# Quantum Gravity from CFT

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1712.08570,1902.00463 with Abl, Lipstein

# Quantum Gravity and QFT

## Two big questions:

- What is quantum gravity? UV completion of GR?

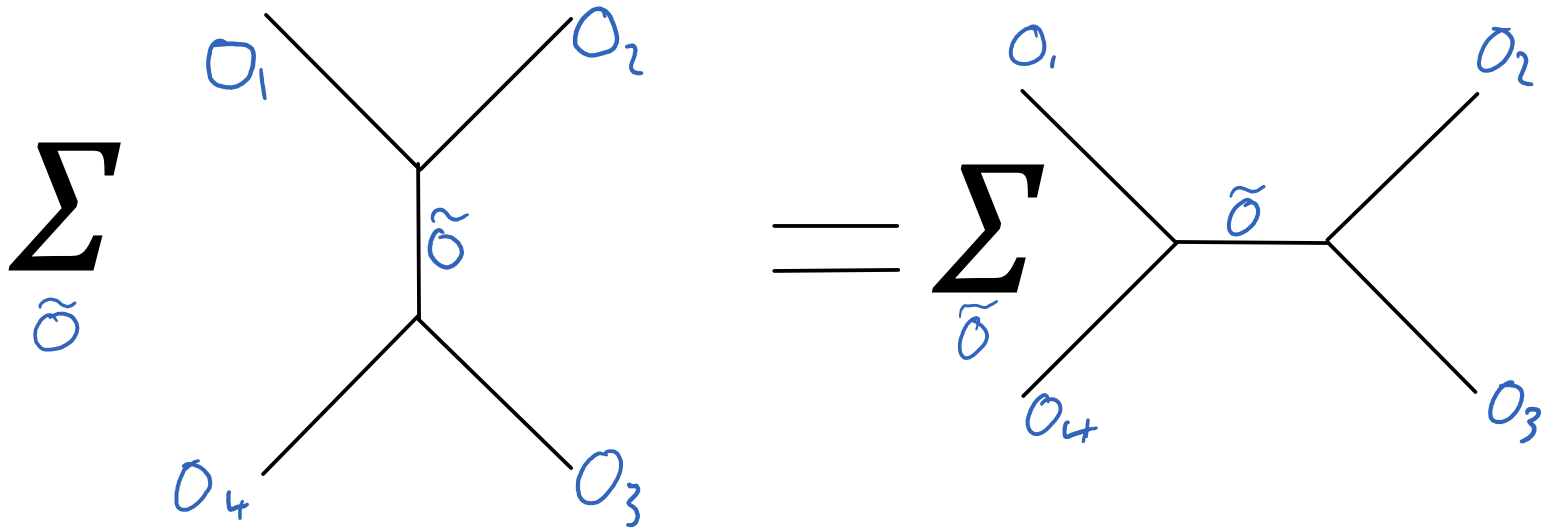
SWAMPLAND

- What is QFT / how to calculate in QFT? Eg for any value of the coupling.

# 2 major developments:

1. **AdS/CFT:** Quantum gravity on AdS = Conformal field theory (on boundary)[Maldacena;Witten;Gubser,Klebanov,Polyakov]

**2. Conformal bootstrap:** old idea with new lease of life, define CFTs as functions of the spectrum + 3-pnt functions: satisfying 4- point crossing symmetry [Polyakov; Ferrara, Gatto, Grillo; Ratazzi,Rychkov,Tonni,Vichi]




- Non-perturbative definition of QFT

> Main idea: Use CFT bootstrap to help understand quantum gravity via AdS/CFT [Heemsderk,Penedones,Polchinski,Sully (2009) ….]

Precision computations in specific theories (N=4 SYM and 6d (2,0))

# AdS/CFT

Gravity loop parameter: $G_N$ <-> $1/N^2$ SU(N) gauge group in CFT

(central charge)

Energy cut off $\Lambda^2$

(or string tension $1/\acute{\alpha}$) $1/\acute{\alpha}$ <-> $\lambda^{1/2}$ $\lambda$= coupling in CFT (tHooft coupling in N=4 SYM)

Gravity understood near:

small $G_N$, small $\acute{\alpha}$

$\Downarrow$

Large N, large $\lambda$

Strong <-> weak duality: therefore we expect:

Perturbative Computations in gravity -> strong coupling results in CFT

# AdS/CFT

Gravity loop parameter: $\boxed{G_N \leftrightarrow 1/N^2}$ SU(N) gauge group in CFT (central charge)

Energy cut off $\Lambda^2$ (or string tension $1/\acute{\alpha}$) $\boxed{\acute{\alpha} \leftrightarrow 1/\lambda^{1/2}}$ $\lambda$= coupling in CFT (tHooft coupling in N=4 SYM)

Gravity understood for:

$$\boxed{\begin{array}{c}\text{small } G_N\text{, small } \acute{\alpha} \\ \Downarrow \\ \text{Large } N\text{, large } \lambda\end{array}}$$

**Strong <-> weak duality:** but instead using bootstrap:

Perturbative Computations in gravity <- strong coupling results in CFT

# Overview: Bootstrap-> Quantum gravity

Focus on: AdS/CFT: N=4 SYM $\longleftrightarrow$ IIB SUGRA (on $AdS_5 x S^5$)

and: 6d (2,0) theory $\longleftrightarrow$ M theory on $AdS_7 x S^4$

bootstrap -> strong coupling results in CFT (N=4 SYM; 6d (2,0) theory)

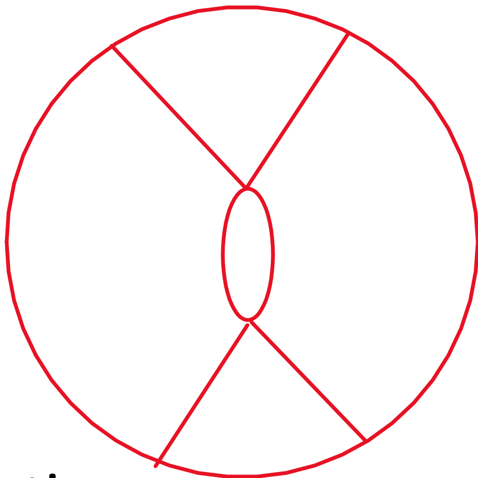
-> many new results in quantum gravity via AdS/CFT

Use strong coupling Gauge theory to understand weak coupling gravity!

Eg. 1 String/ M theory effective action: $R^4$ etc.

Eg. 2 One loop 4 pnt amplitude

(no corresponding direct computation in gravity in AdS)



Also:

- precision data for operators:
- anomalous dimensions + 3-pnt functions

  = binding energies of bound 2-graviton states + their 3-point couplings on gravity side

# AdS/CFT basics: N=4 SYM to IIB

## N=4 SYM

$\boxed{SU(2,2|4) = SO(2,4) \times SO(6) + SUSY}$

4d conformal, flavour

Adjoint rep of $SU(N_c)$: 6 scalars $\Phi_{I=1\cdots 6}$ ($SO(6)$), 4 fermions, Gauge field $A_\mu$

## IIB Sugra/string theory on $AdS_5 \times S^5$

$\boxed{SO(6) = \text{symmetry of } S^6,\ SO(2,4) = \text{symmetry of } AdS_5}$

Graviton etc. (dilaton/axion, p-forms, fermions)

In 10d

---

Gauge invariant ops: (SU(N) traces) $\longleftrightarrow$ States

Eg/

$\boxed{O_2 = Tr(\Phi_{\{I}\Phi_{J\}})}$ SUSY related $\longleftrightarrow$ Graviton compactified on $S^5$, + SUSY

Need precise definition: orthogonal to multi-trace

$\boxed{O_p = Tr(\Phi_{\{I_1}\cdots\Phi_{I_p\}})} + \cdots$ $\longleftrightarrow$ Graviton (p-2) KK mode on $S^5$, + SUSY

[symmetric traceless reps of SO(6)]

$\boxed{O_{p_1}\,\Box^{\tau}\,\partial^{l}\,O_{p_2}}$ $\longleftrightarrow$ Two-particle graviton bound states

# Other operators / states

1) String states. eg $\boxed{\mathrm{Tr}(\Phi_I \Phi_I)}$ = "Konishi operator" $\longleftrightarrow$ massive string state

Invisible in gravity limit $\alpha' \to 0$

2) n-particle bound states: n>2

$$\boxed{\mathcal{O}_{p_1} \Box^{\tau_1} \partial^{l_1} \mathcal{O}_{p_2} \Box^{\tau_2} \partial^{l_2} \mathcal{O}_{p_3}}$$

supressed by additional 1/N corrections than considered here.

- Consider 4-pnt function of single particle operators $\langle O_p\, O_q\, O_r\, O_s \rangle$
- = 4-graviton amplitude (4 Kaluza-Klein gravitons)

QG and string expansion directly from conformal bootstrap

# 4-point functions: the OPE

Focussing on $\langle O_2 O_2 O_2 O_2 \rangle$ at first

primary

OPE: $$\boxed{O_2(x_1)\, O_2(x_2) = \sum_{\hat{O}} C_{22\hat{O}}\, F(x_{12}, \partial_2)\, \hat{O}(x_2)}$$

known

all primary ops

OPE coefficient

of $\hat{O}$

Block decomposition. Insert twice into

$$\boxed{\langle O_2 O_2 O_2 O_2 \rangle = \sum_{\hat{O}} C_{22\hat{O}}^2\, u^{\frac{\Delta_{\hat{O}}}{2}} F_{\hat{O}}(x_1, x_2, x_3, x_4)}$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

primary ops

Conformal partial waves or blocks. Known. Depend on quantum numbers of O only.

# 4-point functions: crossing symmetry

$$\langle O_2(x_1) O_2(x_2) O_2(x_3) O_2(x_4) \rangle = \frac{1}{x_{12}^4 x_{34}^4} G(u,v)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

$$v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- Invariant under permutations of $x_1, x_2, x_3, x_4$

eg. $x_2 \leftrightarrow x_3 \implies \boxed{G(u,v) = u^2 \, G\left(\frac{1}{u}, \frac{v}{u}\right)}$

$x_1 \leftrightarrow x_3 \implies \boxed{G(u,v) = \frac{u^2}{v^2} \, G(v,u)}$

- But OPE says (roughly) $\boxed{G(u,v) = O(u^2)}$ (only twist 4 operators and higher)

crossing

- Very restrictive: eg try polynomial $G(u,v) = u^a v^b = u^{2-a-b} v^b = u^{b+2} v^{a-2}$
- $\implies$ a=4/3, b=-2/3, inconsistent with the OPE (a>=2)

OPE + crossing VERY powerful!

# String expansion

gravity loop corrections

$$\langle 2222\rangle = \langle 2222\rangle + \frac{1}{N^2}\langle 2222\rangle + \frac{1}{N^4}\langle 2222\rangle^{(2,0)} + \cdots$$

$$+ \frac{\alpha'^3}{N^2}\langle 2222\rangle^{(1,1)} + \cdots$$

Tree-level string corrections

for fixed $g_{YM}$ this is lower order than the loop correction 1/N^4

$$\langle 2222\rangle^{(1,1)} = \boxed{\bar{D}_{4444}(u,v)}$$

Dbar functions
=Scalar contact Witten diagram,
= tree-level basis of functions.

($R^4$ effective action)

[Alday, Bissi, Lukowski; Goncalves]

Operators of twist (Δ-l)=2t -> O($u^t$). Twist 2 states absent so t≥2

- Very simple way to derive string tree: $u^a v^b \bar{D}_{pqrs}$ + crossing symmetry u<>v, u<>1/u

OPE -> O($u^2$) $\quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$

Constrains a,b,p,q,r,s

p+q+r+s ≥16

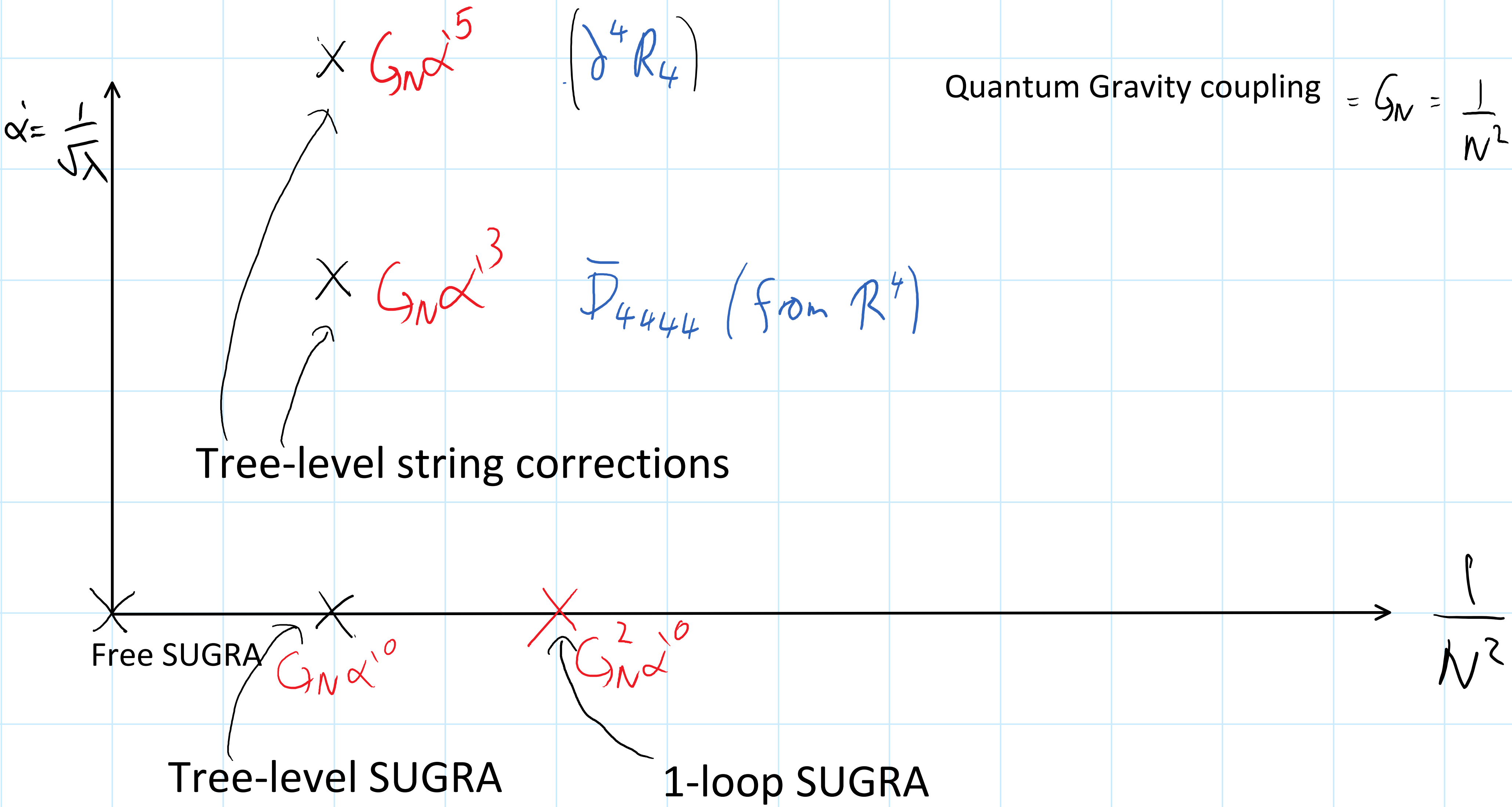
[Heemsderk,Penedones,Polchinski,Sully; Alday,Bissi,Lukowski]

$\bar{D}_{4444}$ minimal case.

