Does Boundary Distinguish Complexities?

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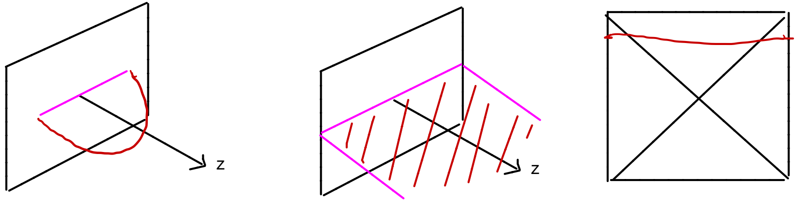
Introduction

In AdS/CFT, geometric quantities in AdS are related with entanglement in CFT.

(e.g. co-dim 2 surface in AdS = entanglement entropy [Ryu-Takayanagi '06])



We can also consider another geometric quantity, co-dim 1 volume in AdS.

Q: What is holographic dual?

1. In two-sided BH, $V \sim t$ at late time
2. "computational complexity" shows similar behaviour

"Complexity = Volume" (CV) conjecture [Susskind '14]

complexity $= \frac{V}{G_N L}$ (V : co-dim 1 maximum volume, L : length scale)

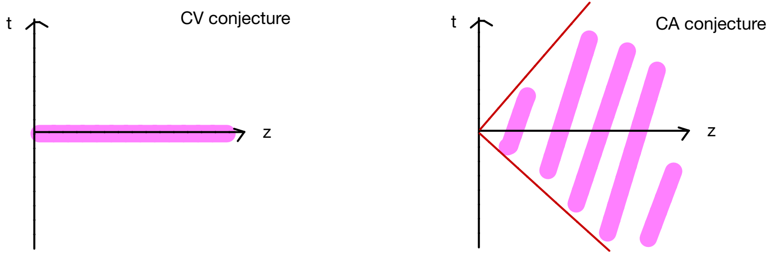
The CV conjecture follows from the observation of the late time behaviour.

- Q: Is it unique?
- A: No.
  The action of Wheeler de-Witt patch also shows the same late time behaviour.

“Complexity = Action” (CA) conjecture [Brown-Roberts-Susskind-Swingle-Zhao ’15]

$$\text{complexity} = \frac{I_{\text{WDW}}}{\pi} \qquad (I_{\text{WDW}} : \text{Wheeler de-Witt patch action})$$

- The CA conjecture does not depend on the length scale introduced by hand.



"Complexity = Volume" (CV) conjecture [Susskind '14]

$$\text{complexity} = \frac{V}{G_N L} \qquad (V : \text{co-dim 1 maximum volume}, L : \text{length scale})$$

"Complexity = Action" (CA) conjecture [Brown-Roberts-Susskind-Swingle-Zhao '15]

$$\text{complexity} = \frac{I_{WDW}}{\pi} \qquad (I_{WDW} : \text{WDW action})$$

- Two proposals basically give us almost the same results, e.g. late time behaviours, divergence structures and so on.
- Can we distinguish them?
  It is really odd that there exist two different holographic duals for one notion in QFT.

Recently, Chapman *et. al.* argue that [Chapman-Ge-Policastro '18]

defect distinguishes action from volume!!

Gravity computation: $AdS_3/DCFT_2$ model [Azeyanagi-Karch-Takayanagi-Thompson '07]

$$\Delta C_{\mathrm{V}}^{\mathrm{defect}} := C_{\mathrm{V}}^{\mathrm{DCFT}} - C_{\mathrm{V}}^{\mathrm{CFT}} \neq 0\,, \qquad \Delta C_{\mathrm{A}}^{\mathrm{defect}} := C_{\mathrm{A}}^{\mathrm{DCFT}} - C_{\mathrm{A}}^{\mathrm{CFT}} = 0$$

CFT computation: Circuit complexities for specific DCFT models vanish,

candidate of definition

$$\Delta C_{\mathrm{circuit}}^{\mathrm{defect}} := C_{\mathrm{circuit}}^{\mathrm{DCFT}} - C_{\mathrm{circuit}}^{\mathrm{CFT}} = 0$$

Their argument: CA complexity is a good holographic dual

However, there are some concerns

- The holographic computation has been done in a special dimension
- There are many candidates of definition of complexity in QFT

Our work

- Our motivation is whether their argument hold in other holographic models or in other definitions of complexity or not.
- We study boundary conformal field theory (BCFT) because DCFT can be constructed from two copies of BCFTs via doubling trick.
- To detect boundary contributions, we study a difference of complexity due to boundary, $\Delta C^{\rm bdy} := C^{\rm BCFT} - C^{\rm CFT}/2$.

