Leptonic mixing, sum rules and the Dirac CP violating phase

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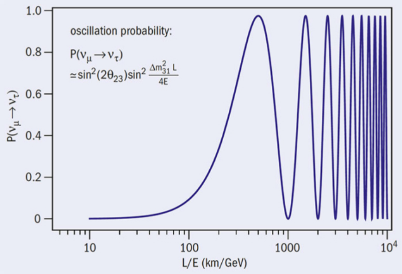
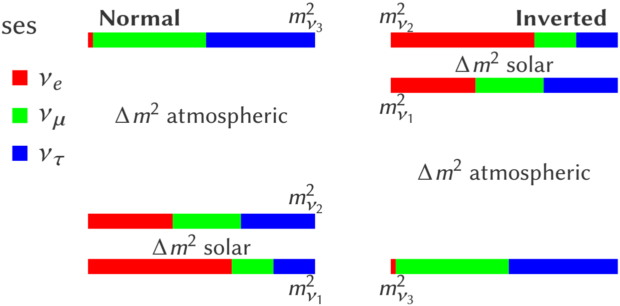
[1]Based on: Delgadillo, Everett, Ramos, Stuart, PRD **97**, 095001 [1801.06377];
Everett, Ramos, Rock, Stuart, **work in progress**.

Massive neutrinos



- Neutrino flavour oscillation
- Mass ordering
- Small neutrino masses



Leptonic mixing

Charged current interactions in mass basis

$$-\mathcal{L}_{\mathrm{CC}} = \frac{g}{\sqrt{2}}(\bar{e}_L, \bar{\mu}_L, \bar{\tau}_L)\gamma^\mu U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} W^+_\mu + \text{h.c.}$$

$$U_{\mathrm{MNSP}} \equiv U = U_e^\dagger U_\nu$$

Most popular parameterization given by the Particle Data Group

$$U^{\mathrm{PDG}} = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & e^{-i\delta}s_{13} \\ -c_{23}s_{12} - c_{12}s_{23}e^{i\delta}s_{13} & c_{12}c_{23} - s_{12}s_{23}e^{i\delta}s_{13} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}e^{i\delta}s_{13} & -c_{12}s_{23} - c_{23}s_{12}e^{i\delta}s_{13} & c_{13}c_{23} \end{pmatrix},$$

$$s_{ij} = \sin\theta_{ij}, \quad c_{ij} = \cos\theta_{ij}$$

Leptonic mixing

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First proof of neutrino oscillation:
"$\sin^2 2\theta > 0.82$ and $5 \times 10^{-4} < \Delta m^2 < 6 \times 10^{-3}$ eV^2"
Super-Kamiokande [hep-ex/9807003]

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Leptonic mixing

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U_{MNSP}

Non-zero θ_{13} mixing angle confirmed: Double Chooz, 1112.6353; Daya Bay, 1203.1669; RENO, 1204.0626

Most popular parameterization given by the Particle Data Group

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Leptonic mixing

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$$-\mathcal{L}_{\rm CC} = \frac{g}{\sqrt{2}}(\bar{e}_L, \bar{\mu}_L, \bar{\tau}_L)\gamma^\mu U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} W_\mu^+ + \text{h.c.}$$

$U_{\rm MNS}$

Most popular parameterization g

A non-zero θ_{13} opens the possibility of measuring the *CP*-violating δ phase

$$U^{\rm PDG} = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & e^{-i\delta}s_{13} \\ -c_{23}s_{12} - c_{12}s_{23}e^{i\delta}s_{13} & c_{12}c_{23} - s_{12}s_{23}e^{i\delta}s_{13} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}e^{i\delta}s_{13} & -c_{12}s_{23} - c_{23}s_{12}e^{i\delta}s_{13} & c_{13}c_{23} \end{pmatrix},$$

$$s_{ij} = \sin\theta_{ij}, \quad c_{ij} = \cos\theta_{ij}$$

Flavor symmetries

High energy discrete flavor symmetry $\longrightarrow$ **residual symmetries of the leptonic mass matrices.**

- **Residual symmetries of the leptonic mass matrices** can be related to **mixing patterns** in diagonalization matrices.
- **Mixing patterns** can be classified according to their leading order predictions for the **solar mixing angle**.
- **Note:** Usually the charged leptons basis is taken **diagonal**.