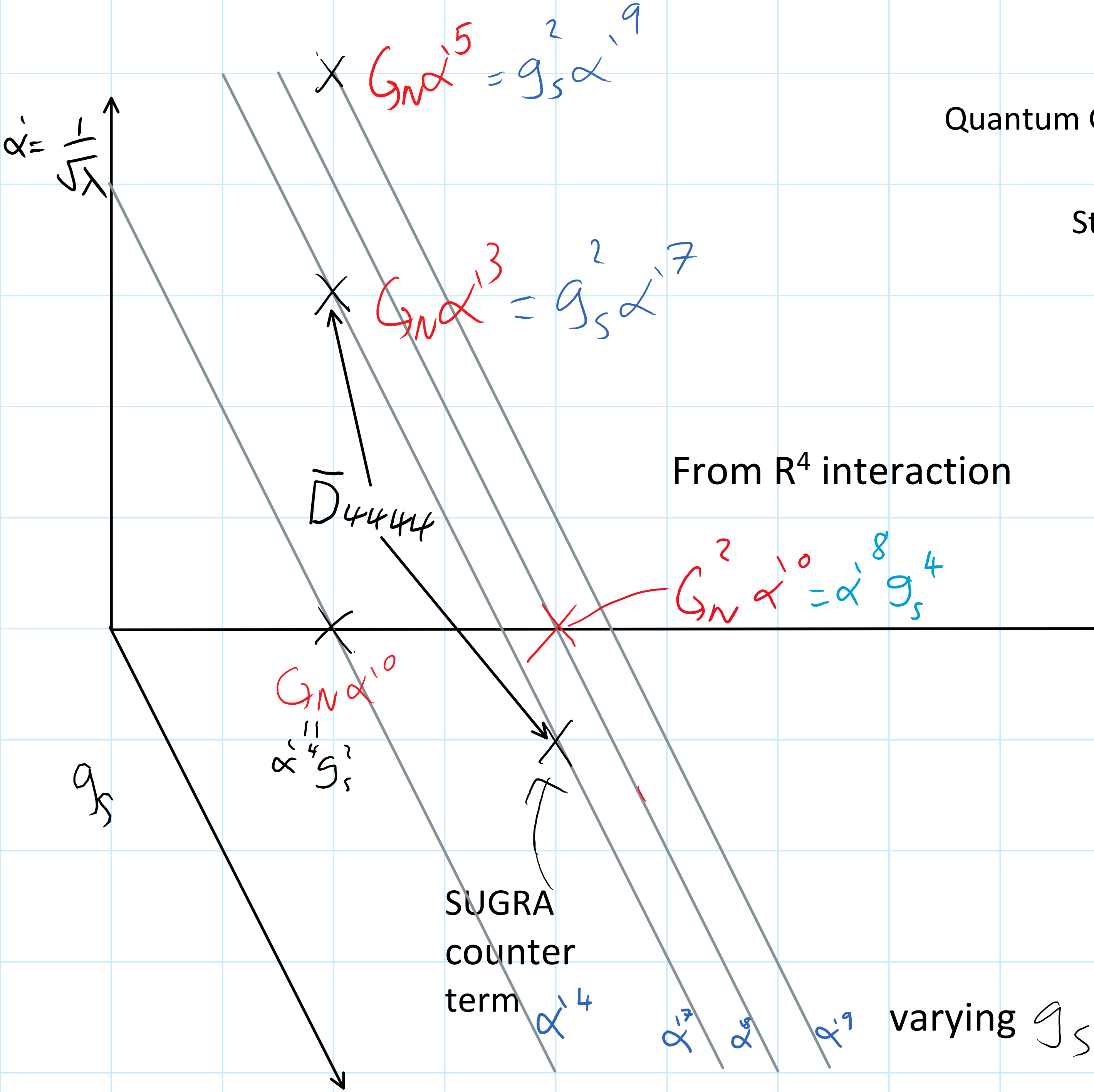
Only spin 0 in the OPE.

Large twist behaviour <> N power

- Another more direct way: solve spin truncated crossing equations. Or Mellin space.



Quantum Gravity coupling $= G_N = \frac{1}{N^2}$
$\alpha' = \frac{1}{\sqrt{\lambda}}$
$G_N \alpha'^5$
$(\partial^4 R_4)$
$G_N \alpha'^3$
$\bar{D}_{4444}$ (from $R^4$)
Tree-level string corrections
Free SUGRA
$G_N \alpha'^0$
$G_N^2 \alpha'^0$
$\frac{1}{N^2}$
Tree-level SUGRA
1-loop SUGRA




Quantum Gravity coupling $= G_N = \frac{1}{N^2}$

String coupling $= g_s = \frac{\lambda}{N} = \frac{\sqrt{G_N}}{\alpha'^2}$

$$G_N^{10d} = \frac{1}{N^2}$$

$$\left[\alpha' = \frac{1}{\sqrt{N}} \frac{1}{\sqrt{g_s}}\right]$$

# AdS/CFT basics: 6d (2,0) to M theory

## 6d (2,0) (M5 brane world volume)

$\boxed{OSp(8|4) = SO(2,6) \times SO(5) + SUSY}$

6d conformal    flavour

- Inherently non-perturbative
- No Lagrangian
- No microscopic description

5 scalars $\Phi_{I=1\cdots5}$ (SO(5))
fermions
2-form gauge field

## 11d Sugra/string theory on $AdS_7 \times S^4$

$\boxed{SO(5) = \text{symmetry of } S^4,\ SO(2,6) = \text{symmetry of } AdS_7}$

Graviton etc.
In 11d

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Gauge invariant ops: ⟷ States

Eg/

$\mathcal{O}_2 = \Phi_{\{I}\Phi_{J\}}$ SUSY related ⟷ Graviton compactified on $S^4$, + SUSY

$\mathcal{O}_p = \Phi_{I_1}\cdots\Phi_{I_p} + SUSY$ ⟷ Graviton (p-2) KK mode on $S^4$, + SUSY

[symmetric traceless reps of SO(5)]

$\boxed{\mathcal{O}_{p_1} \Box^{\tau} \partial^{l} \mathcal{O}_{p_2}}$ ⟷ Two-particle graviton bound states

Unknown microscopic theory    Microscopic details not important for bootstrap therefore proceed anyway!

# M theory: 6d (2,0) theory [Abl,Lipstein,PH]

Only 1/N, no $\alpha'$ coupling

- Structure similar though:

[superblocks: Arutyunov, Dolan, Gallot, Sokatchev; PH]

$$\boxed{\langle 2222\rangle = \langle 2222\rangle\big|_{N^0} + \frac{1}{N^3}\langle 2222\rangle^{(1,0)} + \frac{1}{N^6}\langle 2222\rangle^{(2,0)} + ?\,\langle 2222\rangle^{(1,1)}}$$

[supergravity 1 loop]

Find analogue of the $\frac{\alpha'}{N^2}$ correction.

Same technique. Log u part = O($u^2$) but analytic part = O(u). Crossing symmetry.
(Only short op in $\Phi_1 \otimes \Phi_1$ is the single particle op: no anomalous dimension, but OPE coefficient not protected unlike N=4 SYM)

Minimal case $\boxed{u v \bar{D}_{5755}}$

dimension n

(We've also done the honest computation: solving the spin truncated crossing equations)

$\boxed{\text{Only spin 0 in the OPE}}$

$$\gamma^{\text{spin}-0}_{n,n} = -\frac{(n-1)_8\, n_6}{2240(2n+3)(2n+5)(2n+7)}.$$

- Anomalous dimensions scale like $n^{11}$ for large n (twist)

[Heemsdork, Penedones, Pochinski, Sully]

Large twist behaviour $n^k$
$\Downarrow$
k derivatives in the effective action

- SUGRA anomalous dimensions ~ $n^5$ (m-n=spin fixed)

$$\gamma^{\text{sugra}}_{mn}/2 = -\frac{3}{N^3}\left(1+\frac{(n-2)(n+1)}{2(m+n+4)(m-n+3)}\right)\frac{(n-1)_6}{(m-n+1)(m-n+2)(m+n+5)(m+n+6)},$$

Suppressed by $\left(l_{pl}^{11d}\right)^6 = N^{-2}$ compared to tree SUGRA ($1/N^3$)

$\boxed{= O(N^{-5})}$

- 1st M theory correction
    - Again this should come from $R^4$ term in effective action.
    - Higher order tree-like corrections obtained similarly.
    - Fix coefficient? W-algebra? [Beem, Rastelli, van Rees; Chester, Perlmutter]

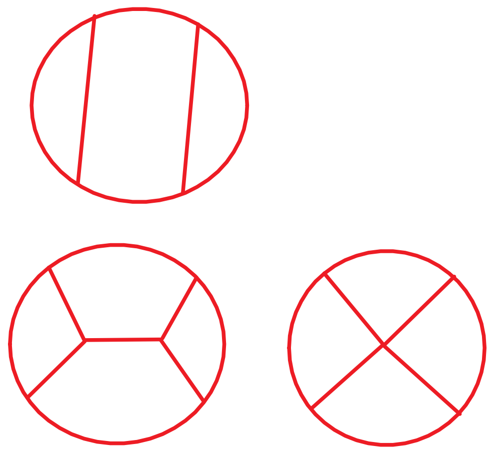
# One loop SUGRA: back to N=4 SYM

**Key point:** Spectrum simplifies at strong coupling

Absence of operators corresponding to string states and higher trace states: hugely simplifies the conformal bootstrap problem

- $O(N^0)$: Free disconnected SYM
- $O(N^{-2})$: Tree-level Witten diagrams:

  SUGRA: [d'Hoker Freedman Mathur Matusis Rastelli; Arutyunov,Frolov; Uruchurtu]
  Bootstrap: [Rastelli, Zhou]



Use OPE to bootstrap $O(N^{-4})$ from these: loop level gravity from CFT

- Both sides depend on the parameter $\boxed{a=\frac{1}{N^2}}$ $(\lambda\to\infty)$
- (Anomalous) dimensions of operators $\hat{O}$ also depend on *a*: $\boxed{\eta_{\hat{O}}(a)}$
- Expanding the block decomposition in *a*:

① $$\langle O_2 O_2 O_2 O_2\rangle\Big|_{a^0} = \sum_{\hat{O}} C_{22\hat{O}}^2\Big|_{a^0} F_{\hat{O}}(x_1,x_2,x_3,x_4)$$

② $$\langle O_2 O_2 O_2 O_2\rangle\Big|_{a^1} = \sum_{\hat{O}} C_{22\hat{O}}^2\Big|_{a^0} \eta_{\hat{O}}^{(1)} \log u\, F_{\hat{O}}(x_1,x_2,x_3,x_4) + \text{no log}$$

known SUGRA results

[d' Hoker, Freedman, Mathur,Matusis,Rastelli; Arutyunov, Frolov; Rastelli,Zhou]

Bootstrapped!

lower order data

③ $$\langle O_2 O_2 O_2 O_2\rangle\Big|_{a^2} = \sum_{\hat{O}} C_{22\hat{O}}^2\Big|_{a^0} \frac{1}{2}\left(\eta_{\hat{O}}^{(1)}\right)^2 \log^2 u\, F_{\hat{O}}(x_1,x_2,x_3,x_4) + \log u + \text{no log}.$$

Idea: Extract $C_{22\hat{O}}$, $\eta_{\hat{O}}$ from ①,② plug into ③ → full double log u piece of 1 -loop amplitude

# Technical details

**Problem 1.** More than one scalar. Solution: auxiliary variable $Y^I$ $\varphi^I(x) \Downarrow \varphi(x,Y) = \varphi^I(x) Y^I$

(analytic superspace (GIKOS; Howe Hartwell; Howe PH. Details suppressed.)

