The Probe of Curvature in the Lorentzian AdS_2/CFT_1 Correspondence

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OPE Block

The OPE block $B_l^{jk}(x_1, x_2)$ follows from the operator product expansion (OPE) of the operators $\mathcal{O}_j(x_1)$ and $\mathcal{O}_k(x_2)$

$$\mathcal{O}_j(x_1)\mathcal{O}_k(x_2) = \sum_l c_{jkl}(x_1 - x_2, \partial)\mathcal{O}_l(x_2), \quad (1)$$

in which c_{jkl} takes into account the contribution from the descendants of operator $\mathcal{O}_k(x_1, x_2)$. The OPE block is defined as the contribution from a particular channel of primary operator to the OPE of the operators $\mathcal{O}_j(x_1)$ and $\mathcal{O}_k(x_2)$

$$\mathcal{O}_j(x_1)\mathcal{O}_k(x_2) \equiv |x_1 - x_2|^{-\Delta_j - \Delta_k} \sum_l C_{jkl} B_l^{jk}(x_1, x_2), \quad (2)$$

where Δ_j and Δ_k are conformal dimensions of the operators, $\mathcal{O}_j$ and $\mathcal{O}_k$, and C_{jkl} are the OPE coefficients.

Randon Transformation

- The Randon transformation (Bulk field $\phi(x) \to \hat{\phi}(\gamma)$)

$$\hat{\phi}(\gamma) \equiv \int_{x\in\gamma} d\sigma(\gamma)\ \phi(x), \tag{3}$$

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- The field is then given by the OPE block associated to the primary operator O_k

$$\hat{\phi}(\gamma) \sim B_l^{jk}. \tag{4}$$

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- The Lorentzian AdS_2 metric is

$$ds_{2l}^2 = \frac{-dt^2 + dz^2}{z^2}. \tag{5}$$

The light cone of a boundary point becomes a single light ray in the bulk. Therefore, the past light ray of the boundary point τ_2 and the future light ray of the boundary point τ_1 (assuming $\tau_2 > \tau_1$) meet at the following bulk point in the Lorentzian AdS_2 metric. In general, the bulk point is determined by:

$$t = \frac{1}{2}(\tau_1 + \tau_2), \qquad z = \frac{1}{2}|\tau_1 - \tau_2|. \tag{6}$$

- In summary, the OPE block or the codimension-two surface operator from two boundary operators at τ_1 and τ_2 becomes a bulk local operator, whose position is uniquely determined by (6).

Reference of the OPE Block

- B. Czech, L. Lamprou, S. McCandlish, B. Mosk and J. Sully, "A Stereoscopic Look into the Bulk," JHEP **1607**, 129 (2016) [arXiv:1604.03110 [hep-th]].

Modular Hamiltonian

- Since the modular Hamiltonian $H_{\rm mod}$ is hermitian, this can be diagonalized as $H_{\rm mod} \equiv U^\dagger \mathcal{D} U$, where U is unitary, and $\mathcal{D}$ is a diagonal matrix.

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- The modular Berry transport is

$$\begin{aligned} \frac{\partial H_{\rm mod}}{\partial\lambda} &= U^\dagger\left(\frac{\partial\mathcal{D}}{\partial\lambda}\right)U + \left[\left(\frac{\partial U^\dagger}{\partial\lambda}\right)U, H_{\rm mod}\right], \\ P_0\left(\frac{\partial U^\dagger}{\partial\lambda}U\right) &= 0, \end{aligned} \tag{7}$$

where the projection P_0 is onto the zero-modes (Hermitian operators that commute with $H_{\rm mod}$).

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  where the projection P_0 is onto the zero-modes (Hermitian operators that commute with $H_{\rm mod}$).
- The second equation says that the transport is parallel when the tangent vector is along the horizontal subspace.

Modular Hamiltonian in CFT_1

- The modular Hamiltonian in CFT_1 can be expressed in terms of the SL(2) generators

$$H_{\mathrm{mod}} = s_1 L_1 + s_0 L_0 + s_{-1} L_{-1}, \tag{8}$$

where

$$L_{-1} \equiv i\partial_\tau, \qquad L_0 \equiv -\tau\partial_\tau, \qquad L_1 \equiv -i\tau^2\partial_\tau, \tag{9}$$

$$\begin{aligned} s_1 &\equiv \frac{2\pi}{\tau_2 - \tau_1}, \qquad s_0 \equiv \frac{-2\pi i(\tau_1 + \tau_2)}{\tau_2 - \tau_1}, \\ s_{-1} &\equiv \frac{-2\pi\tau_1\tau_2}{\tau_2 - \tau_1}. \end{aligned} \tag{10}$$

- Since one modular Hamiltonian can be mapped to other modular Hamiltonian from the conformal transformation, the equation can reduce to

$$\frac{\partial H_{\mathrm{mod}}}{\partial \lambda} = \left[\frac{\partial U^\dagger}{\partial \lambda} U, H_{\mathrm{mod}}\right], \qquad P_0\left(\frac{\partial U^\dagger}{\partial \lambda} U\right) = 0. \tag{11}$$