Flavor symmetries

Let's start with a neutrino mixing matrix with **vanishing reactor angle at leading order**

$$U_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23}^\nu & s_{23}^\nu \\ 0 & -s_{23}^\nu & c_{23}^\nu \end{pmatrix} \begin{pmatrix} c_{12}^\nu & s_{12}^\nu & 0 \\ -s_{12}^\nu & c_{12}^\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_{12}^\nu & s_{12}^\nu & 0 \\ -s_{12}^\nu c_{23}^\nu & c_{12}^\nu c_{23} & s_{23}^\nu \\ s_{12}^\nu s_{23}^\nu & -c_{12}^\nu s_{23} & c_{23}^\nu \end{pmatrix}$$

where

$$s_{ij}^\nu = \sin\theta_{ij}^\nu, \quad c_{ij}^\nu = \cos\theta_{ij}^\nu$$

Flavor symmetries: Leading order classification

Tribimaximal (TBM): $(s_{12}^{\nu})^2 = 1/3$

$$U_{\nu} = \begin{pmatrix} \sqrt{\frac{2}{3}} & s_{12}^{\nu} & 0 \\ -\frac{1}{\sqrt{3}} c_{23}^{\nu} & \sqrt{\frac{2}{3}} c_{23} & s_{23}^{\nu} \\ \frac{1}{\sqrt{3}} s_{23}^{\nu} & -\sqrt{\frac{2}{3}} s_{23} & c_{23}^{\nu} \end{pmatrix}$$

Flavor symmetries: Leading order classification

Bimaximal (BM): $(s_{12}^{\nu})^2 = 1/2$

$$U_\nu = \begin{pmatrix} \frac{1}{\sqrt{2}} & s_{12}^{\nu} & 0 \\ -\frac{1}{\sqrt{2}} c_{23}^{\nu} & \frac{1}{\sqrt{2}} c_{23} & s_{23}^{\nu} \\ \frac{1}{\sqrt{2}} s_{23}^{\nu} & -\frac{1}{\sqrt{2}} s_{23} & c_{23}^{\nu} \end{pmatrix}$$

Flavor symmetries: Leading order classification

Hexagonal (HEX): $(s_{12}^{\nu})^2 = 1/4$

$$U_{\nu} = \begin{pmatrix} \frac{\sqrt{3}}{2} & s_{12}^{\nu} & 0 \\ -\frac{1}{2}c_{23}^{\nu} & \frac{\sqrt{3}}{2}c_{23} & s_{23}^{\nu} \\ \frac{1}{2}s_{23}^{\nu} & -\frac{\sqrt{3}}{2}s_{23} & c_{23}^{\nu} \end{pmatrix}$$

Flavor symmetries: Leading order classification

Golden ratio 1 (GR1): $(s_{12}^{\nu})^2 = (2+\phi)^{-1} = (5-\sqrt{5})/10 \approx (0.53)^2$

$$U_\nu \approx \begin{pmatrix} 0.85 & s_{12}^{\nu} & 0 \\ -0.53 c_{23}^{\nu} & 0.85 c_{23} & s_{23}^{\nu} \\ 0.53 s_{23}^{\nu} & -0.85 s_{23} & c_{23}^{\nu} \end{pmatrix}$$

Flavor symmetries: Leading order classification

Golder ratio 2 (GR2): $(s^{\nu}_{12})^2 = (3-\phi)/4 = (5-\sqrt{5})/8 \approx (0.59)^2$

$$U_{\nu} = \begin{pmatrix} 0.64 & s^{\nu}_{12} & 0 \\ -0.59 c^{\nu}_{23} & 0.64 c_{23} & s^{\nu}_{23} \\ 0.59 s^{\nu}_{23} & -0.64 s_{23} & c^{\nu}_{23} \end{pmatrix}$$

Flavor symmetries: Leading order classification

Classification according to the leading order value of the solar angle:

Mixing pattern	BM	TBM	HEX	GR1	GR2
$(s_{12}^{\nu})^2$	$1/2$	$1/3$	$1/4$	$(5-\sqrt{5})/10$	$(5-\sqrt{5})/8$

(!) A **non-zero reactor angle** θ_{13} **is incompatible** with these scenarios at **leading order**,

- **However**, sufficient corrections to the mixing angle predictions, e.g., from **perturbations to the charged lepton mixing matrix** U_e, can correct this incompatibility.