**Problem 2.** Blocks $F_{\hat{O}}$ only depend on conformal rep. of $\hat{O}$:
many ops have same free theory rep. DEGENERACY

**Solution** (part 1). Expand in superblocks. [Dolan, Osborn; Bissi, Lukowski; Doobary, PH]
Separate contributions from each supermultiplet (super primary)

**But:** Still degeneracy supermultiplets with the same quantum numbers:
Eg 2 twist 6 singlet multiplets for each spin: $O_3 \partial^l O_3$, $O_2 \Box \partial^l O_2$

**Solution** (part 2). Consider more general correlators and perform unmixing:

$$\boxed{\langle O_p O_p O_q O_q \rangle}$$

# twist 6 unmixing

$$\boxed{\underbrace{O_3 \partial_\mu^l O_3}_{K_1}, \quad \underbrace{O_2 \Box \partial_\mu^l O_2}_{K_2}}$$

2 operators for each spin l

(not pure, mix together)

- Superblock decomposition of $\langle 2222 \rangle$ gives:

$$C_{22K_1}^2 + C_{22K_2}^2 = \frac{2}{5}(l+1)(l+8) \times \frac{(l+4)!^2}{(2l+8)!}$$

$$C_{22K_1}^2 \eta_{K_1} + C_{22K_2}^2 \eta_{K_2} = -96 \times \frac{(l+4)!^2}{(2l+8)!}$$

known

Problem:

Not enough info:
2 equations, 4 unknowns...

$$\begin{pmatrix} C_{22K_1}, & C_{22K_2} \\ \eta_{K_1}, & \eta_{K_2} \end{pmatrix}$$

[Rastelli,Zhou; see also Dolan, Nirschl, Osborn; Uruchurtu

We convert from Mellin space + fix normalization (via light-like limit)

Solution:

Consider <2233> and <3333> correlators too:
6 equations, 6 unknowns!

$$\begin{pmatrix} C_{33K_1}, & C_{33K_2} \end{pmatrix}$$

# Twist 6 data.

$$\eta_1 = -\frac{240}{(l+1)(l+2)}, \qquad \eta_2 = -\frac{240}{(l+7)(l+8)},$$

$$c_{21} = -\sqrt{\frac{2(l+1)(l+2)(l+8)}{5(2l+9)}}, \qquad c_{22} = -\sqrt{\frac{2(l+1)(l+7)(l+8)}{5(2l+9)}},$$

$$c_{31} = \sqrt{\frac{9(l+1)(l+2)(l+7)^2(l+8)}{40(2l+9)}}, \qquad c_{32} = -\sqrt{\frac{9(l+1)(l+2)^2(l+7)(l+8)}{40(2l+9)}}.$$

$\eta$ = rational function of linear factors in spin l. No square root!

$C$'s $= \sqrt{\text{rational function of linear factors in l}}$

- Persists at higher twists T: T/2-1 operators
  - # Unknowns T/2-1 + (T/2-1)^2 $\quad \eta_K \quad C_{ppK}$
  
  = # Equations (T/2-1)(T/2)/2 * 2 $\quad \langle ppqq\rangle|_{N^0}, \ \langle ppqq\rangle_{N^{-2}}$

  (2 symmetric matrices)

## Anomalous dimensions (binding energy) of all 2-particle operators

$$\mathcal{O}_{pq} = \mathcal{O}_p \partial^l \Box^{\frac{1}{2}(\tau-p-q)} \mathcal{O}_q \,, \qquad (p \le q)$$

Twist 2t+2a-b Spin l SU(4) rep [a,b,a]

$$\boxed{\Delta_{pq} = \tau + l - \frac{2}{N^2} \frac{2M_t^{(4)} M_{t+l+1}^{(4)}}{\left(l + 2p - 2 - a - \frac{1+(-)^{a+l}}{2}\right)_6}}$$

$$M_t^{(4)} \equiv (t-1)(t+a)(t+a+b+1)(t+2a+b+2)$$

# Residual Degeneracy: Independent of q

- Reason for simplicity, rationality, degeneracy?
  10 d conformal symmetry [Caron-Huot, Trinh]
- All single particle ops part of a single 10d massless field.
- All double trace ops, 10d higher spin currents, one for each even (10d) spin L
- Anomalous dimension: $\Delta^{(8)}/L_6$
- Above structure arises from decomposing 10d conf group SO(2,10) -> SO(2,4)xSO(6)

# Back to <2222> one loop $O(1/N^4)$

Recall:

$$\langle 2222\rangle\Big|_{1/N^4}\Big|_{\log^2 u} = \sum_0 C_{220}^2\, \frac{\eta_0^2}{2} \times \text{superblock}_0$$

$$u = x\bar{x} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

$$v = (1-x)(1-\bar{x}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

We now have this data analytically.

All tree amplitudes have the structure:

(weight 2 and below)

$$\bar{D}_{pqrs} = \bigotimes_{\,s\;\;r}^{\,p\;\;q} = \frac{1}{(x-\bar{x})^{\#}} \sum \text{Polynomial}(x,\bar{x})\ \text{Polylog}(x,\bar{x})$$

Perform the sum to high order.

Therefore match to similar ansatz: works!!

$$\boxed{\frac{\sum\ \text{Polynomial}(x,\bar{x})\ \ \text{polylogs (weight 2 and below)}}{(x-\bar{x})^n}}$$

n some high integer.

By inspection n=15

# Uplift to full function.

- Look for a crossing symmetric function whose $\log^2 u$ coefficient is the one given.

Ansatz:

$$\frac{\sum \text{Polynomial}(x,\bar{x}) \ \text{polylogs (weight 4 and below)}}{(x-\bar{x})^n}$$

Impose:

1. crossing symmetry
2. Finite as $x \to \bar{x}$
3. $\text{Log}^2 u$ part matches

$\Big\}$ solution with 1 free parameter

# Ambiguity

Ambiguity $= \alpha \frac{1}{uv}[(1+u\partial_u+v\partial_v)u\partial_u v\partial_v]^2\Phi^{(1)}(u,v)$ $\left(= \bar{D}_{4444}\right)$ $\frac{1}{(x-\bar{x})^{13}}$

- $R^4$ term again! One loop counter term $\frac{1}{N^4}\frac{1}{\alpha'}$ and ambiguity $\frac{1}{N^4}\alpha'^0$
- Appears in 3 places! (tree level string correction too $\frac{1}{N^2}\alpha'^3$ )
- Coefficient (all three) fixed by localization [Chester]

Full 1-loop quantum gravity 4-pnt amplitude

# Higher charge correlators at 1 loop $O(1/N^4)$

- Recently developed an algorithm to obtain any SUGRA 4 point function at one loop <p q r s >
- MANY new subtleties:
- First subtlety: Degeneracy means lack of analytic control of unmixing problem needed for double logarithm. Not really needed. In fact:
  - Can use 10d symmetry [Caron-Huot,Trinh].

    Double disc of any correlator directly given by (with k=2):

$$\mathcal{H}^{(k)}_{pqrs}(x,\bar{x},y,\bar{y})\Big|_{\log^k u} = \left[\Delta^{(8)}\right]^{k-1}\cdot \mathcal{D}_{pqrs}\cdot \mathcal{D}_{(3)}\cdot h^{(k)}(\boldsymbol{x}).$$

# Higher charge correlators at 1 loop $O(1/N^4)$

- Second subtlety: Completion of double disc to full function has many remaining free coefficients
  - Solution: There is further information from lower loops, single log and non log pieces "below threshold" (ie for small powers of u). (3-point functions of low twist operators are $O(1/N^2)$)
- Third subtlety: minimal ansatz based on the form of the double log, doesn't (quite) work
- Contribution from free theory $O(1/N^4)$ spoils resulting predictions
- Solution: Cancel low twist free theory contributions with a "generalised tree" contribution at $O(1/N^4)$
- Final subtlety: non-renormalised semi-short operators appear at $O(1/N^4)$. There OPE coefficients can be determined non-trvially, but purely within free theory.
- Amazingly: the minimal one loop ansatz with generalised tree correctly reproduces these.

# One loop SUGRA results so far

<2222>, <22pp>, <3333>, <3324>, <2424>

Also <3344>,<3335>,<2444>,<2435>,<2444>,<4444>,<4435>,<3535> At 1-loop! Now done!

- In all cases the only remaining ambiguities have CPW expansion truncated to spin 0 (or 1) only.
- Mellin amplitudes of ambiguities are linear in Mellin variables

# Some explicit results + Δ$^{(8)}$ structure

"can we pull out of our minimal one-loop functions the operators $\widehat{\mathcal{D}}_{pqrs}$ and $\Delta^{(8)}$?"

For <22pp> we can.... almost:

$$\mathcal{H}^{(2)}_{2222} = \tfrac{1}{u^2}\Delta^{(8)}\mathcal{L}^{(2)}_{2222} + 4u^2\overline{D}_{2422} \tag{226}$$

$$\mathcal{H}^{(2)}_{2233} = \tfrac{1}{u^2}\Delta^{(8)}\mathcal{L}^{(2)}_{2233} - \tfrac{2u^2}{v^2} + 4u^2(\overline{D}_{1423} + \overline{D}_{1432}) + 24u^2\overline{D}_{2422} - 30u^3\overline{D}_{3522} \tag{227}$$

$$\mathcal{H}^{(2)}_{2244} = \tfrac{1}{u^2}\Delta^{(8)}\mathcal{L}^{(2)}_{2244} - \tfrac{8u^2}{v^2} + 24u^3(\overline{D}_{2523} + \overline{D}_{2532}) + 40u^2\overline{D}_{2422} + 48u^3(\overline{D}_{3522} - u\overline{D}_{4622})$$

$$\Delta^{(8)} = \frac{x\bar{x}y\bar{y}}{(x-\bar{x})(y-\bar{y})}\prod_{i,j=1}^{2}\left(\mathbf{C}_{x_i}^{[+\alpha,+\beta,0]} - \mathbf{C}_{y_j}^{[-\alpha,-\beta,0]}\right)\frac{(x-\bar{x})(y-\bar{y})}{x\bar{x}y\bar{y}}$$

where $\alpha = p_{21}/2$, $\beta = p_{34}/2$ and $\mathbf{C}_x^{[\alpha,\beta,\gamma]}$ is the elementary $2d$ casimir

$$\mathbf{C}_x^{[\alpha,\beta,\gamma]} \;=\; x^2(1-x)\partial_x^2 + x(\gamma - (1+\alpha+\beta)x)\partial_x - \alpha\beta x\ .$$

# $L_{2222}$ at one loop

$$L_{2222} = \frac{W_4 + \cdots + W_0}{(x-\bar{x})^7}$$

$$\phi^{(\ell)} = \sum_{r=0}^{\ell} \frac{(-)^r(2\ell-r)!}{l!(l-r)!r!}(\log u)^r(\mathrm{Li}_{2\ell-r}(x) - \mathrm{Li}_{2\ell-r}(\bar{x}))$$

Then, we can write the basis in the following form

$$\begin{aligned}
W_{4-} &= h_1\,\phi^{(2)}(x_1', x_2') + h_2\,\phi^{(2)}(x,\bar{x}) + h_3\,\phi^{(2)}(1-x,1-\bar{x})\\
W_{3-} &= h_4\, x\partial_x\phi^{(2)}(x,\bar{x}) + h_5\,(x-1)\partial_x\phi^{(2)}(1-x,1-\bar{x}) - (x\leftrightarrow\bar{x})\\
W_{3+} &= (x-\bar{x})\Big(h_6\,\partial_v\phi^{(2)}(x,\bar{x}) + h_7\,\partial_u\phi^{(2)}(1-x,1-\bar{x})\Big) + h_8\,\zeta_3\\
W_{2+} &= h_9\log(u)\log(v) + h_{10}\log^2 v + h_{11}\log^2 u
\end{aligned}$$

and

$$W_{2-} = h_\Box\,\phi^{(1)}(x,\bar{x}) \qquad W_0 = h_0$$
$$W_{1u} = h_u\log u \qquad W_{1v} = h_v\log v$$

$$\begin{aligned}
h_1 &= +4u^4v^2(u-1+v)\,,\\
h_2 &= -4u^4(u+1-v)\,,\\
h_3 &= -4v(u^5 - 5u^4Y_+ - Y_-^4Y_+ + 5u^3(2Y_+^2 - v) + uY_-^2(5Y_+^2-2v) + u^2Y_+(10Y_+^2-19v)\,,\\
h_4 &= -\tfrac{1}{3}u^3(u^4 + Y_-^4 - 4u^3Y_+ + 8uY_-^2Y_+ - 2u^2(3Y_-^2-2v)) + \,,\\
h_5 &= +\tfrac{1}{2}h_4 - \tfrac{1}{6}Y_-(5u^6 - 39u^5Y_+ + 3Y_-^4(Y_+^2+4v) + u^4(73Y_+^2-4v)\\
&\qquad - uY_-^2Y_+(19Y_+^2+32v) - 2u^3Y_+(37Y_+^2-8v) + u^2(51Y_+^4 - 56vY_+^2 - 88v^2))\,,\\
h_6 &= -\tfrac{1}{3}u^3Y_-(u^2+Y_-^2+10uY_+)\,,\\
h_7 &= -\tfrac{5}{12}h_8 + \tfrac{1}{6}(4u^5Y_-^2 + 3Y_-^4(Y_+^2+4v) + u^4(19+48v+99v^2) - 4uY_-^2Y_+(4Y_+^2+11v) +\\
&\qquad - 4u^3(9+39v+45v^2+7v^3) + 2u^2(17Y_+^2 + 11vY_+ - 64v^2))\,,\\
h_8 &= -\tfrac{2}{5}u^2(u^4 + Y_-^4 - 2u(2+7v)(u^2+Y_-^2) + u^2(6+24v-94v^2))\,,\\
h_9 &= -\tfrac{1}{3}(14u^5 - 2u^6 - 3Y_-^5Y_+ - 14u^4(2+v-3v^2) + 2uY_-^3(8Y_+^2-5v) +\\
&\qquad + u^3Y_-(38Y_- + 135v) - 7u^2(5-v+v^3-5v^4))\,,\\
h_{10} &= -\tfrac{1}{3}v(7u^5 - 3Y_-^4(3+v) - u^4(37+35v) + uY_-^2(43+49v+16v^2) +\\
&\qquad + u^3(78+99v+38v^2) - u^2(82+60v+75v^2+35v^3))\,,\\
h_{11} &= -\tfrac{1}{3}u^4(2u^2 - 7Y_-^2 - 7uY_+)\,,
\end{aligned} \tag{230}$$

for weight four, three, and two symmetric, and

$$\begin{aligned}
\widetilde{h}_\Box &= +\tfrac{1}{180}(527u^7 - 2939u^6Y_+ + 234Y_-^6Y_+ - 3uY_-^4(379Y_+^2-4v) + u^5(5295Y_+^2-2320v)\\
&\qquad + 9u^2Y_-^2Y_+(193Y_+^2-64v) - 4u^4Y_+(974Y_+^2-2087v) + 2u^3(67Y_+^4 - 1322vY_+^2 + 400v^2)\\
\widetilde{h}_u &= -\tfrac{1}{5}u(29u^5 - 114u^4Y_+ - 10Y_-^4Y_+ + 2uY_-^2(31Y_+^2-34v)\\
&\qquad + u^3(193Y_+^2-174v) - 10u^2Y_+(16Y_+^2-43v))\\
\widetilde{h}_v &= -\tfrac{1}{2}\widetilde{h}_u - \tfrac{1}{90}Y_-(317u^5 - 1115u^4Y_+ - 90Y_-^4Y_+ + uY_-^2(569Y_+^2-656v)\\
&\qquad + 2u^3(879Y_+^2-917v) - u^2(1439+451v+451v^2+1439v^3))\\
\widetilde{h}_0 &= \tfrac{1}{45}u(47u^3 - 79uY_+ - 15Y_-^2Y_+ + u(77-4v+77v^2))
\end{aligned}$$