Our results

- Path-integral optimization ← another definition [Caputa *et. al.* '17]

$$\Delta C_{\rm L}^{\rm bdy} := C_{\rm L}^{\rm BCFT} - \frac{1}{2} C_{\rm L}^{\rm CFT} = \frac{c}{6\pi} \alpha \log\left(\frac{z_\infty}{\epsilon}\right)$$

- $\mathrm{AdS}_{d+1}/\mathrm{BCFT}_d$ [Takayanagi '11] : $\Delta C_{\rm V}^{\rm bdy} \sim \Delta C_{\rm A}^{\rm bdy} \propto \frac{1}{\epsilon^{d-2}}$
- $\mathrm{AdS}_3/\mathrm{BCFT}_2$: $\Delta C_{\rm V}^{\rm bdy} = \frac{\alpha L}{G_{\rm N}} \log\left(\frac{z_\infty}{\epsilon}\right)$, $\Delta C_{\rm A}^{\rm bdy} = \frac{L}{4\pi G_{\rm N}} \left(\sqrt{1+\alpha^2} - 1\right)$

Plan of my talk

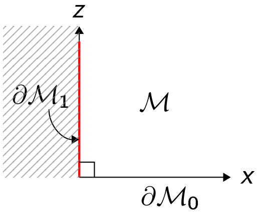
1. Introduction
2. Path-integral Optimization
3. AdS/BCFT model
    - CV conjecture
    - CA conjecture
4. Conclusion

Path-integral Optimization

[Caputa *et. al.* '17]

- Ground state wave functional

$$\Psi^{\text{BCFT}}_{\delta_{ab}}[\tilde{\varphi}(x)] = \int_{\mathcal{M}} \mathcal{D}\varphi\, \mathrm{e}^{-S_{\text{BCFT}}[\varphi]} \prod_{x>0} \delta(\varphi(\epsilon, x) - \tilde{\varphi}(x))$$



- We can obtain the same wave functional $\Psi^{\text{BCFT}}_{\delta_{ab}}$ by reducing some high-energy degrees of freedom in the deep region of $\mathcal{M}$ in path integral.

- To reduce such degrees of freedom, we deform the background metric with a boundary condition keeping the wave functional.

- In CFT_2, it can be realized by Weyl transformation,

$$\delta_{ab} \;\to\; \mathrm{e}^{2\phi}\delta_{ab} \quad \Rightarrow \quad \Psi^{\mathrm{BCFT}}_{\mathrm{e}^{2\phi}\delta_{ab}}[\tilde{\varphi}(x)] = \mathrm{e}^{S_{\mathrm{L}}[\phi]-S_{\mathrm{L}}[0]}\,\Psi^{\mathrm{BCFT}}_{\delta_{ab}}[\tilde{\varphi}(x)]$$

with

$$S_{\mathrm{L}}[\phi] = \frac{c}{24\pi}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\left(R\phi + (\partial\phi)^2 + \mu\mathrm{e}^{2\phi}\right) + \frac{c}{12\pi}\sum_i \int_{\partial\mathcal{M}_i} \mathrm{d}s\sqrt{h}\left(K\phi + \mu_{\mathrm{B}}^{(i)}\mathrm{e}^{\phi}\right)$$

- The overall factor reflects how much redundant degrees of freedom (or lattice sites) can be reduced.

Optimized complexity: $C_{\mathrm{L}} = S_{\mathrm{L}}\big|_{\text{on-shell}}$ where S_{L} is Liouville action

- The Liouville action leads
  - Equation of motion:

$$-2\partial^2\phi + 2\mu\mathrm{e}^{2\phi} = 0$$

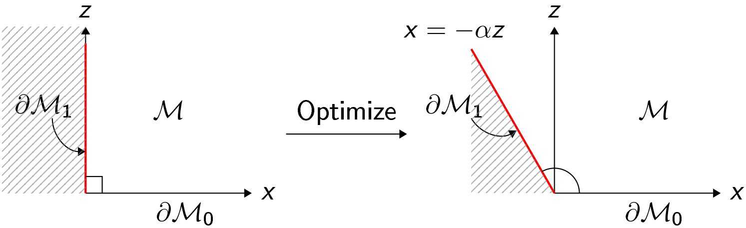
  - Boundary condition:

$$n\cdot\partial\phi + \mu_{\mathrm{B}}^{(i)}\mathrm{e}^{\phi} = 0$$

- The path-integral optimization leads the time slice of the Poincaré AdS,

$$ds^2 = e^{2\phi}\delta_{ab}dx^a dx^b = L^2 \frac{dz^2 + dx^2}{z^2}$$

with boundary $x = -\alpha z$ in the radial direction.