Flavor symmetries: Corrections from U_e

Consider, for example, the non-diagonal U_e

$$U_e = U_{12}^e = \begin{pmatrix} c_{12}^e & s_{12}^e e^{-i\delta_{12}^e} & 0 \\ -s_{12}^e e^{i\delta_{12}^e} & c_{12}^e & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $s_{ij}^e = \sin\theta_{ij}^e$, $c_{ij}^e = \cos\theta_{ij}^e$, then

$$U_{\mathrm{MNSP}} = U_e^\dagger U_\nu = \begin{pmatrix} c_{12}^e & -s_{12}^e e^{-i\delta_{12}^e} & 0 \\ s_{12}^e e^{i\delta_{12}^e} & c_{12}^e & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{12}^\nu & s_{12}^\nu & 0 \\ -s_{12}^\nu c_{23}^\nu & c_{12}^\nu c_{23} & s_{23}^\nu \\ s_{12}^\nu s_{23}^\nu & -c_{12}^\nu s_{23} & c_{23}^\nu \end{pmatrix}$$

Flavor symmetries: Corrections from U_e

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where $s_{ij}^e = \sin\theta_{ij}^e$, $c_{ij}^e = \cos\theta_{ij}^e$, then

$$U_{\rm MNSP} = U_e^\dagger U_\nu = \begin{pmatrix} c_{12}^e & \boxed{-s_{12}^e e^{-i\delta_{12}^e}} & 0 \\ s_{12}^e e^{i\delta_{12}^e} & c_{12}^e & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{12}^\nu & s_{12}^\nu & 0 \\ -s_{12}^\nu c_{23}^\nu & c_{12}^\nu c_{23} & \boxed{s_{23}^\nu} \\ s_{12}^\nu s_{23}^\nu & -c_{12}^\nu s_{23} & c_{23}^\nu \end{pmatrix}$$

$$(U_{\rm MNSP})_{e3} = -s_{23}^\nu s_{12}^e e^{-i\delta_{12}^e}$$

Sum rules

First we require that

$$U_{MNSP} = U^{PDG}$$

then we require that all the ratios

$$\frac{\left|(U^{PDG})_{lj}\right|}{\left|(U^{PDG})_{l'j'}\right|} = \frac{\left|(U_{MNSP})_{lj}\right|}{\left|(U_{MNSP})_{l'j'}\right|}$$

hold simultaneously, **all the** $9!/(7!2!) = 36$

Sum rules

The ratios result in relations of the form:

$$|t_{12}^{\nu}| = \frac{|s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}|}{|c_{12}s_{23} + c_{23}s_{12}s_{13}e^{i\delta}|}, \quad s_{12}^{\nu}t_{23}^{\nu} = \frac{|s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}|}{|c_{13}c_{23}|}, \quad (t_{12}^{e})^2 = cs_{23}^2 t_{13}^2$$

$$c_{12}^{\nu}t_{23}^{\nu} = \frac{|c_{12}s_{23} + c_{23}s_{12}s_{13}e^{i\delta}|}{|c_{13}c_{23}|}, \quad (s_{12}^{e})^2(t_{23}^{\nu})^2 = sc_{23}^2 t_{13}^2, \quad (c_{12}^{e})^2(t_{23}^{\nu})^2 = t_{23}^2,$$

$$\ldots$$

That can be put into equations like

$$\begin{aligned}
2\cos\delta &= (c_{12}^{\nu})^2 \left[cs_{13}ct_{12}t_{23}(t_{12}^2 - (t_{12}^{\nu})^2) + ct_{23}s_{13}t_{12}(ct_{12}^2 - (t_{12}^{\nu})^2)\right], \\
2\cos\delta &= ct_{23}\left[ct_{12}s_{13} - c_{13}cs_{12}ct_{13}sc_{12}(s_{12}^{\nu})^2(t_{23}^{\nu})^2\right] + cs_{13}t_{12}t_{23}, \\
2\cos\delta &= ct_{23}\left[c_{13}cs_{12}ct_{13}sc_{12}(c_{12}^{\nu})^2(t_{23}^{\nu})^2 - s_{13}t_{12}\right] - cs_{13}ct_{12}t_{23}, \\
&\ldots
\end{aligned}$$

$$\begin{aligned}
2\cos(\delta_{12}^{e}) &= t_{12}^{e}\left(c_{12}^2 ct_{13}^2 cs_{12}^{\nu}s_{23}^{\nu}sc_{12}^{\nu}t_{23}^{\nu} - c_{23}^{\nu}t_{12}^{\nu}\right) - ct_{12}^{e}ct_{12}^{\nu}sc_{23}^{\nu}, \\
2\cos(\delta_{12}^{e}) &= c_{23}^{\nu}ct_{12}^{\nu}t_{12}^{e} - ct_{13}^2 s_{12}^2 cs_{12}^{\nu}t_{12}^{e}s_{23}^{\nu}sc_{12}^{\nu}t_{23}^{\nu} + ct_{12}^{e}sc_{23}^{\nu}t_{12}^{\nu}, \\
&\ldots
\end{aligned}$$