$(Y_\pm \equiv 1\pm v)$:

Now for the leading log part we have [Caron-Huot, Trinh]

$$\mathcal{L}^{(2)}_{22pp}\Big|_{\log^2 u} = \widehat{\mathcal{D}}_{22pp}\mathcal{F}^{(2)}_{2222}$$

$$\widehat{\mathcal{D}}_{2233} \;=\; \tfrac{3}{4\times 2}(4-u\partial_u)$$

For the full pre-amplitude we find a simple correction to this, eg:

$$\mathcal{L}^{(2)}_{2233} - \widehat{\mathcal{D}}_{2233}\mathcal{L}^{(2)}_{2222} = \left[\tfrac{1}{2}R_3 \;+ R_5\,(x-1)\partial_x\right]\phi^{(2)}(1-x,1-\bar{x}) - (x\leftrightarrow\bar{x}) +$$
$$+(x-\bar{x})R_7\,\partial_u\phi^{(2)}(1-x,1-\bar{x}) - 32\zeta_3 + R_9\,\log(u)\log(v) + R_{10}\,\log^2 v\,,$$

$$\begin{aligned}
R_3 &= -\tfrac{2}{(x-\bar{x})^3}(u^3+u(7Y_+^2+2v)-5u^2Y_+-3Y_+^3)\\
R_5 &= +\tfrac{1}{(x-\bar{x})^3}Y_-(7u^2-15uY_++8(Y_+^2-v))\\
R_7 &= +\tfrac{1}{(x-\bar{x})^2}(5u^2-13uY_++8(Y_+^2-v))\\
R_9 &= +\tfrac{1}{(x-\bar{x})^2}Y_-(6Y_--5u)\\
R_{10} &= +\tfrac{1}{2(x-\bar{x})^2}(5u(5+3v)+60v-13u^2-12)
\end{aligned}$$

Hints of deeper structure, (10d conformal symmetry) but no clear conclusion

# Extracting-loop data: from <2222>

- One loop correlator contains info about one loop anomalous dimensions

- Problem: operator mixing with other double trace ops

- Also with triple trace operators

$$\boxed{O_2 \cdot \partial^l O_2}$$

- Twist 4 avoids both problems!

- Extract the one loop $O(1/N^4)$

$$\boxed{\eta_l^{(2)} = \begin{cases} \frac{1344(l-7)(l+14)}{(l-1)(l+1)^2(l+6)^2(l+8)} - \frac{2304(2l+7)}{(l+1)^3(l+6)^3} \\ \frac{9}{14}\alpha + \frac{1148}{3} \end{cases}} \qquad \begin{matrix} l = 2, 4, \ldots \\ l = 0 \end{matrix}$$

[l=2,4 found independently by Alday, Bissi]

# Conclusions + Future work.

- We gave an algorithm for entire 4-point 1 loop (all KK modes) gravity amplitude
  [See also [Alday, Zhou] for Mellin amplitude]
- 2-loops or higher? Basis? $\text{Log}^3 u$: $\boxed{\sum_0^2 C_{220}\, {\eta_0^{(1)}}^3}$ (Caron-Huot, Trinh give summed up formula)
- Higher points [Gonçalves, Pereira, Zhou]
- New basis for free N=4 SYM single particle half BPS ops: $\boxed{O_4 = \mathrm{Tr}(\varphi^4) - \frac{2N^2-3}{N(N^2+1)}\mathrm{Tr}(\varphi^2)^2}$
- Explore string corrections, string + loop corrections [Drummond,Nandan,Paul; Alday,Bissi,Perlmutter]
- M theory: 6d (2,0) <2222> tree-level $D^kR^4$ corrections. Coefficients? No flat space limit-> string theory, but chiral algebra? [Abl,Lipstein,PH;Chester,Perlmutter; Binder,Chester,Pufu,Wang]
- M theory Loops? Superblocks beyond <2222> v tricky! (but not impossible - beautiful relation to Calogero-Moser and symmetric polynomials Jacobi BCn). [Isachenkov,Schomerus]

# Multi-particle SUGRA states

- Only other states remaining when $\alpha' \to 0$ = $\boxed{\text{bound states of these}}$

can have derivatives $\partial_\mu^l \Box^n$

2-particle states $\longrightarrow$ Double trace $Tr(\varphi^n)\,Tr(\varphi^m) + \frac{1}{N}\cdots$

3-particle states $\longrightarrow$ Triple trace $Tr(\varphi^n)\,Tr(\varphi^m)\,Tr(\varphi^p) + \frac{1}{N}$

etc.

can be different scalars

More precise definition of single trace ops: Operators dual to single-particle states are defined as those orthogonal to all multi-particle ops.

eg. $Tr(\varphi^2)$ is a single particle op. $\Rightarrow$ $Tr(\varphi^2)\,Tr(\varphi^2)$ is 2-particle op.

$$\Rightarrow \boxed{O_4 = Tr(\varphi^4) - \frac{2N^2-3}{N(N^2+1)}\,Tr(\varphi^2)^2}$$

is wt. 4 single particle op. $\boxed{\langle O_4\,Tr(\varphi^2)^2\rangle = 0}$

Dual of single trace: formula Tom Brown

# More on single particle ops

The single particle operators we have defined are proportional to the dual of the single trace operators, $\xi(\tau, Z)$, in Tom Brown's notation, where $\tau = (1..n)$ is a full $n$ cycle. The dual operators are given in terms of the trace basis as (see (15) of Brown)

$$\xi(\tau, Z) = \sum_J \langle \xi(\sigma_J, Z^\dagger)\xi(\tau, Z)\rangle \, \mathrm{Tr}(\sigma_J Z) \tag{1}$$

(where the sum is over the different conjugacy classes $J$ of $S_n$) whereas we define the single particle operators to have unit coefficient in front of the single trace operator. Therefore our single particle operators are normalised as:

$$\mathcal{O}_n = \frac{\xi(\tau, Z)}{\langle \xi(\tau, Z^\dagger)\xi(\tau, Z)\rangle} \quad . \tag{2}$$

Thus the two point function of our single trace ops is:

$$\langle \mathcal{O}_n^\dagger \mathcal{O}_n \rangle = \frac{1}{\langle \xi(\tau, Z^\dagger)\xi(\tau, Z)\rangle} \; . \tag{3}$$

Now ((14) of Brown)

$$\langle \xi(\tau, Z^\dagger)\xi(\tau, Z)\rangle = \frac{1}{n^2}\sum_R \frac{1}{f_R}\chi_R(\tau)^2 \tag{4}$$

Observation: the character of the $n$ cycle has a particularly simple form

$$\chi_R((1..n)) = \begin{cases} \pm 1 & R = \text{ hook YT} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

# More on single particle ops

. Thus (4) reduces simply to

$$\begin{aligned}
\langle \xi(\tau, Z^\dagger)\xi(\tau, Z)\rangle &= \frac{1}{n^2}\sum_{R=\text{ hook}}\frac{1}{f_R} = \frac{1}{n^2}\sum_{k=0}^{n-1}\frac{1}{(N)_{n-k}(N-k)_{k-1}} \\
&= \frac{1}{n^2 N}\left(\prod_{k=1}^{n-1}\frac{1}{(N^2-k^2)}\right)\sum_{k=0}^{n-1}(N{+}n{-}k)_k(N{-}n{+}1)_{n-k-1} \\
&= \frac{1}{n^2 N}\left(\prod_{k=1}^{n-1}\frac{1}{(N^2-k^2)}\right)\frac{(N)_n - N(N{-}n{+}1)_{n-1}}{n-1}\ . \qquad (6)
\end{aligned}$$

Then from (3) the two point function of our single particle ops is just the inverse of this:

$$\langle \mathcal{O}_n^\dagger \mathcal{O}_n\rangle = n^2\left(\prod_{k=1}^{n-1}(N^2-k^2)\right)\frac{n-1}{(N+1)_{n-1}-(N{-}n{+}1)_{n-1}}\ . \qquad (7)$$

In fact one can simplify this a bit further by noting that $\prod_{k=1}^{n-1}(N^2-k^2) = (N+1)_{n-1}(N{-}n{+}1)_{n-1}$ to

$$\begin{aligned}
\langle \mathcal{O}_n^\dagger \mathcal{O}_n\rangle &= n^2(n-1)\frac{1}{1/(N{-}n{+}1)_{n-1} - 1/(N+1)_{n-1}} \qquad (9) \\
&= n^2(n-1)\frac{1}{\frac{1}{(N-1)^{\underline{n-1}}} - \frac{1}{(N+1)^{\overline{n-1}}}}\ . \qquad (10)
\end{aligned}$$

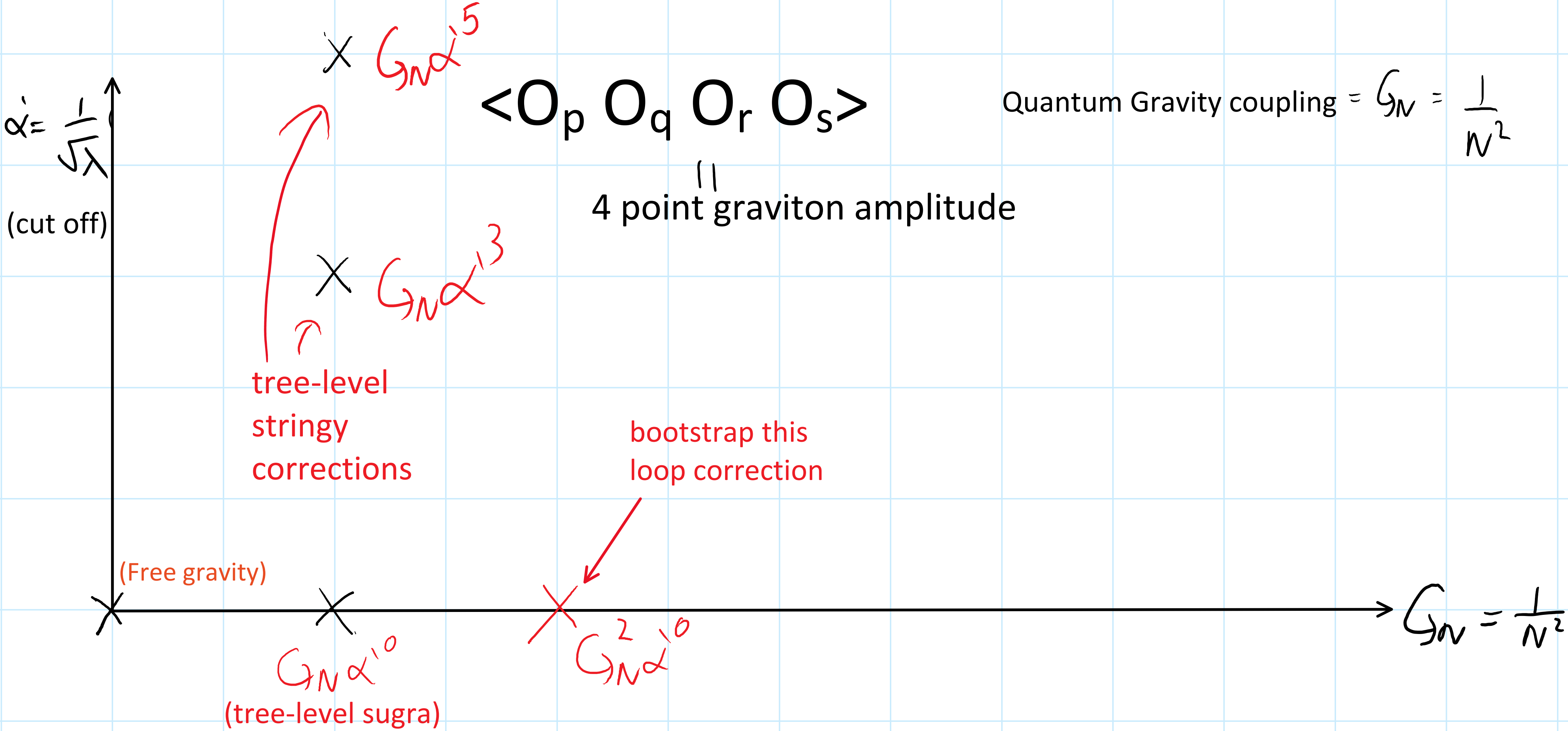
where we switch to the notation $x^{\overline{n}}$ for the Pochhammer (rising factorial) and $x^{\underline{n}}$ for the falling factorial in order to manifest the $N \to -N$ symmetry.

