- Other bases of orthogonal functions work too
- Boundary conditions $\rho^h(0) = 0 \text{ and } \rho^h(1) = a$
- Tune Λ, m_V, f_V, $b_0 \text{ and } b_1$ to minimize difference between two sides
- Many terms as you want. No duality at low s

Best fit

- Inputs κ = 2-4

$$\langle m_q \bar{q} q \rangle = 0.007 \times (-0.246)^3 \ \mathrm{GeV}^4, \ \ \langle \alpha_s G G \rangle = 0.07 \ \mathrm{GeV}^4$$

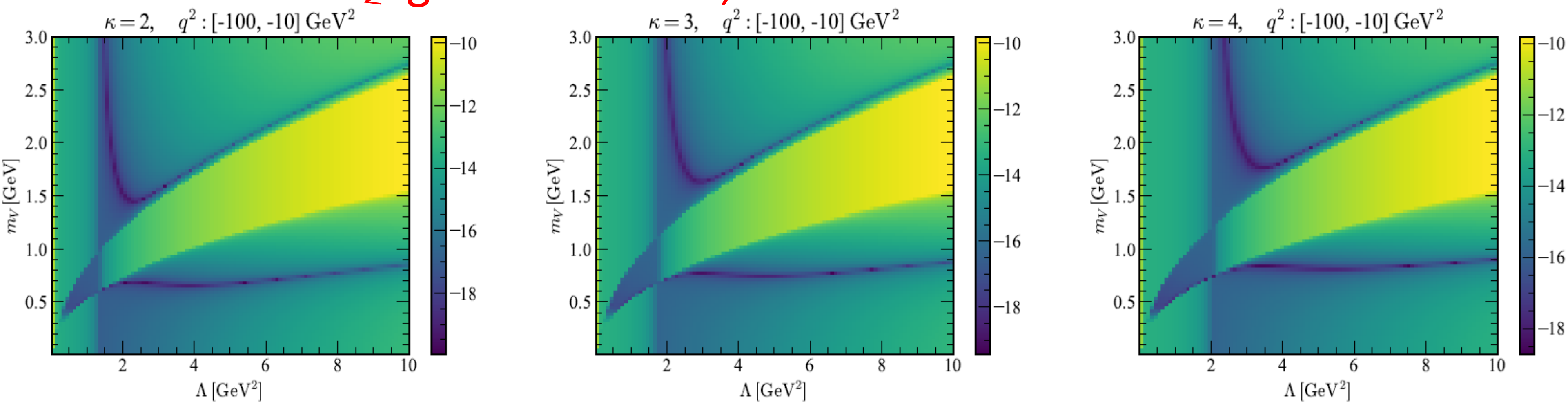
$$\alpha_s \langle \bar{q} q \rangle^2 = 1.49 \times 10^{-4} \ \mathrm{GeV}^6, \ \ \alpha_s = 0.5$$

- Minimize residual sum of square (RSS)