Sum rules

In the end, everything is reduced to one relation that can be expressed in terms of θ_{23}^{ν}, θ_{12}^{ν}, θ_{12} and θ_{13}

$$\cos\delta = \frac{1}{s'_{12} s_{13} |c_{23}^{\nu}| \sqrt{(s_{23}^{\nu})^2 - s_{13}^2}} \Big[((s_{23}^{\nu})^2 - s_{13}^2) s_{12}^2 + s_{13}^2 c_{12}^2 (c_{23}^{\nu})^2 - (s_{12}^{\nu})^2 (s_{23}^{\nu})^2 c_{13}^2 \Big].$$

Additionally, we have the well-known relations

$$s_{13}^2 = |U_{e3}|^2 = (s_{12}^e)^2 (s_{23}^{\nu})^2$$

$$s_{23}^2 = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} = \frac{(c_{12}^e)^2 (s_{23}^{\nu})^2}{1 - (s_{12}^e)^2 (s_{23}^{\nu})^2}$$

$$s_{12}^2 = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \frac{(c_{12}^{\nu})^2 (c_{23}^{\nu})^2 (s_{12}^e)^2 + (c_{12}^e)^2 (s_{12}^{\nu})^2 - 2 c_{12}^e c_{12}^{\nu} c_{23}^{\nu} \cos(\delta_{12}^e) s_{12}^e s_{12}^{\nu}}{1 - (s_{12}^e)^2 (s_{23}^{\nu})^2}.$$

Sum rules

Simpler notation:

$$a \equiv \left(s_{12}^{e}\right)^2, \qquad b \equiv \left(s_{23}^{\nu}\right)^2, \qquad c \equiv \cos\left(\delta_{12}^{e}\right), \qquad z_0 \equiv \left(s_{12}^{\nu}\right)^2$$

$$\begin{aligned}
x &\equiv s_{13}^2 = ab \\
y &\equiv s_{23}^2 = \frac{(1-a)b}{1-ab} \\
z &\equiv s_{12}^2 = z_0 - \frac{2c\sqrt{a(1-a)(1-b)z_0(1-z_0)}}{1-ab} + \frac{a(1-b)(1-2z_0)}{1-ab}.
\end{aligned}$$

Moreover, given the small error bars for the reactor angle

$$s_{13}^2 = 0.02241^{+0.00066}_{-0.00065} \quad \text{(NuFIT 4.1, 2019)}$$

we will fix the value of $s_{13}^2 = x = x_0 = 0.02241$. We will assume **normal ordering**.

Probability densities

If we assume all the **model parameter distributions to be completely independent**, the probability distributions can be related as

$$P_x(x) = \int d\ell_x P_a(a) P_b(b),$$

$$P_y(y) = \int d\ell_y P_a(a) P_b(b),$$

$$P_z(z) = \int dA_z P_a(a) P_b(b) P_c(c),$$

where $d\ell_x$ and $d\ell_y$ are line elements in a-b space for fixed x and y, respectively. For $P_z(z)$, dA_z is an area element in a-b-c space for a fixed value of z.

Probability densities

However, considering **correlations between the model parameters due to *x*, *y* and *z*** we have to consider conditional probability distributions

$$P_x(x) = \int d\ell_x P_{a|b}(a) P_b(b) = \int d\ell_x P_a(a) P_{b|a}(b),$$
$$P_y(y) = \int d\ell_y P_{a|b}(a) P_b(b) = \int d\ell_y P_a(a) P_{b|a}(b)$$
$$P_z(z) = \int dA_z P_{a|b}(a) P_{b|c}(b) P_c(c),$$

where $P_{\alpha|\beta}(\alpha)$ represents the probability distribution of some variable α for a given value of some other variable β.