$\langle O_p\, O_q\, O_r\, O_s \rangle$

$\parallel$

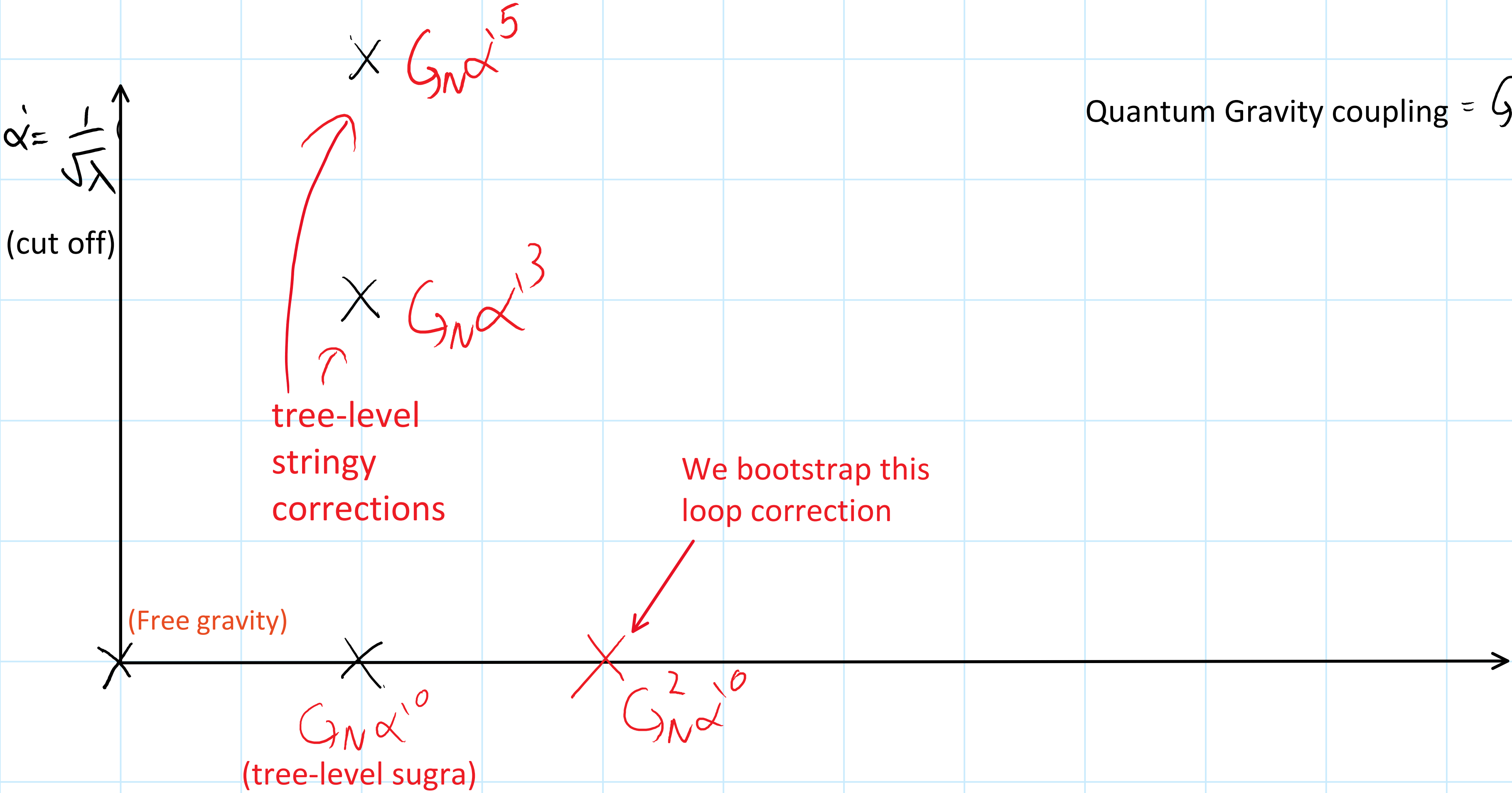
4 point graviton amplitude

Quantum Gravity coupling $= G_N = \frac{1}{N^2}$




$$\boxed{\text{Free gravity} + \frac{1}{N^2} \times (\text{tree level}) + \frac{1}{N^4} \times (\text{1 loop gravity}) + \frac{\alpha'^3}{N^2} (\text{tree-level string corrections}) + \cdots}$$

Quantum Gravity coupling $= G_N = \dfrac{1}{N^2}$

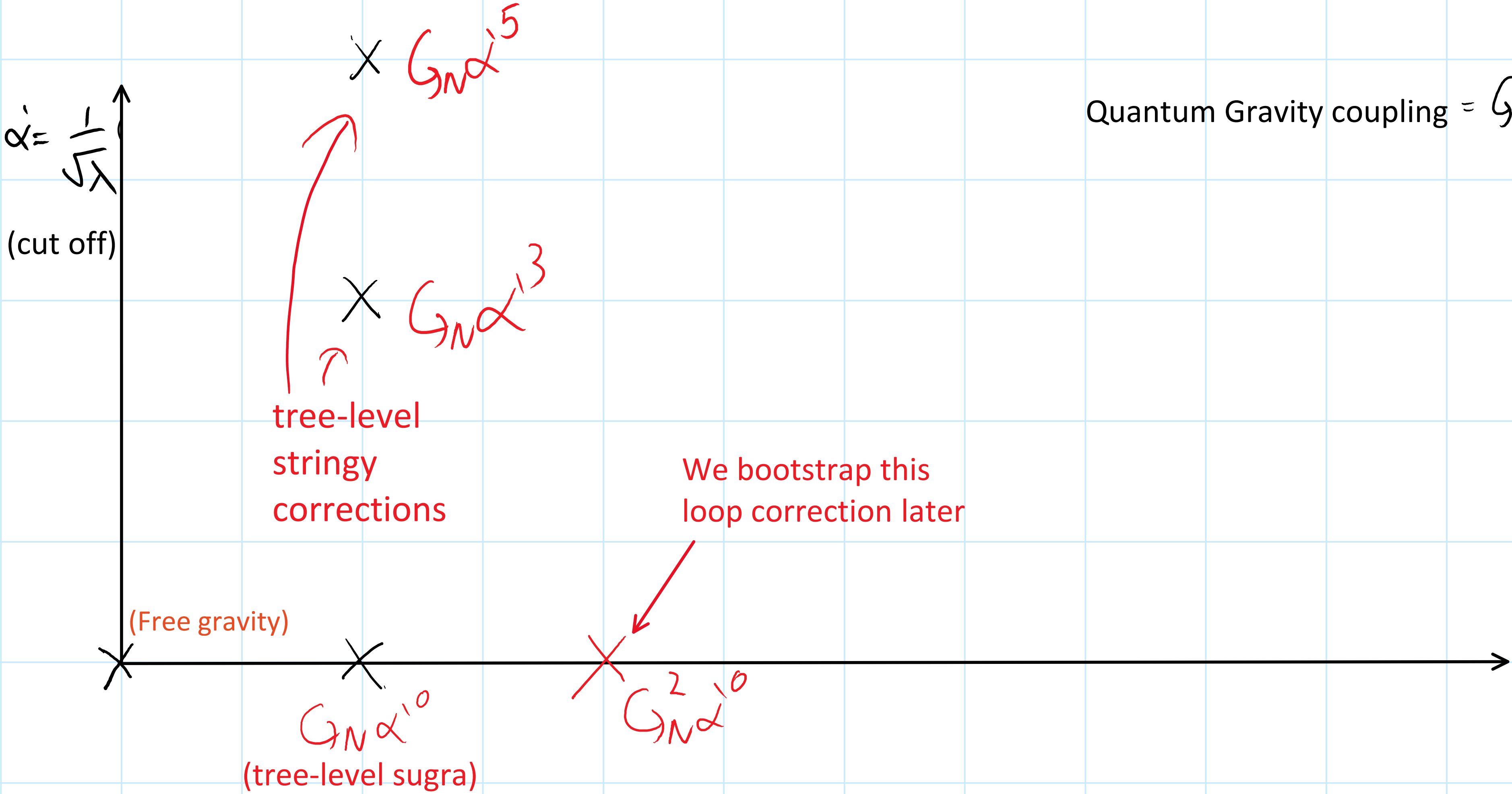



$$\text{Free gravity} + \frac{1}{N^2} \times (\text{tree level}) + \frac{1}{N^4} \times (\text{1 loop gravity})$$

(R⁴ term)

$$+ \frac{\alpha'^3}{N^2} (\text{tree-level string corrections}) + \cdots$$

Quantum Gravity coupling $= G_N = \frac{1}{N^2}$




$$\text{Free gravity} + \frac{1}{N^2} \times (\text{tree level}) + \frac{1}{N^4} \times (\text{1 loop gravity})$$

$$+ \frac{\alpha'^3}{N^2} (\text{tree-level string corrections}) + \cdots$$

(1 loop amplitude)

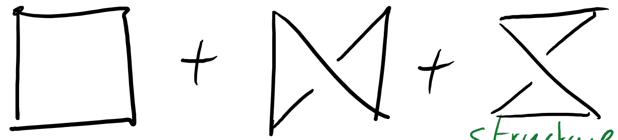
# Supercorrelators

[disconnected free theory]

$$\langle 2222 \rangle = g_{12}^2 g_{34}^2 + g_{13}^2 g_{24}^2 + g_{14}^2 g_{23}^2$$

$= $ 

$$+ \frac{1}{N^2}\left( g_{12} g_{34} g_{13} g_{24} + \text{crossing} \right)$$

$= $ 

structure [Eden, Schubert, Sokatchev]

$$+ \frac{1}{N^2} g_{13}^2 g_{24}^2 (x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y}) \times \bar{D}_{2422} + O\left(\frac{1}{N^4}\right)$$

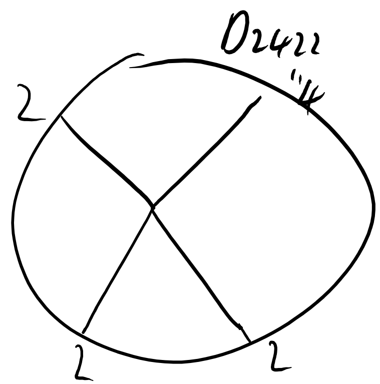
structure free $+ (x-y)\ldots$ generalizes to higher charges

[d'Hoker, Freedman; Arutyunov, Frolov]

$$\left[ g_{ij} = \frac{Y_i \cdot Y_j}{x_{ij}^2} \right]$$

$$u = x\bar{x} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = (1-x)(1-\bar{x}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$y\bar{y} = \frac{Y_1 \cdot Y_2 \; Y_3 \cdot Y_4}{Y_1 \cdot Y_3 \; Y_2 \cdot Y_4} \qquad (1-y)(1-\bar{y}) = \frac{Y_1 \cdot Y_4 \; Y_2 \cdot Y_3}{Y_1 \cdot Y_3 \; Y_2 \cdot Y_4}$$




$$\boxed{\bar{D}_{2422} = -4 \partial_u \partial_v \left( 1 + u\partial_u + v\partial_v \right) \mathrm{Box}(x,\bar{x})}$$

degree 4

$$= \frac{\text{polynomial}(x,\bar{x}) \times \text{wt.2 polylogs in}(x,\bar{x}) + \text{poly}(x,\bar{x}) \times \text{wt1} + \text{poly}(x,\bar{x})\,\text{wt.0}}{(x-\bar{x})^7}$$

# Long superblocks

Simple $\boxed{= (x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y}) \times \text{bosonic block}(x,\bar{x}) \times \text{internal block}(y,\bar{y})}$

$= 1$ for $\langle 2222 \rangle$

- Superblock expansion of the dynamical part of the correlator is (relatively) straightforward, only contains long ops.

- Expansion of free part is more complicated. (short ops - more complicated superblocks.)

- Explicit superblocks known [Dolan, Osborn; Bissi, Lukowski; Doobary, PH]

- Alternatively lift across to a "bosonic superblock".

Simple determinantal blocks in p bosonic variables SU(2,2|4)-> SU(p,p)
(Doobary, PH; Aprile, Drummond, Hynek, PH.)

# Free theory CPW coeffs via "bosonised superblocks"

$SU(2,2|4) \to SU(m,m|2n)$

$\downarrow$

$SU(m,m)$

- Free theory:

$$\prod_{i<j} g_{ij}^{d_{ij}} = \mathcal{P}^{(\mathrm{OPE})}_{\{p_i\}} \times \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac{1}{2}(\gamma - p_4 + p_3)} \times \left(\frac{g_{14}g_{23}}{g_{13}g_{24}}\right)^{d_{23}}$$

$$\left(\frac{g_{14}g_{23}}{g_{13}g_{24}}\right)^{d_{23}} = \left(\frac{(1-y_1)(1-y_2)}{(1-x_1)(1-x_2)}\right)^{d_{23}} = \mathrm{sdet}^{-d_{23}}(1-Z) \qquad Z \sim \left(\begin{array}{cc|cc} x_1 & & & \\ & x_2 & & \\ \hline & & y_1 & \\ & & & y_2 \end{array}\right)$$

$$= \sum A_{\gamma\underline{\lambda}} F^{\alpha\beta\gamma\underline{\lambda}}(Z)$$

superblocks above

$SU(2,2|4)$

$\downarrow$

$SU(m,m)$

simple bosonised blocks

$\downarrow$

$$Z \to \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_m \end{pmatrix}$$

$$F^{\alpha\beta\gamma\underline{\lambda}}(\underline{x}) = \frac{\det\left(x_i^{\lambda_j+m-j}\,{}_2F_1(\lambda_j+1-j+\alpha, \lambda_j+1-j+\beta; 2\lambda_j+2-2j+\gamma; x_i)\right)_{1\le i,j\le m}}{\det\left(x_i^{m-j}\right)_{1\le i,j\le m}}$$

# Superblocks. [Dolan, Osborn; Bissi, Lukowski; Doobary P.H.]

[0 p 0] SU(4) rep [Y·Y = 0]

$$\boxed{\mathcal{O}_p(X) := \mathrm{Tr}\left(\left(\varphi_I(x)\, Y_I\right)^p\right) + \dots}$$

$X = (x, Y)$ internal co-ordinate

deal with all 6 scalars.

(in fact all ops in multiplet for free - analytic superspace)

Then the superblock decomposition is:

$$\langle \mathcal{O}^{p_1}(X_1)\mathcal{O}^{p_2}(X_2)\mathcal{O}^{p_3}(X_3)\mathcal{O}^{p_4}(X_4)\rangle$$
$$= \sum_{\gamma,\underline{\lambda}} A_{\gamma\underline{\lambda}}^{p_1p_2p_3p_4}\, g_{12}^{\frac{p_1+p_2}{2}}\, g_{34}^{\frac{p_3+p_4}{2}} \left(\frac{g_{24}}{g_{14}}\right)^{\frac{1}{2}p_{21}} \left(\frac{g_{14}}{g_{13}}\right)^{\frac{1}{2}p_{43}} \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac{1}{2}\gamma} F^{\alpha\beta\gamma\underline{\lambda}}(Z),$$

$$\alpha = \tfrac{1}{2}(\gamma - p_{12}) \quad \beta = \tfrac{1}{2}(\gamma + p_{34})\,,$$

where $\boxed{g_{ij} = \frac{Y_i \cdot Y_j}{x_{ij}^2}}$ superpropagator

$\gamma = \Delta - l$ = twist of multiplet (ws)

$$\boxed{A^{p_1p_2p_3p_4}_{\gamma\underline{\lambda}} = \sum_{\mathcal{O}} C_{p_1p_2\mathcal{O}_{\gamma\underline{\lambda}}}\, C_{p_3p_4\mathcal{O}_{\gamma\underline{\lambda}}}}$$

Multiplets with same quantum numbers.