- This is the same geometry appearing in Takayanagi's AdS/BCFT model.
- Optimized complexity

$$\Delta C_{\mathrm{L}}^{\mathrm{bdy}} = C_{\mathrm{L}}^{\mathrm{BCFT}} - \frac{1}{2} C_{\mathrm{L}}^{\mathrm{CFT}} = \frac{c}{6\pi}\alpha \log\left(\frac{z_\infty}{\epsilon}\right)$$

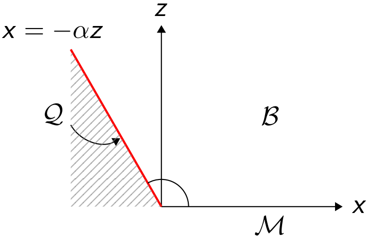
AdS/BCFT model [Takayanagi '11]

- Metric : $\mathrm{d}s^2 = G_{MN}\mathrm{d}X^M\mathrm{d}X^N = L^2\frac{\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}{z^2}$
- Action :

$$8\pi G_N I = \frac{1}{2}\int_{\mathcal{B}}\sqrt{-G}\left(R + \frac{d(d-1)}{L^2}\right) + \int_{\mathcal{Q}}\sqrt{-\hat{G}}\,(K - T) + \int_{\mathcal{M}}\sqrt{-\hat{G}}\,K$$

We introduce the boundary $\mathcal{Q}$ with a brane of tension $T = \frac{d-1}{L}\frac{\alpha}{\sqrt{1+\alpha^2}}$.

$\Longrightarrow$ the isometry of the metric is reduced from $SO(2,d)$ to $SO(1,d)$.



CV complexity

CV conjecture [Susskind '14]

$$C_{\mathrm{V}} = \frac{V}{G_{\mathrm{N}} L} \qquad (V : \text{co-dim maximal volume}, \ L : \text{length scale})$$

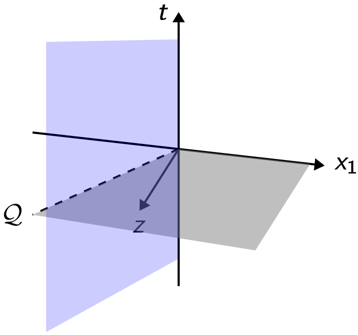
The the maximum volume, V, at $t = 0$ is a $t = 0$ time-slice,

$$V = \int_{\epsilon}^{\infty} \mathrm{d}z \int_{-\alpha z}^{\infty} \mathrm{d}x_1 \int \prod_{i=2}^{d-1} \mathrm{d}x_i \frac{L^d}{z^d}$$

$$= \frac{1}{2} V_{d-1} L^d \int_{\epsilon}^{\infty} \frac{\mathrm{d}z}{z^d} + \alpha \frac{L^d V_{d-2}}{(d-2)\epsilon^{d-2}}$$

$$\Longrightarrow \Delta C_{\mathrm{V}}^{\mathrm{bdy}} = C_{\mathrm{V}}^{\mathrm{BCFT}} - \frac{1}{2} C_{\mathrm{V}}^{\mathrm{CFT}}$$

$$= \alpha \frac{L^{d-1} V_{d-2}}{(d-2) G_{\mathrm{N}} \epsilon^{d-2}}$$

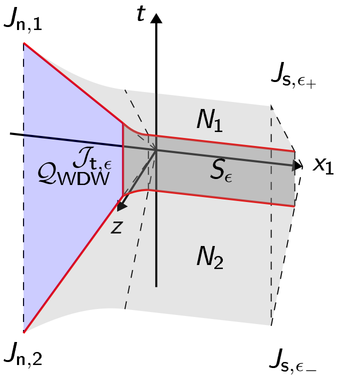


CA complexity

CA conjecture [Brown-Roberts-Susskind-Swingle-Zhao '15]

$$C_A = \frac{I_{WDW}}{\pi}$$

- Wheeler de-Witt patch = region surrounded by light rays emanating from $t = 0$ & $z = 0$.
- For $x_1 \geq 0$, WDW patch is surrounded by null surfaces, $z = \pm t$.
- For $x_1 < 0$, WDW patch is surrounded by null surfaces, $\sqrt{z^2 + x_1^2} = \pm t$.