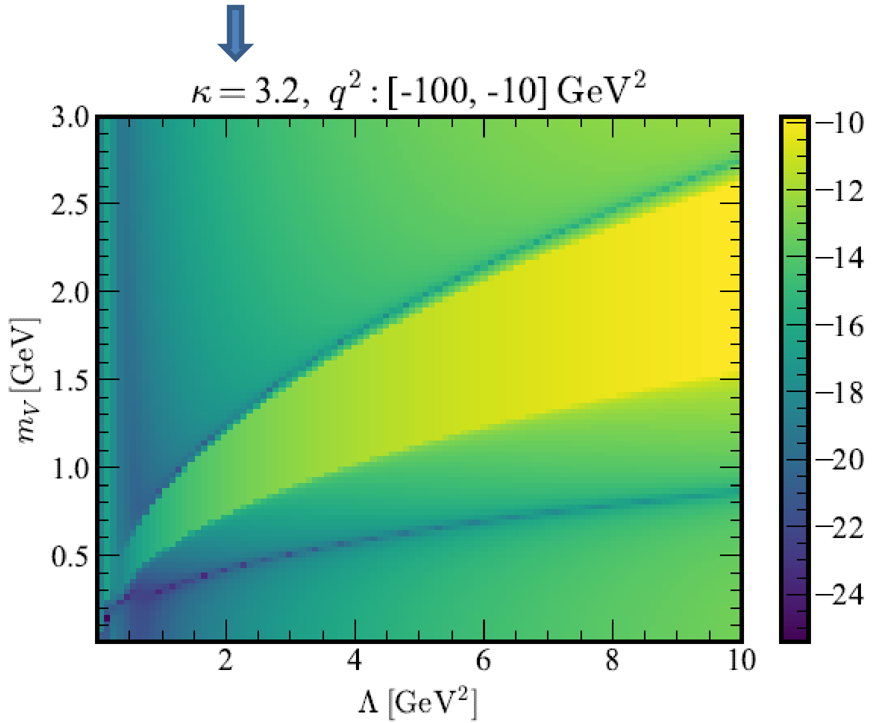
$$\sum_{i=1}^{20} \left| \frac{f_V^2}{m_V^2 - q_i^2} + \int_0^{\Lambda} ds \frac{\rho^h(s)}{s - q_i^2} - \omega(q_i^2) \right|^2$$

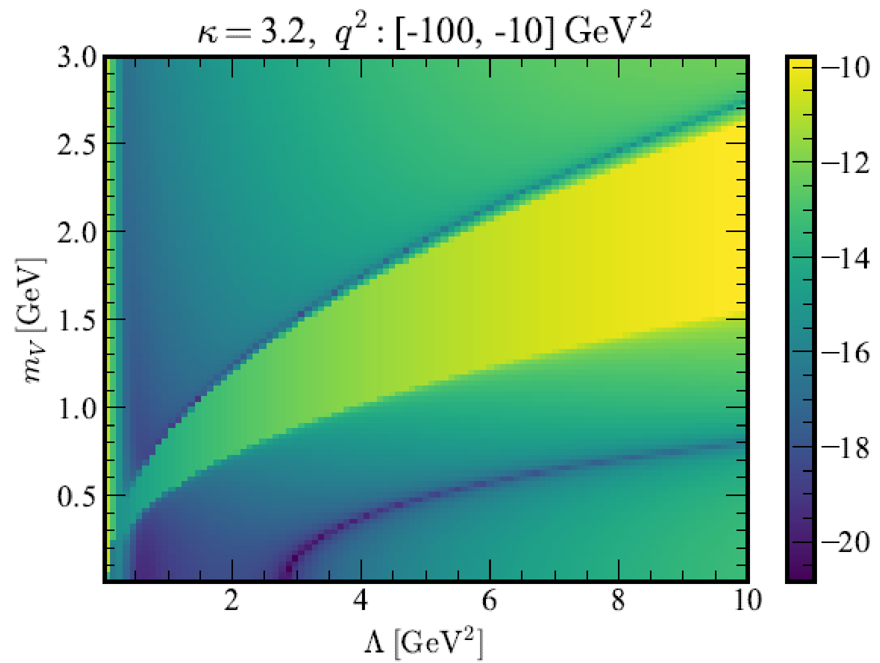
ground state mass 0.78 GeV appears automatically

guarantee OPE, no need of Borel



Which term in OPE establishes the ground state?