$\underline{\lambda}$ other quantum numbers.
[SU(4) rep, spin $l$]

# Superblocks (continued)

[Doobary, P.H.]

$$F^{\alpha\beta\gamma\underline{\lambda}}(x|y) = \delta_{\underline{\lambda};0} + D^{-1}\left[\left(\frac{f(x_2,y_2)}{x_1-y_1} - y_1 \leftrightarrow y_2\right) - x_1 \leftrightarrow x_2\right] + \boxed{D^{-1} f(x_1,x_2,y_1,y_2).} \qquad (7$$

where here

$$D^{-1} = \frac{(x_1-y_1)(x_1-y_2)(x_2-y_1)(x_2-y_2)}{(x_1-x_2)(y_1-y_2)}\,. \qquad (7$$

$$x_1 x_2 = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} = u$$

$$(1-x_1)(1-x_2) = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2} = v$$

$$y_1 y_2 = \frac{Y_1\cdot Y_2\ Y_3\cdot Y_4}{Y_1\cdot Y_3\ Y_2\cdot Y_4}$$

$$(1-y_1)(1-y_2) = \frac{Y_1\cdot Y_4\ Y_2\cdot Y_3}{Y_1\cdot Y_3\ Y_2\cdot Y_4}$$

The functions are given explicitly as

$\lambda_2 > 1$ **(long)** :

$f(x,y) = 0$

$$\boxed{f(x_1,x_2,y_1,y_2) = (-1)^{\lambda_1'+\lambda_2'}\left(F^{\alpha\beta\gamma}_{\lambda_1}(x_1)\,F^{\alpha\beta\gamma}_{\lambda_2-1}(x_2) - x_1 \leftrightarrow x_2\right)\left(G^{\alpha\beta\gamma}_{\lambda_1'}(y_1)\,G^{\alpha\beta\gamma}_{\lambda_2'-1}(y_2) - y_1 \leftrightarrow y_2\right)}$$

$\lambda_2 = 0,1$ **(semi-short / quarter BPS)** :

$f(x,y) = (-1)^{\lambda_1'} F^{\alpha\beta\gamma}_{\lambda_1}(x) G^{\alpha\beta\gamma}_{\lambda_1'}(y)$

$$f(x_1,x_2,y_1,y_2) = \sum_{j=\lambda_1'+1}^{p} (-1)^{\lambda_1'}\left(F^{\alpha\beta\gamma}_{1-j}(x_2)\,F^{\alpha\beta\gamma}_{\lambda_1}(x_1) - (x_1 \leftrightarrow x_2)\right)\left(G^{\alpha\beta\gamma}_{j}(y_2)\,G^{\alpha\beta\gamma}_{\lambda_1'}(y_1) - (y_1 \leftrightarrow y_2)\right)$$
$$+ \sum_{j=2}^{\lambda_1'} (-1)^{\lambda_1'}\left(F^{\alpha\beta\gamma}_{2-j}(x_2)\,F^{\alpha\beta\gamma}_{\lambda_1}(x_1) - (x_1 \leftrightarrow x_2)\right)\left(G^{\alpha\beta\gamma}_{j-1}(y_2)\,G^{\alpha\beta\gamma}_{\lambda_1'}(y_1) - (y_1 \leftrightarrow y_2)\right)$$

$\underline{\lambda} = 0$ **(half BPS)** :

$$f(x,y) = -\sum_{i=1}^{p} F^{\alpha\beta\gamma}_{1-i}(x) G^{\alpha\beta\gamma}_{i}(y)$$

$$f(x_1,x_2,y_1,y_2) = \sum_{1\le i<j\le p}\left(F^{\alpha\beta\gamma}_{1-i}(x_2)\,F^{\alpha\beta\gamma}_{1-j}(x_1) - F^{\alpha\beta\gamma}_{1-i}(x_1)\,F^{\alpha\beta\gamma}_{1-j}(x_2)\right)\left(G^{\alpha\beta\gamma}_{i}(y_1)\,G^{\alpha\beta\gamma}_{j}(y_2) - G^{\alpha\beta\gamma}_{i}(y_2)\,G^{\alpha\beta\gamma}_{j}(y_1)\right)$$

where we have defined the functions

$$F^{\alpha\beta\gamma}_{\lambda}(x) := [x^{\lambda-1}{}_2F_1(\lambda+\alpha,\lambda+\beta;2\lambda+\gamma;x)]$$
$$G^{\alpha\beta\gamma}_{\lambda'}(y) := y^{\lambda'-1}{}_2F_1(\lambda'-\alpha,\lambda'-\beta;2\lambda'-\gamma;y)$$

# Degeneracy of multiplets – Unmixing.

- Taking the $\langle 2222 \rangle$ $N^0$, $\frac{1}{N^2}$ result (log $u$ piece) and decomposing into superblocks gives the combinations:

$N^0$: $\boxed{\sum_{\mathcal{O}} \left(C_{22\mathcal{O}}\right)^2}$ $\qquad$ $\frac{1}{N^2}$: $\boxed{\sum_{\mathcal{O}} \left(C_{22\mathcal{O}}\right)^2 \gamma_{\mathcal{O}}}$

Sum over all (long) multiplets with the same quantum numbers (superconf. rep.)

- Only $SU(4)$ singlets appear in $\langle 2222 \rangle$
- triple and higher trace ops $\frac{1}{N}$ suppressed.

$\Rightarrow$ Double trace singlet operators

# Double trace singlet long multiplets:

- $\boxed{O_2 (\partial_\mu)^l O_2}$ twist $(= \Delta - l) = \boxed{4}$, spin $l$.

- Only one operator for each (even) spin with
No degeneracy at twist 4.

- Superblock decomposition gives:

$$C_{220}^2 = \frac{4}{3}(l+1)(l+6) \times \frac{(l+3)!^2}{(2l+6)!} \; ; \; C_{220}^2 \eta_0 = -64 \times \frac{(l+3)!^2}{(2l+6)!}$$

[Dolan, Osborn]

$$\Rightarrow \boxed{\eta = -\frac{48}{(l+1)(l+6)}}$$

- Very useful feature: $\hat{A} = \begin{pmatrix} \langle 2222 \rangle & \langle 2233 \rangle \\ \langle 3322 \rangle & \langle 3333 \rangle \end{pmatrix}\Bigg|_{N^0, \text{CPW} \text{ orels}} \to 0$ is diagonal

Why? $\langle ppqq \rangle \big|_{N^0} = \begin{cases} g_{12}^p\, g_{34}^q & p \neq q \\ g_{12}^p\, g_{34}^p + g_{13}^p g_{24}^p + g_{14}^p g_{23}^p & p = q \end{cases}$ $\longrightarrow$ $\emptyset$ contrib to long ops.

contribute to $\frac{1}{2}$ BPS ops $O_{|p-2|}$ only in OPE.

- Define $\tilde{c}(t|l)$.

$$\boxed{\tilde{c}\,\tilde{c}^T = \mathrm{Id}_{t-1},} \qquad C = \hat{A}^{\frac{1}{2}} \cdot \tilde{c}(t|l)$$

$\tilde{c}$ orthonormal

- eg twist 4: $\tilde{c} = 1$

twist 6:
$$\tilde{c}(3|l) = \begin{pmatrix} \sqrt{\frac{l+2}{2l+9}} & \sqrt{\frac{l+7}{2l+9}} \\ -\sqrt{\frac{l+7}{2l+9}} & \sqrt{\frac{l+2}{2l+9}} \end{pmatrix}$$

$\tilde{c}$ orthonormal

# Higher twist examples.

## Twist 8.

$$\tilde{c}(4|l) = \begin{pmatrix} \sqrt{\frac{7(l+2)(l+3)}{6(2l+9)(2l+11)}} & \sqrt{\frac{5(l+3)(l+8)}{3(2l+9)(2l+13)}} & \sqrt{\frac{7(l+8)(l+9)}{6(2l+11)(2l+13)}} \\ -\sqrt{\frac{2(l+2)(l+8)}{(2l+9)(2l+11)}} & -\sqrt{\frac{35}{(2l+9)(2l+13)}} & \sqrt{\frac{2(l+3)(l+9)}{(2l+11)(2l+13)}} \\ \sqrt{\frac{5(l+8)(l+9)}{6(2l+9)(2l+11)}} & -\sqrt{\frac{7(l+2)(l+9)}{3(2l+9)(2l+13)}} & \sqrt{\frac{5(l+2)(l+3)}{6(2l+11)(2l+13)}} \end{pmatrix},$$

$$\eta_{4,l,i} = \left\{ -\frac{720(l+7)}{(l+1)(l+2)(l+3)}, -\frac{720}{(l+3)(l+8)}, -\frac{720(l+4)}{(l+8)(l+9)(l+10)} \right\}.$$

## Twist 10.

$$\tilde{c}(5|l) = \begin{pmatrix} \sqrt{\frac{3}{2}\frac{(2)(3)(4)}{[9][11][13]}} & \sqrt{\frac{5}{2}\frac{(3)(4)(9)}{[9][13][15]}} & \sqrt{\frac{5}{2}\frac{(4)(9)(10)}{[11][13][17]}} & \sqrt{\frac{3}{2}\frac{(9)(10)(11)}{[13][15][17]}} \\ -\sqrt{\frac{27}{8}\frac{(2)(3)(9)}{[9][11][13]}} & -\sqrt{\frac{5}{8}\frac{(l+18)(3)}{[9][13][15]}} & \sqrt{\frac{5}{8}\frac{(l-5)(10)}{[11][13][17]}} & \sqrt{\frac{27}{8}\frac{(4)(10)(11)}{[13][15][17]}} \\ \sqrt{\frac{5}{2}\frac{(2)(9)(10)}{[9][11][13]}} & -\sqrt{\frac{3}{2}\frac{(l-3)(10)}{[9][13][15]}} & -\sqrt{\frac{3}{2}\frac{(l+16)(3)}{[11][13][17]}} & \sqrt{\frac{5}{2}\frac{(3)(4)(11)}{[13][15][17]}} \\ -\sqrt{\frac{5}{8}\frac{(9)(10)(11)}{[9][11][13]}} & \sqrt{\frac{27}{8}\frac{(2)(10)(11)}{[9][13][15]}} & -\sqrt{\frac{27}{8}\frac{(2)(3)(11)}{[11][13][17]}} & \sqrt{\frac{5}{8}\frac{(2)(3)(4)}{[13][15][17]}} \end{pmatrix} \qquad (n) = \sqrt{l+n}\,, \qquad [n] = \sqrt{2l+n}$$

$$\eta_{5,l,i} = \left\{ -\frac{1680(l+7)(l+8)}{(l+1)(l+2)(l+3)(l+4)}, -\frac{1680}{(l+3)(l+4)}, -\frac{1680}{(l+9)(l+10)}, -\frac{1680(l+5)(l+6)}{(l+9)(l+10)(l+11)(l+12)} \right\}$$

## Higher twist, $\Delta - l = 2t$

- Basis of singlet (double trace long) operators of twist $2t$, spin $l$:

$$\boxed{K^{\text{free}}_{t,l,i} = \mathcal{O}_{i+1} \Box^{t-i-1} \partial^l \mathcal{O}_{i+1} + \ldots \qquad i = 1 \cdots t-1}$$

- These mix, $K_{t,l,i}$ pure operators at strong coupling
- Matrix of 3-point functions

$$C(t|l) = \begin{pmatrix} C_{22K_{t,l,1}} & C_{22K_{t,l,2}} & \cdots & C_{22K_{t,l,t-1}} \\ C_{33K_{t,l,1}} & C_{33K_{t,l,2}} & \cdots & \\ \cdots & & & \\ C_{ttK_{t,l,1}} & & & \end{pmatrix}$$

- Superblock decomposition of $N^0$ correlator then gives the eq$^n$:

$$\boxed{C C^T = \hat{A}} \qquad \hat{A} \text{ block coeffs of } \left.\begin{pmatrix} \langle 2222 \rangle & \langle 2233 \rangle & \cdots & \langle 22tt \rangle \\ \langle 3322 \rangle & \langle 3333 \rangle & \cdots & \langle 33tt \rangle \\ \vdots & \vdots & & \vdots \\ \langle tt22 \rangle & \langle tt33 \rangle & \cdots & \langle tttt \rangle \end{pmatrix}\right|_{N^0}$$

# General Formulae

- By computing many higher twist results, find pattern and deduce general formulae: t=T/2

- Anomalous dimensions of all double trace singlets:

$$\boxed{\eta^{[0,0,0]}_{t,l,i} = -\frac{2(t-1)_4(t+l)_4}{(l+2i-1)_6}}$$

- OPE coeffs have following structure: remarkably remaining coefficients fixed uniquely (up to signs) by demanding orthonormality of this matrix

$$\tilde{c}^{[0,0,0]}_{pi} = \sqrt{\frac{2^{1-t}(2l+4i+3)\left((l+i+1)_{t-i-p+1}\right)^{\sigma_1}\left((t+l+p+2)_{i-p+1}\right)^{\sigma_2}}{\left(l+i+\frac{5}{2}\right)_{t-1}}} \times \sum_{k=0}^{\min(i-1,p-2,t-i-1,t-p)} l^k a^{[0,0,0]}_{(p,i,k)}.$$

$$\sigma_1 = \mathrm{sgn}(t-p-i+1)\,, \qquad \sigma_2 = \mathrm{sgn}(i-p+1)$$

# Even more general formulae

- One can go even further and consider completely general double trace operators (general reps of SU(4)).
- Need to consider all single particle 4 point functions <pqrs>

Operators: Twist $\tau$, spin $l$ SU(4) rep [a,b,a]

$$\boxed{\mathcal{O}_{pq} = \mathcal{O}_p \partial^l \Box^{\frac{1}{2}(\tau-p-q)} \mathcal{O}_q \,,} \qquad (p \le q)$$