- With the help of the following algebra:

$$\begin{aligned} [H_{\rm mod}, H_{\rm mod}] &= 0, \\ [H_{\rm mod}, \partial_{\tau_1} H_{\rm mod}] &= -2\pi i \partial_{\tau_1} H_{\rm mod}, \\ [H_{\rm mod}, \partial_{\tau_2} H_{\rm mod}] &= 2\pi i \partial_{\tau_2} H_{\rm mod}, \end{aligned} \tag{12}$$

we can solve the modular Berry curvature equation, and this leads to

$$\partial_\lambda H_{\rm mod} = [V_{\delta\lambda}, H_{\rm mod}], \tag{13}$$

where

$$V_{\delta\lambda} \equiv \frac{1}{2\pi i}\big((\partial_\lambda \tau_1)(\partial_{\tau_1} H_{\rm mod} - (\partial_\lambda \tau_2)(\partial_{\tau_2} H_{\rm mod})\big). \tag{14}$$

- Therefore, we define the covariant derivative

$$D_\lambda H \equiv \partial_\lambda H - [V_{\delta\lambda}, H] \tag{15}$$

and the commutator for derivatives along directions $\lambda = \tau_1$ and $\lambda = \tau_2$ reads

$$[D_{\tau_1}, D_{\tau_2}]H = \frac{i}{\pi(\tau_2 - \tau_1)^2}[H_{\mathrm{mod}}, H], \tag{16}$$

which leads to the Berry curvature tensor

$$\mathcal{R}_{\tau_1\tau_2} \equiv \frac{i}{\pi(\tau_2 - \tau_1)^2} H_{\mathrm{mod}}. \tag{17}$$

This also provides the following curvature

$$\mathcal{R}_{z\tau} = -\frac{i}{2\pi z_0^2} H_{\mathrm{mod}}. \tag{18}$$

Here we define $\tau_0 \equiv (\tau_1 + \tau_2)/2$ and $z_0 \equiv (\tau_2 - \tau_1)/2$. The subscript 0 of τ_0 and z_0 means that we fix the variables.

- Now we extend the SL(2) generators from the boundary to the bulk for getting the AdS_2 Riemann curvature tensor:

$$\begin{aligned} L_1 &= -2i\tau z\partial_z - i(\tau^2 + z^2)\partial_\tau, \\ L_0 &= -z\partial_z - \tau\partial_\tau, \\ L_{-1} &= i\partial_\tau. \end{aligned} \tag{19}$$

- From the following commutator relations:

$$\begin{aligned} [L_1, \partial_z] &= 2i\tau\partial_z + 2iz\partial_\tau, \qquad [L_0, \partial_z] = \partial_z, \\ [L_{-1}, \partial_z] &= 0, \end{aligned} \tag{20}$$

$$\begin{aligned} [L_1, \partial_\tau] &= 2iz\partial_z + 2i\tau\partial_\tau, \qquad [L_0, \partial_\tau] = \partial_\tau, \\ [L_{-1}, \partial_\tau] &= 0, \end{aligned} \tag{21}$$

we can find that the diagonal entries of modular Hamiltonian (as a two by two matrix on the tangent vectors) vanish at the point (τ_0, z_0), and the off-diagonal ones are symmetric and are $2\pi i$. Therefore, the Riemann curvature at the point z_0 is:

$$\mathcal{R}_{z\tau} = -\frac{i}{2\pi z_0^2} H_{\mathrm{mod}}\Big|_{z=z_0} = \frac{1}{z_0^2} = -\mathcal{R}_{\tau z}. \tag{22}$$

- The AdS$_2$ Riemann curvature tensor:

$$\begin{aligned} R^{\rho}{}_{\sigma\mu\nu} &\equiv \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}, \\ \Gamma^{\mu}{}_{\nu\delta} &\equiv \frac{1}{2} g^{\mu\lambda}(\partial_\delta g_{\lambda\nu} + \partial_\nu g_{\lambda\delta} - \partial_\lambda g_{\nu\delta}) \end{aligned} \tag{23}$$

exactly corresponds to the curvature $\mathcal{R}$ at the point z_0:

$$\mathcal{R}_{z\tau} \to R^{z}{}_{\tau z \tau} = \frac{1}{z_0^2}, \qquad \mathcal{R}_{\tau z} \to R^{z}{}_{\tau\tau z} = -\frac{1}{z_0^2}. \tag{24}$$

Reference of the Curvature

- B. Czech, L. Lamprou, S. Mccandlish and J. Sully, "Modular Berry Connection for Entangled Subregions in AdS/CFT," Phys. Rev. Lett. **120**, no. 9, 091601 (2018) [arXiv:1712.07123 [hep-th]].
- B. Czech, J. De Boer, D. Ge and L. Lamprou, "A modular sewing kit for entanglement wedges," JHEP **1911**, 094 (2019) [arXiv:1903.04493 [hep-th]].

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- In the CFT_1 case, the OPE block is a bulk local operator because the co-dimensional two surface is a point.
- We probed the AdS_2 Riemann curvature tensor using the holonomy of the modular Hamiltonian. Because this tensor only has one physical degree of freedom, and we can directly study the AdS_2 space, we explicitly confirmed the relation between the modular Berry transport and the curvature.