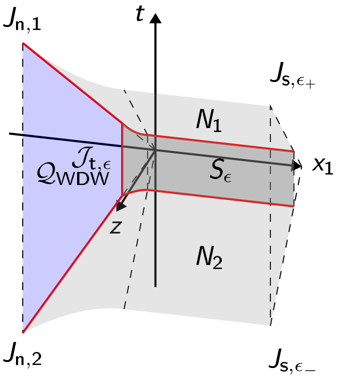


CA complexity

CA conjecture [Brown-Roberts-Susskind-Swingle-Zhao '15]

$$C_A = \frac{I_{\mathrm{WDW}}}{\pi}$$

$$
\begin{aligned}
8\pi G_N I_{\mathrm{WDW}} = {} & \frac{1}{2}\int_{\mathcal{B}_{\mathbf{WDW}}} \sqrt{-G}\left(R + \frac{d(d-1)}{L^2}\right) \\
& + \int_{\mathcal{Q}_{\mathbf{WDW}}} \sqrt{-\hat{G}}\,(K - T) + \sum_{i=\epsilon,\infty} \int_{S_i} \mathrm{d}^d X \sqrt{-\hat{G}}\, K \\
& + \sum_{i=1}^{2} \epsilon_\kappa \left(\int_{N_i} \mathrm{d}\lambda \mathrm{d}\boldsymbol{x} \sqrt{\gamma}\kappa + \int_{N_i} \mathrm{d}\lambda \mathrm{d}\boldsymbol{x} \sqrt{\gamma}\Theta \log(\ell_{\mathrm{ct}}|\Theta|) \right) \\
& + \sum_{J} \epsilon_a \int_J \mathrm{d}^{d-1} X \sqrt{h} a + \sum_{\mathcal{J}} \epsilon_\phi \int_{\mathcal{J}} \mathrm{d}^{d-1} X \sqrt{-h}\phi \,.
\end{aligned}
$$



- $\underline{d > 2}$:

$$\Delta C_A^{\text{bdy}} = \frac{L^{d-1} V_{d-2}}{8\pi^2 G_N \epsilon^{d-2}} (d-2) \left(\alpha \sqrt{1+\alpha^2} + \operatorname{arcsinh} \alpha \right) + \frac{L^{d-1} V_{d-2}}{4\pi^2 G_N \epsilon^{d-2}} \log \left(\frac{\ell_{\text{ct}}(d-2)}{L} \right) \operatorname{arcsinh} \alpha + \frac{L^{d-1} V_{d-2}}{4\pi^2 G_N \epsilon^{d-2}} \left(\sqrt{1+\alpha^2} \arccos \left(\frac{\alpha}{\sqrt{1+\alpha^2}} \right) - \frac{\pi}{2} \right)$$

Contributions from $z = \infty$ are ignored.

For $d > 2$, the divergence structure is the same as CV.

- $\underline{d = 2}$:

$$\Delta C_A^{\text{bdy}} = \frac{L}{4\pi G_N} \left(\sqrt{1+\alpha^2} - 1 \right)$$

The boundary complexity does not diverge.

Conclusion

- We study the boundary complexity $\Delta C^{\text{bdy}} = C^{\text{BCFT}} - C^{\text{CFT}}/2$.
- By applying the path-integral optimization to BCFT_2, we obtained

$$\Delta C_{\text{L}}^{\text{bdy}} = \frac{c}{6\pi}\alpha \log\left(\frac{z_\infty}{\epsilon}\right)$$

$\Rightarrow$ imply that vanishment of ΔC^{bdy} (or ΔC^{defect}) depends on the definition of the complexity or models in BCFT (or DCFT).

- In $\text{AdS}_{d+1}/\text{BCFT}_d$ model, we showed

$$\Delta C_{\text{V}}^{\text{bdy}} \sim \Delta C_{\text{A}}^{\text{bdy}} \propto 1/\epsilon^{d-2}$$

- Especially, in $\text{AdS}_3/\text{BCFT}_2$ model,

$$\Delta C_{\text{V}}^{\text{bdy}} = \frac{\alpha L}{G_{\text{N}}} \log\left(\frac{z_\infty}{\epsilon}\right), \qquad \Delta C_{\text{A}}^{\text{bdy}} = \frac{L}{4\pi G_{\text{N}}}\left(\sqrt{1+\alpha^2}-1\right)$$

$\Longrightarrow$ Boundary (or defect) can not detect the definite difference of CV and CA except a special case.

Thank you for your attention!