$p, q \in$

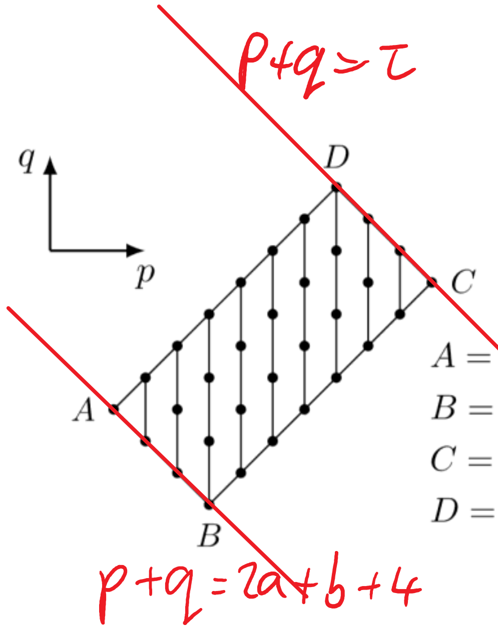
$$\mathcal{D}^{\text{long}}_{\tau,l,a,b}$$

$$p = i + a + 1 + r\,, \qquad q = i + a + 1 + b - r$$

$$i = 1, \ldots, (t-1)\,, \qquad r = 0, \ldots, (\mu - 1)\,,$$

so that $d = \mu(t-1)$ with

$$t \equiv (\tau - b)/2 - a\,, \qquad \mu \equiv \begin{cases} \left\lfloor \frac{b+2}{2} \right\rfloor & a + l \text{ even,} \\ \left\lfloor \frac{b+1}{2} \right\rfloor & a + l \text{ odd.} \end{cases}$$




$A = (a+2, a+b+2)$;
$B = (a+1+\mu, a+b+3-\mu)$;
$C = (a+\mu+t-1, a+b+1+t-\mu)$;
$D = (a+t, a+b+t)$;

# Fixing the remaining OPE coeffs without calculating.

- Remarkably the remaining coefficients $\boxed{a^{[000]}_{p,i,k}}$ are always uniquely fixed by orthonormality of $\tilde{C}$
- Orthonormality $\Rightarrow$ $\boxed{\text{linear equations in } a^{[000]}_{p,i,k}}$ (or $a^2$)
- Unique (up to signs) solution always exists!
- Allows quick solution. Complete data up to $\boxed{\text{twist } 48}$
  (without needing tree SUGRA result!)
- Analytic formula for $a^{[000]}_{p,i,k}$ ?

$$\boxed{a^{[0,0,0]}_{(2,i,0)} = \frac{2^{t-1}(2i+2)!(t-2)!(2t-2i+2)!}{3(i-1)!(i+1)!(t+2)!(t-i-1)!(t-i+1)!},}$$

$\langle O_2 O_2 K_{t,l,i} \rangle$ $\Leftarrow$ We originally wanted!

$$\boxed{\text{What are those orthonormal matrices?}}$$

Ideas welcome!

- Superblock decomposition of $\frac{1}{N^2}$ correlator then gives the eq$^n$:

$$\boxed{C \eta C^T = \hat{M}}$$

$\hat{M}$ blockcoeffs of $\left.\begin{pmatrix} \langle 2222\rangle & \langle 2233\rangle & \cdots & \langle 22tt\rangle \\ \langle 3322\rangle & \langle 3333\rangle & \cdots & \langle 33tt\rangle \\ \vdots & \vdots & & \vdots \\ \langle tt22\rangle & \langle tt33\rangle & \cdots & \langle tttt\rangle \end{pmatrix}\right|_{\frac{1}{N^2}}$

$$\eta = \begin{pmatrix} \eta_{K_1} & & & \\ & \eta_{K_2} & & \\ & & \ddots & \\ & & & \eta_{K_{t-1}} \end{pmatrix}$$

(from SUGRA results $\boxed{\text{Rastelli, Zhou}}$)

- <u>Unknowns</u>: $C : (t-1)\times(t-1)$
$+\ \eta : t-1$

$= \boxed{(t-1)t}$

<u>Equations</u>: $\hat{A} : \frac{(t-1)t}{2}$ (symmetric)
$+\ \hat{M} : \frac{(t-1)t}{2}$

$= \boxed{(t-1)t}$

unique solution.

## ~~1-loop correlator.~~

Full result:

$$\langle 2222\rangle = \langle 2222\rangle_{\text{free}} + g_{13}^2 g_{24}^2 s(x,\bar{x};y,\bar{y}) F(u,v)$$

$$F(u,v) = aF^{(1)}(u,v) + a^2 F^{(2)}(u,v) + O(a^3)$$

$$F^{(1)}(u,v) = -4\partial_u\partial_v(1+u\partial_u+v\partial_v)\Phi^{(1)}(u,v)$$

$$F^{(2)}(u,v) = \frac{1}{uv}\left[f(u,v) + \frac{1}{u}f\left(\frac{1}{u},\frac{v}{u}\right) + \frac{1}{v}f\left(\frac{1}{v},\frac{u}{v}\right)\right] + \text{ambiguity}$$

$$f(u,v) = \Delta^{(4)} g(u,v)\,, \qquad \Delta^{(4)} = (x-\bar{x})^{-1} uv \partial_x^2 \partial_{\bar{x}}^2 (x-\bar{x})$$

$$g = (x-\bar{x})^{-10}[g^{(4)} + g^{(3)} + g^{(2)} + g^{(1)} + g^{(0)}]$$

$$\begin{aligned}
g^{(4)}(u,v) =& P_-^{(4)}(u,v)\Phi^{(2)}(u,v)\\
g^{(3)}(u,v) =& P_+^{(3)}(u,v)\Psi(u,v) + P_-^{(3)}(u,v)\log(uv)\Phi^{(1)}(u,v)\\
g^{(2)}(u,v) =& P_+^{(2)}(u,v)\log u\log v + P_-^{(2)}(u,v)\Phi^{(1)}(u,v)\\
g^{(1)}(u,v) =& P_+^{(1)}(u,v)\log(uv)\\
g^{(0)}(u,v) =& P_+^{(0)}(u,v)\,.
\end{aligned}$$

$$\Phi^{(l)}(u,v) = -\frac{1}{x-\bar{x}}\phi^{(l)}\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right),$$

l-loop
ladder

$$\phi^{(l)}(x,\bar{x}) = \sum_{r=0}^{l}(-1)^r\frac{(2l-r)!}{r!(l-r)!l!}\log^r(x\bar{x})(\mathrm{Li}_{2l-r}(x)-\mathrm{Li}_{2l-r}(\bar{x}))$$

$$\begin{aligned}\Psi(u,v) &= (x-\bar{x})(u\partial_u + v\partial_v)[(x-\bar{x})\Phi^{(2)}(u,v)]\\ &= [x(1-x)\partial_x - \bar{x}(1-\bar{x})\partial_{\bar{x}}]\phi^{(2)}\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right)\end{aligned}$$

$$\begin{aligned}P_-^{(4)}(u,v) &= 96p^2\bar{s}[\bar{s}^4 + 20p\bar{s}^2 + 30p^2]\,,\\ P_+^{(3)}(u,v) &= \frac{8}{5}p^2[137\bar{s}^4 + 1214p\bar{s}^2 + 512p^2]\,,\\ P_-^{(3)}(u,v) &= 336p^2[\bar{s}(1-\bar{s})(6-6\bar{s}+\bar{s}^2) + 2p(3-14\bar{s}+4\bar{s}^2) - 16p^2]\,,\\ P_+^{(2)}(u,v) &= 2[(1-\bar{s})^2\bar{s}^6 - 2p\bar{s}^4(20-33\bar{s}+14\bar{s}^2)\\ &\quad + 8p^2(756-1323\bar{s}+601\bar{s}^2-54\bar{s}^3+30\bar{s}^4)\\ &\quad - 32p^3(583-25\bar{s}+26\bar{s}^2) + 1024p^4]\,,\end{aligned}$$

$$\begin{aligned}P_-^{(2)}(u,v) &= 56p^2[-\bar{s}^2(2-\bar{s})(18-18\bar{s}+5\bar{s}^2)\\ &\quad + 2p(108-144\bar{s}+128\bar{s}^2-11\bar{s}^3) - 8p^2(63-\bar{s})]\,,\\ P_+^{(1)}(u,v) &= \frac{1}{3}[5\bar{s}^7(2-3\bar{s}) - 2p\bar{s}^5(158-193\bar{s})\\ &\quad + 16p^2\bar{s}(378-567\bar{s}+233\bar{s}^2-147\bar{s}^3)\\ &\quad + 32p^3(378-139\bar{s}+129\bar{s}^2) + 256p^4]\,,\\ P_+^{(0)}(u,v) &= \frac{2}{15}(x-\bar{x})^2[20(1-\bar{s})\bar{s}^6 - 5p\bar{s}^4(102-75\bar{s}-4\bar{s}^2)\\ &\quad + 8p^2(630-630\bar{s}+481\bar{s}^2-255\bar{s}^3-30\bar{s}^4)\\ &\quad - 16p^3(217-215\bar{s}-60\bar{s}^2) - 1280p^4]\,.\end{aligned}$$

$$\bar{s} = 1-u-v\,, \qquad p = uv$$

# Semi-short sector

- Key point 1: In $O_{p1}$ $O_{p2}$ OPE: No semishorts with twist > $p_1+p_2$
- This means twist $p_1+p_2$ semishort can not combine with higher twist operator.

genuine semishort CPW coefficient

Naive free semi-short coefficient

Long recombined coefficient

$$\underbrace{S^{\{p_i\}}_{\tau,[l+1,1^a]}}_{\text{genuine semishort CPW coefficient}} = \underbrace{A^{\{p_i\}}_{\tau,[l+1,1^a]}}_{\text{Naive free semi-short coefficient}} - \underbrace{L^{\{p_i\}}_{\tau-2,l+1,[a-1,b,a-1]}}_{\text{Long recombined coefficient}} \qquad \text{for} \quad \tau = 2a+b+2 = \underset{\min(p_1+p_2,\,p_3+p_4)}{\gamma_{\max}}(p_i), \quad a \geq 1$$

$$S^{\{p_i\}}_{\tau,[l+1]} = A^{\{p_i\}}_{\tau,[l+1]} \qquad \text{for} \quad \tau = b+2 = \gamma_{\max}(p_i), \quad .$$

Known (Free CPW)

obtained in previous step (recursively)

- Key point 2: there are μ-1 (double trace) semi-short operators:
- $\mathcal{O}_{pq} = \mathcal{O}_p \partial^l \mathcal{O}_q \quad \tau = p + q = 2a + b + 2$
- $p = a + 1 + r + \delta_{a,0}, \qquad q = a + 1 + b - r - \delta_{a,0} \qquad r = 0, \ldots, \mu-1$

# Semi-short sector

Consider the matrix of semishort coeffs:

$$M_{rs} = S_{\tau,[l+2,1^a]}^{\{p_1(r)p_2(r)p_3(s)p_4(s)\}} \quad r = 0,..,\mu, \; s = 0,..,\mu$$

Where

$$p_1(r) = p_3(r) = a + 1 + r + \delta_{a,0} \qquad p_2(r) = p_4(r) = a + 1 + b - r - \delta_{a,0} \qquad r < \mu$$

$$p_i(\mu) = p_i \; .$$

- Claim M has zero determinant. Only has rank μ-1

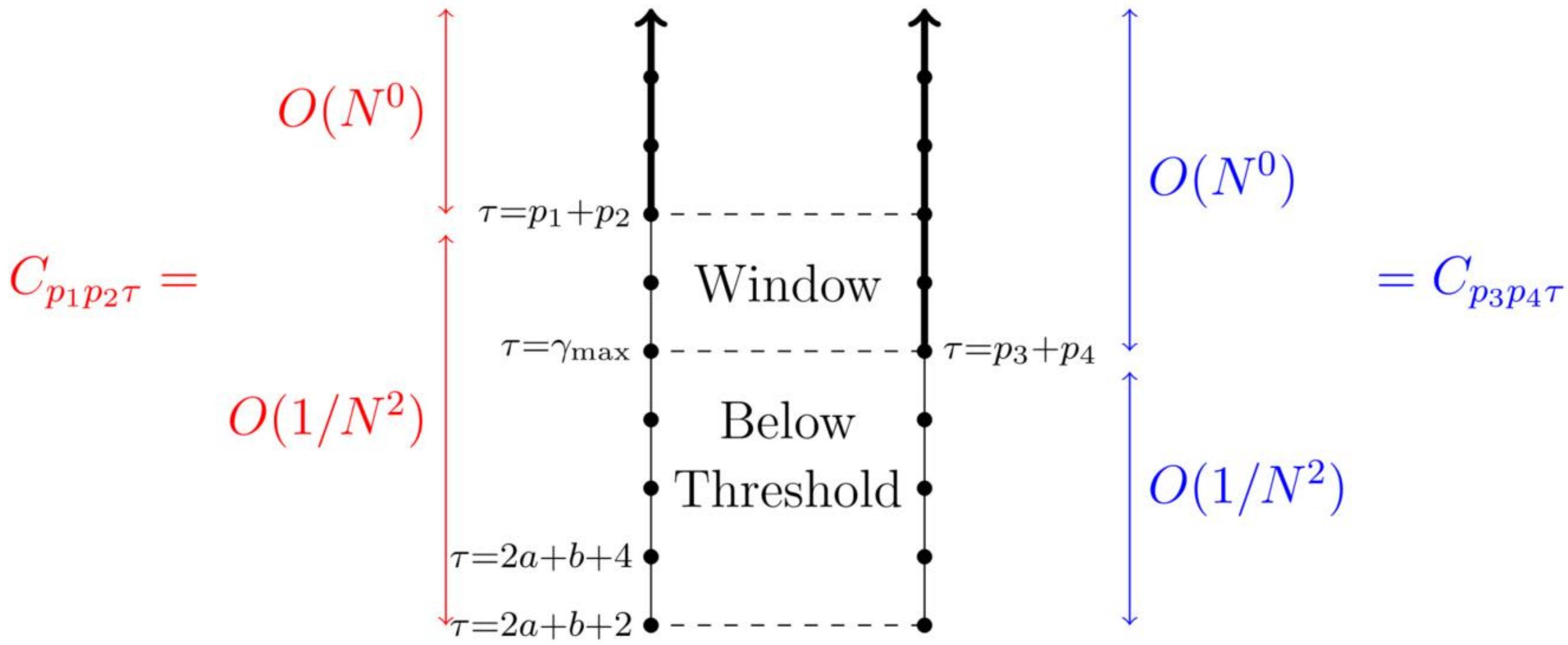
$M = C_{12} C_{34}^T$ where $C_{12}, C_{34}$ are both $(\mu + 1) \times (\mu)$ matrices of OPE coefficients

- Therefore this gives a non trivial eqn for the semishort coeff of <p1 p2 p3 p4 > correlator in terms of those with obtained in previous step

Remarkably the bootstrapping of the one loop correlator (with minimal ansatz) is consistent with this semishort prediction!