Probability densities

To obtain the distributions $P_y(y)$ and $P_z(z)$ we take the one dimensional χ^2 projections available in NuFIT's webpage (`www.nu-fit.org`), then assume

$$P_y(y) = N_y \exp(-\chi^2(y)/2)$$
$$P_z(z) = N_z \exp(-\chi^2(z)/2)$$

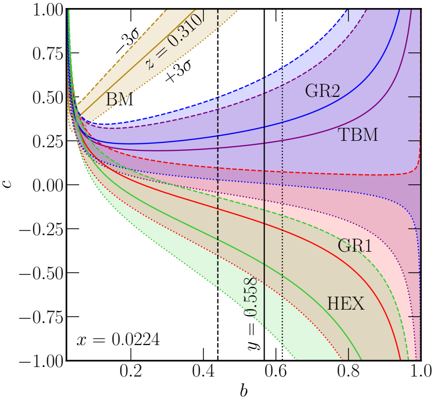
where $N_{y,z}$ is some normalization factor. Since we fixed the value of $x = x_0 = 0.02241$, we consider

$$P_x(x) = \delta(x - x_0)$$

for completely **independent** a and b this means they also **follow a δ-function probability distribution**. However, that is **not the case when we consider correlations**.

Probability densities

Why model parameter distributions?



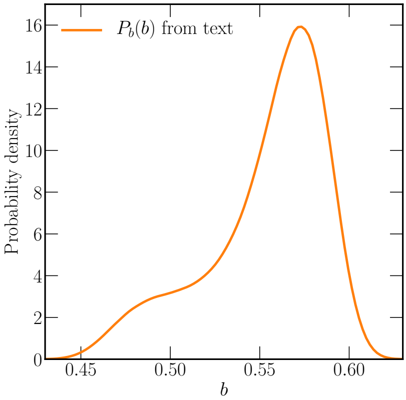
The possible y-z combinations that are possible within the allowed ranges of b and c **depend on the mixing pattern**.

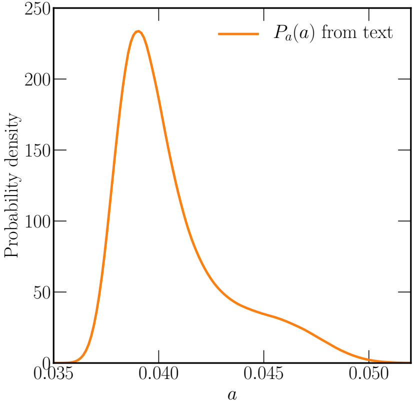
We need to go back to the model parameters and make sure that **we are still working inside the limits of our model**.

Probability densities

From the integral of $P_y(y)$ we can find $P_a(a)$ and $P_b(b)$ when we consider correlations.

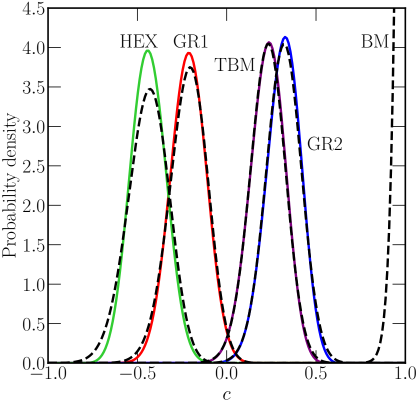
$$P_b(b) = \frac{1}{1-x_0} P_y\left(\frac{b-x_0}{1-x_0}\right) \qquad P_a(a) = \frac{x_0}{a^2(1-x_0)} P_y\left(\frac{x_0(1-a)}{a(1-x_0)}\right)$$





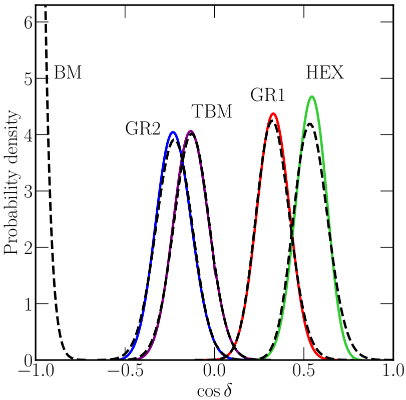
Probability densities

The distribution of c is more involved. It has to be **guessed (dashed lines)** or **marginalized** from a two-dimensional distribution obtained from a two-dimensional χ^2 (**solid colored lines**)



Predictions for $\cos\delta$

Once all the probability distributions for the model parameters have been determined we can integrate them for fixed values of $\cos\delta$ to find the predictions and their distribution



Summary

- Massive neutrinos are a clear indication of beyond SM physics.
- We need tools to discriminate between the enormous amount of theoretical scenarios.
- In this work we developed a process to go from mixing matrices to CP-violating phase predictions, taking into account as many details as possible.
- This level of detail eases the identification of weak points in theoretical models.
- In particular, the BM pattern has problems to reproduce current measured mixing angles.
- It would be interesting to see how these results change with new data.