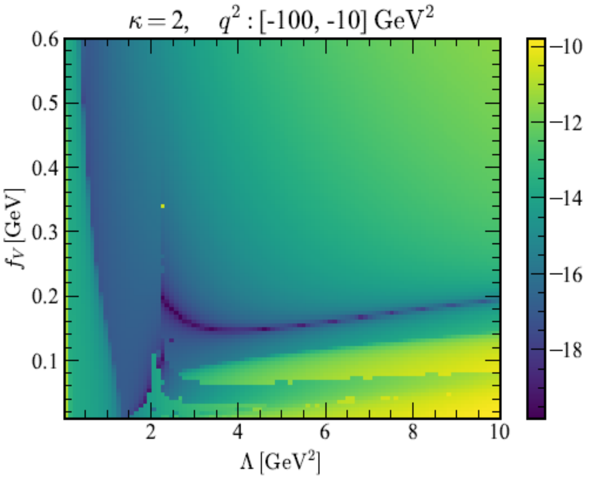


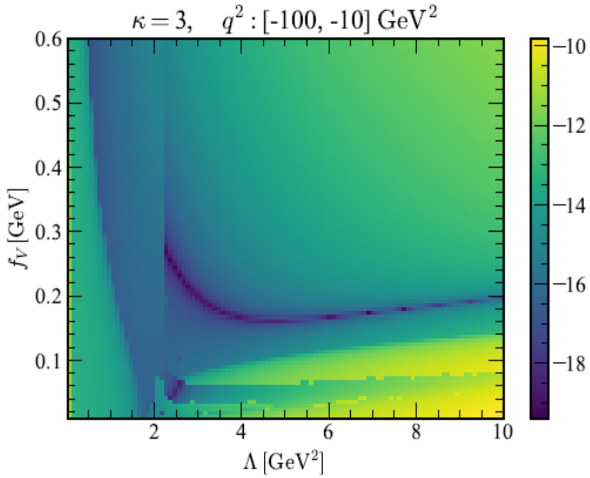
pert term only

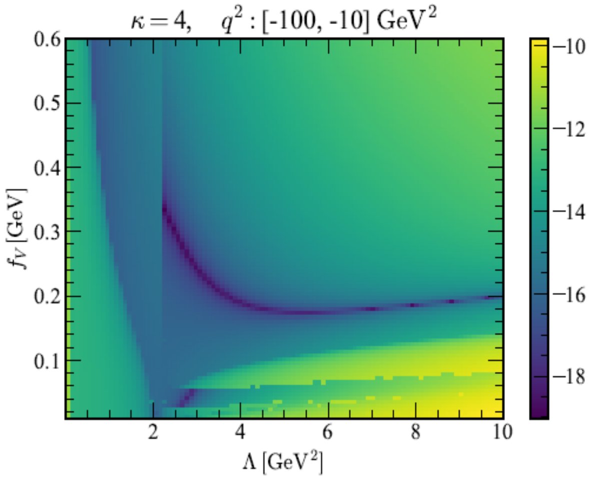
pert + $1/(q^2)^2$

four-quark condensate is more crucial for its emergence

Decay constant and width