# Higher charge correlators at 1 loop $O(1/N^4)$

- new feature: below threshold tree-level data prediction.

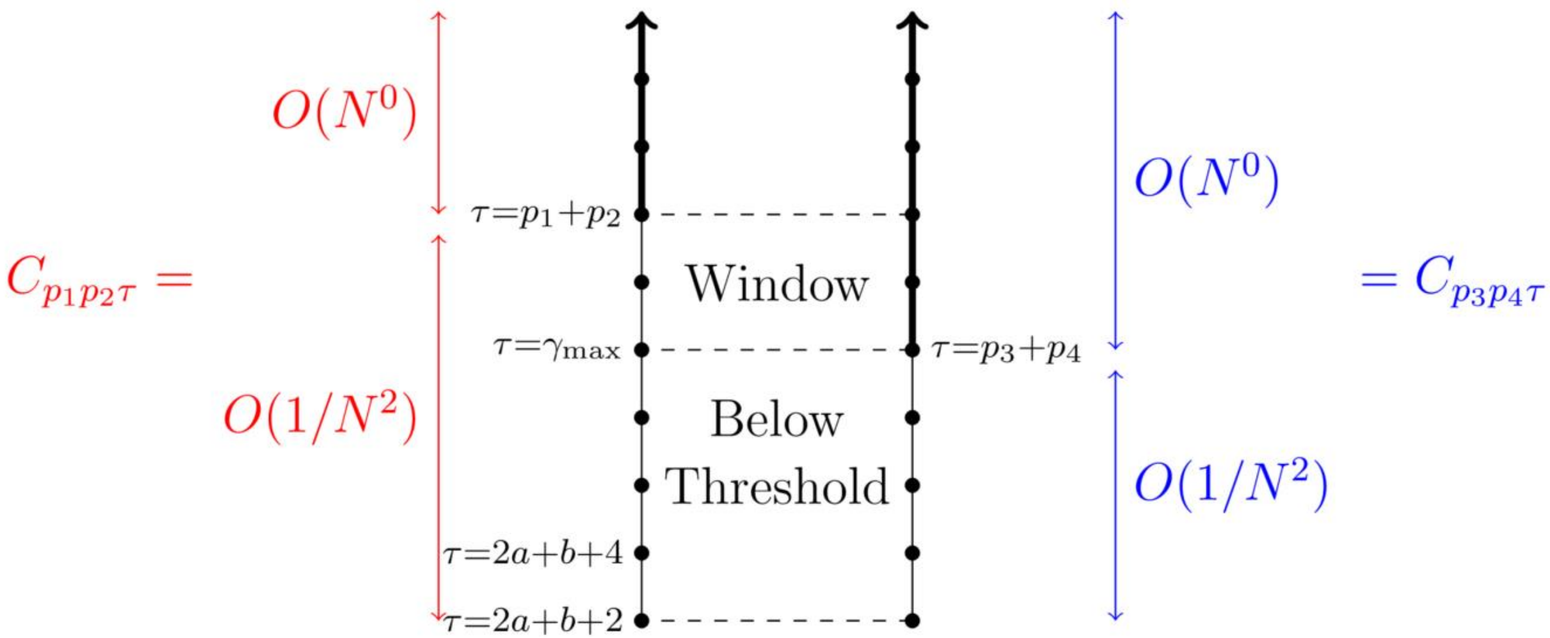



For $\tau < \min(p1+p2, p3+p4)$ $C_{p1\ p2\ O\tau} \sim C_{p3\ p4\ O\tau} \sim 1/N^2$ -> prediction at $O(1/N^4)$

# Higher charge correlators at 1 loop O(1/N$^4$)

Predictions for single disc in the "window":

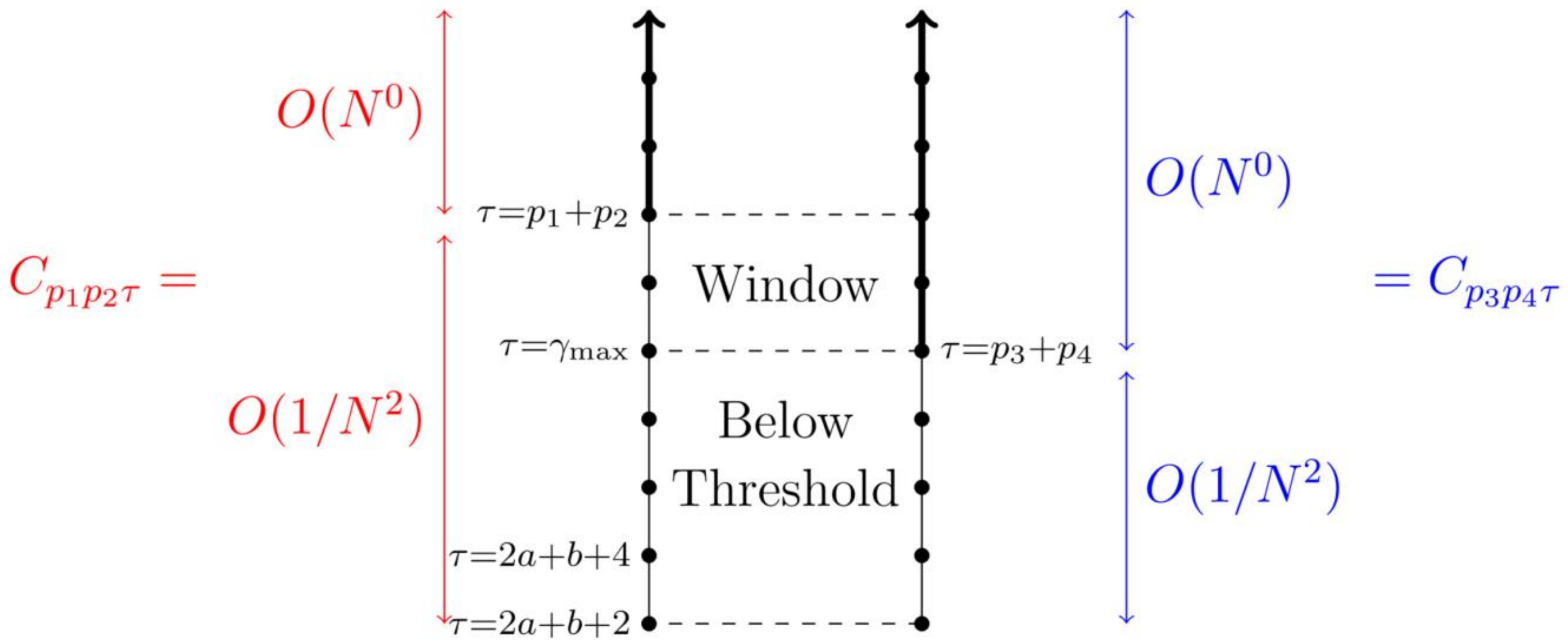
$$(\mathcal{P}\mathcal{I})\,\mathcal{H}^{(2)}_{\vec{p}} \supset \log^1 u \sum_{\substack{\tau\in\mathcal{W}_{\vec{p}} \\ (pq)\in\mathcal{R}_{\tau,l,a,b}}} C^{(1)}_{p_1p_2K_{pq}}\eta_{K_{pq}}C^{(0)}_{p_3p_4K_{pq}}\,\mathbb{L}_{K_{pq}}$$



# Higher charge correlators at 1 loop O(1/N⁴)

Predictions for analytic part below threshold

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle \supset \frac{\log^0 u}{N^4} \sum_{\tau\geq 4+2a+b}^{\min(p_1+p_2,p_3+p_4)-2} \left[ \sum_{(pq)\in\mathcal{R}_{\tau,l,a,b}} C^{(1)}_{p_1p_2K_{pq}} C^{(1)}_{p_3p_4K_{pq}} \mathbb{L}_{K_{pq}} \right]$$



# Derivation of tree amplitudes

- Problem: Minimal ansatz based on ddisc --> consistent with below threshold predictions, only if free theory is not present at O(1/N^4)

○ Solution 1: Widen the ansatz (higher powers of 1/v). -> [Very (too) big ansatz]
○ Solution 2: Use "generalised tree" to cancel free theory below threshold.
○ Rastelli Zhou tree amplitudes unique solution to the following problem:

Bloch-Wigner dilog

$$\mathcal{H}^{(1)} = \frac{\mathbf{P}_{\square}(x,\bar{x},y,\bar{y})}{(x-\bar{x})^{\mathbf{d}_1-1}}\phi^{(1)}(x,\bar{x}) + \frac{\mathbf{P}_u(x,\bar{x},y,\bar{y})}{(x-\bar{x})^{\mathbf{d}_1-2}}\frac{\log(u)}{v^{\mathbf{d}_2}} + \frac{\mathbf{P}_v(x,\bar{x},y,\bar{y})}{(x-\bar{x})^{\mathbf{d}_1-2}}\log(v) + \frac{\mathbf{P}_1(x,\bar{x},y,\bar{y})}{(x-\bar{x})^{\mathbf{d}_1-4}}\frac{1}{v^{\mathbf{d}_2}}$$

$$\mathbf{d}_1 = p_1+p_2+p_3+p_4$$
$$\mathbf{d}_2 = p_3 - 1 + \min(0, \tfrac{p_1+p_2-p_3-p_4}{2})$$

Obeying:

1. Consistent under crossing (sets the degree of the polynomials)
2. Non singular as x->$\bar{x}$
3. Cancellation of below threshold operators from free theory at $O(1/N^2)$

[Alternative way to derive all tree-level correlators]

# Generalised tree amplitudes

To obtain "generalised tree" amplitudes, increase the ansatz by two powers of $x-\bar{x}$

Box

$$
\begin{aligned}
\mathcal{H}^{(1)} &= \frac{\mathbf{P}_{\square}(x,\bar{x},y,\bar{y})}{(x-\bar{x})^{\mathbf{d}_1+1}}\phi^{(1)}(x,\bar{x}) + \\
&\quad \frac{\mathbf{P}_u(x,\bar{x},y,\bar{y})}{(x-\bar{x})^{\mathbf{d}_1}}\frac{\log(u)}{v^{\mathbf{d}_2}} + \frac{\mathbf{P}_v(x,\bar{x},y,\bar{y})}{(x-\bar{x})^{\mathbf{d}_1}}\log(v) + \\
&\quad + \frac{\mathbf{P}_1(x,\bar{x},y,\bar{y})}{(x-\bar{x})^{\mathbf{d}_1-2}}\frac{1}{v^{\mathbf{d}_2}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{d}_1 &= p_1+p_2+p_3+p_4 \\
\mathbf{d}_2 &= p_3-1+\min(0,\tfrac{p_1+p_2-p_3-p_4}{2})
\end{aligned}
$$

Obeying:

1. Crossing
2. Non singular as $x\to\bar{x}$
3. Cancellation of below threshold operators from entire all N free theory

New tree-like functions linked to any free theory (can leave arbitrary coeffs in front of free theory propagator structures)

- Note below threshold ops are not absent at $O(1/N^4)$!?

- But quite remarkably, if we add the generalised tree at $O(1/N^4)$ (thus cancelling the below threshold operators from free theory)
- then:

Function obtained from "minimal ansatz consistent with ddisc" is consistent with $O(1/N^4)$ below threshold corrections

Why? Meaning of generalised tree?

# Semi-short sector

Final subtlety:

- Operators $O_{\tau,l,[aba]}$ with $\tau$=2a+b+2 are in semi-short multiplets
- Ambiguity due to potential recombination: at superblock level:
- $$\mathbb{L}^{\{p_i\}}_{\tau,l,[a,b,a]} = \mathbb{S}^{\{p_i\}}_{\tau,[l+2,1^a]} + \mathbb{S}^{\{p_i\}}_{\tau+2,[l+1,1^{a+1}]} \qquad \tau = 2a+b+2$$

Long superblock

HWS: τ,l,[a,b,a] (semishort)

HWS: τ+2,l-1,[a+1,b,a+1] (semishort)

- If recombination takes place at weak coupling, at strong coupling long operator disappears from spectrum (string state)
- BUT there can be (and are) left over semishort operators
- One can in fact reconcile the ambiguity purely from the free theory
- Intricate recursive (in twist) algorithm involving many free correlators

Bootstrapping the one loop correlator (with minimal ansatz) consistent with semishort prediction!

# Properties of the data.

- All data satisfies a non-trivial symmetry as analytic functions of $\ell$. Trajectories in $\ell$. [swap even & odd spin trajectories and order of degenerate ops $i \to t-i$]

$$\boxed{\ell \to -\ell - T(\ell;a) - 3}$$

Full (anomalous) twist

Equivalent to reciprocity [Basso, Korchemsky; Alday, Bissi]

Eg 2 twist 6 singlets swap under the symmetry

$$\eta_1^{(2)}(-\ell-9)|_{[0,0,0]} = \eta_2^{(2)}(\ell)|_{[0,0,0]} - \frac{\partial}{\partial\ell}\left(\eta_2^{(1)}(\ell)|_{[0,0,0]}\right)^2,$$

$$\eta_2^{(2)}(-\ell-9)|_{[0,0,0]} = \eta_1^{(2)}(\ell)|_{[0,0,0]} - \frac{\partial}{\partial\ell}\left(\eta_1^{(1)}(\ell)|_{[0,0,0]}\right)^2.$$