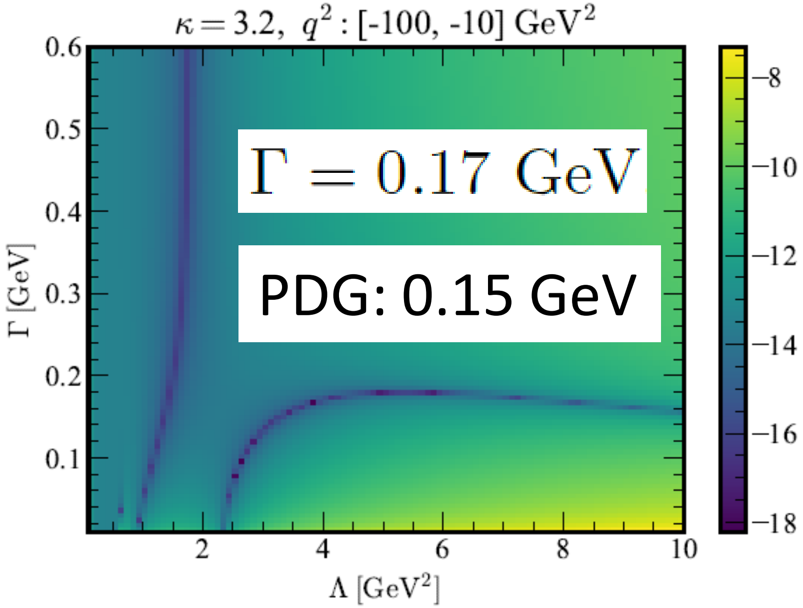


$f_{\rho(770)} = 0.22$ GeV   data : 209 MeV from tau decay

$$\frac{1}{24\pi}\frac{m_V^4 + m_V^2\Gamma^2}{(q^2 - m_V^2)^2 + m_V^2\Gamma^2}$$

$$+\pi\rho^h(q^2)$$

pole replaced by BW form

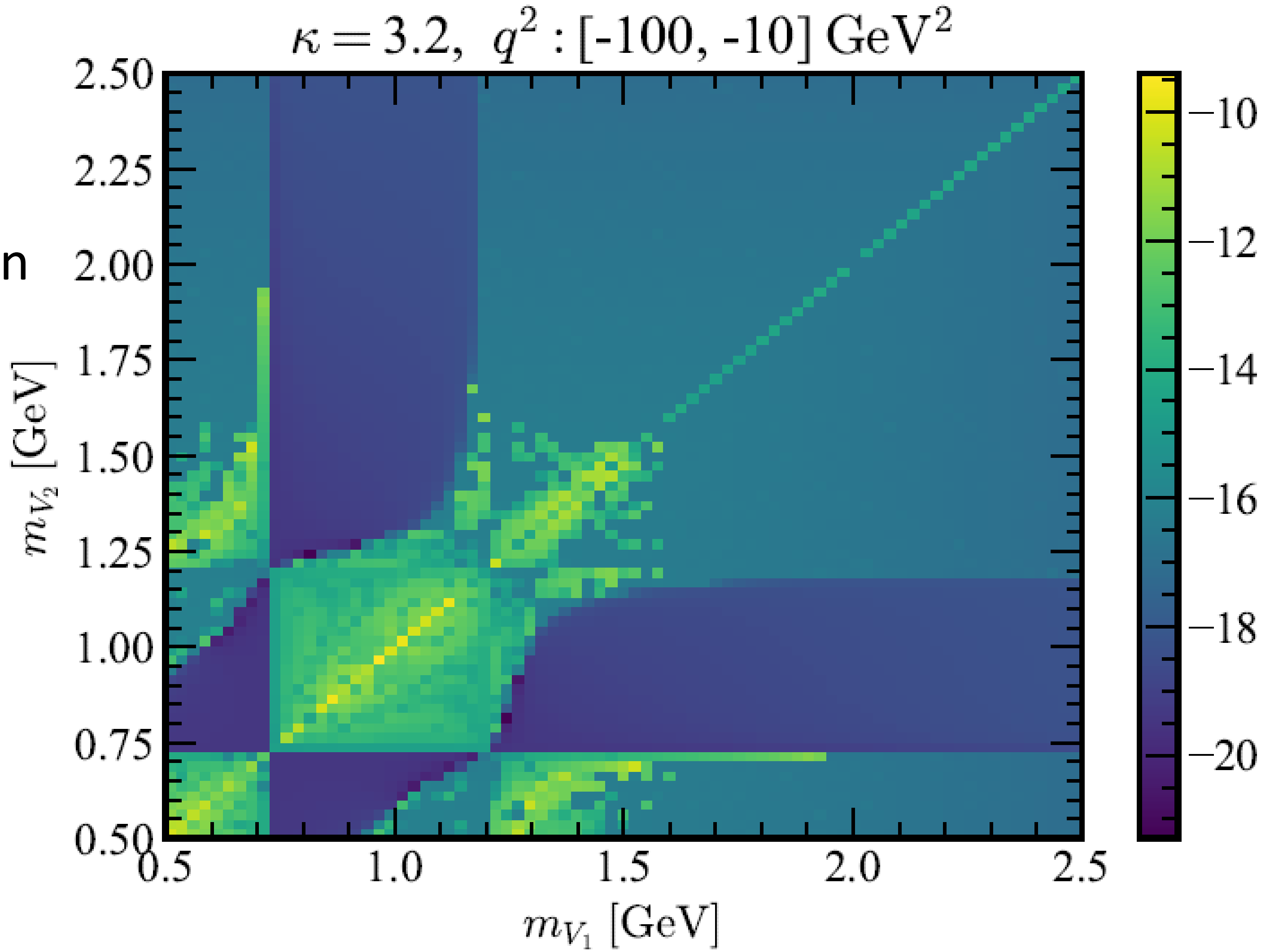


Excited state?

- Upper branch of RSS minima means excited state?

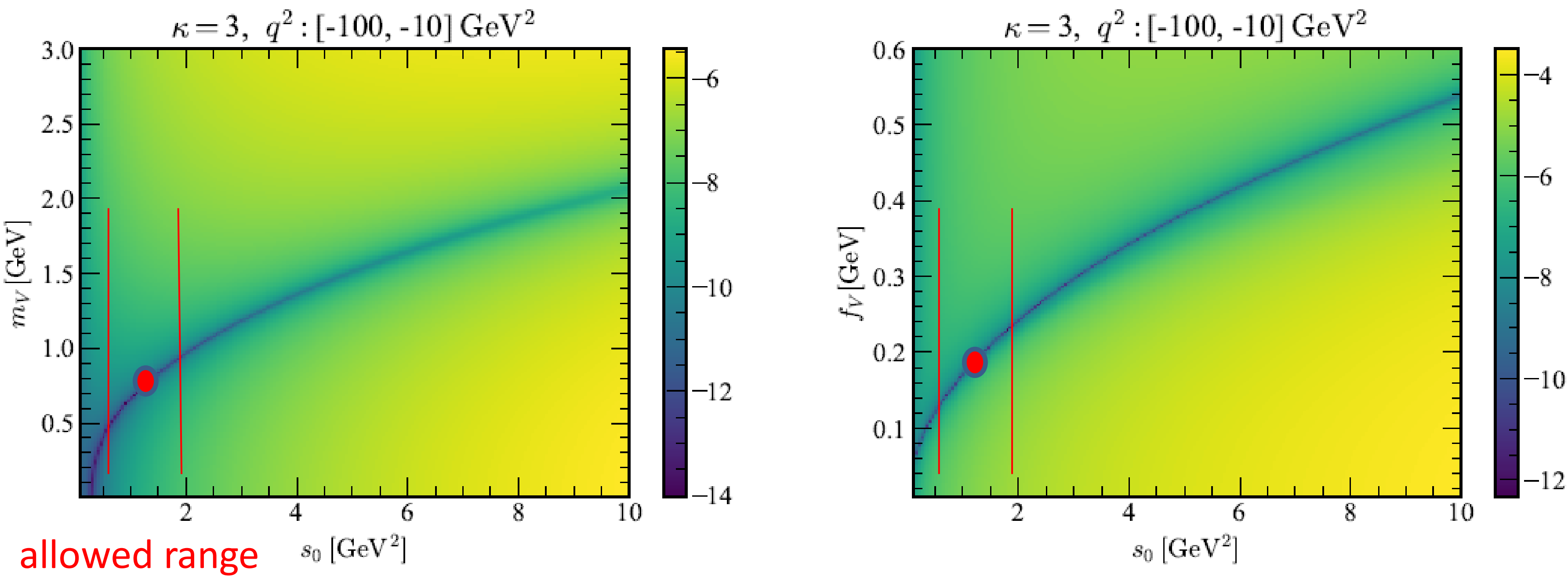
double pole parametrization

OPE prefers two-state solution $m_{V_1} \approx m_{V_2}$ excluded



Conventional sum rues

- Treating conventional sum rules with duality assumption as inverse problem



- No plateau. Need to know m_V first
- Difficult to study excited states

Summary

- Conventional QSR relies on quark-hadron duality at low energy, which may break, and induce uncontrollable uncertainty
- Turn QSR into inverse problem. Solve for nonpert observables and spectral density
- No Borel transform, no stability analysis
- Bound states appear automatically
- Successful for studying series of rho resonances. Many applications are expected

$$m_{\rho(770)}\,(m_{\rho(1450)}, m_{\rho(1700)}, m_{\rho(1900)}) \approx 0.78\ (1.46,\ 1.70,\ 1.90)\ \mathrm{GeV}$$

$$f_{\rho(770)}\,(f_{\rho(1450)}, f_{\rho(1700)}, f_{\rho(1900)}) \approx 0.22\ (0.19,\ 0.14,\ 0.14)$$