Classical spacetime from quantum scattering

(messages from the on-shell world)

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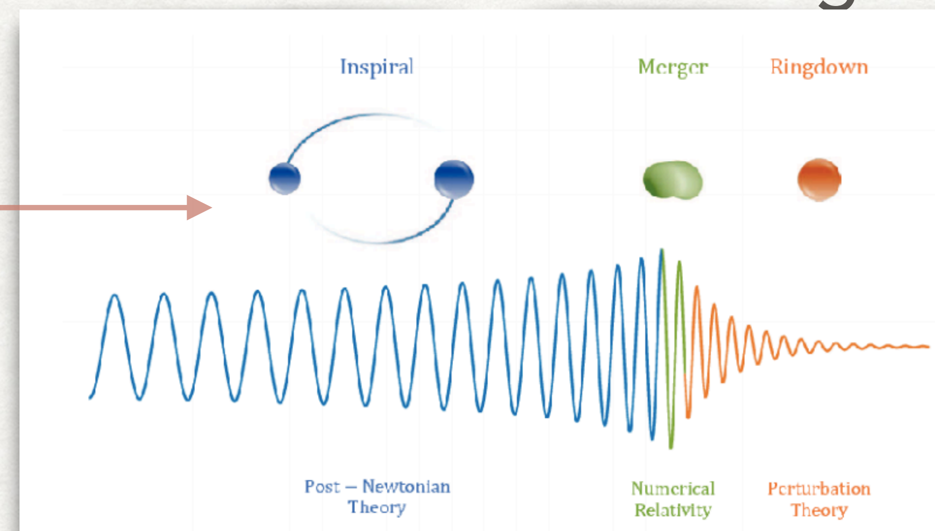
2019 NCTS annual theory workshop

- We are used to discussing classical space time in terms of the metric $g_{\mu\nu}(x)$ which is not gauge invariant (coordinate dependent)
- What is gauge invariant is the physical observables, deflection angle, impulse …
- If space-time is emergent, we should be able to derive these observables without ever introducing $g_{\mu\nu}(x)$
- Certainly this can be done for gravity as perturbation around flat spacetime, what about classical solutions, i.e Black Holes ?

BLACK HOLES <=> QUANTUM PARTICLES

- No hair theorem tells us: Outside the event horizon, a BH is characterized by (M, Q, |S|), with no reference to the origin of its makeup

At large distances, BHs are particles !?



- But: For sufficient large distances aren't all objects essentially point particles ? No!

$$S = \int d\sigma \left\{ -m\sqrt{u^2} + c_E E^2 + c_B B^2 + \cdots \right\}$$

$$E_{\mu\nu} := R_{\mu\alpha\nu\beta} u^\alpha u^\beta$$
$$B_{\mu\nu} := \frac{1}{2} \epsilon_{\alpha\beta\gamma\mu} R^{\alpha\beta}{}_{\delta\nu} u^\gamma u^\delta$$

Tidal Love numbers: vanishing for BHs but not Neutron stars (Naturalness problem?)

BLACK HOLES <=> QUANTUM PARTICLES

- **No hair theorem tells us:** Outside the event horizon, a BH is characterized by (M, Q, |S|), with no reference to the origin of its makeup

- **But:** For sufficient large distances aren't all objects essentially point particles ? No!

If we consider spinning objects

$$S = \int d\sigma \left\{ -m\sqrt{u^2} - \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + L_{SI}\left[u^\mu, S_{\mu\nu}, g_{\mu\nu}(y^\mu)\right] \right\}$$

$$L_{SI} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{\mathrm{ES}^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \cdots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \cdots S^{\mu_{2n-1}} S^{\mu_{2n}}$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{\mathrm{BS}^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \cdots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \cdots S^{\mu_{2n}} S^{\mu_{2n+1}}$$

The worldline action are distinctly different for distinct stellar objects!

BLACK HOLES <=> QUANTUM PARTICLES

The Wilson coefficients are derived from the linearized metric.

$$\bar{h}^{\mathrm{Kerr}}_{\mu\nu}(x) = u_\mu u_\nu \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(-(a\cdot\partial)^2\right)^n \frac{4Gm}{r} + u_{(\mu}\epsilon_{\nu)\alpha\beta\gamma} u^\alpha a^\beta \partial^\gamma \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(-(a\cdot\partial)^2\right)^n \frac{4Gm}{r}$$

$$a^\mu := \tfrac{1}{m} S^\mu.$$

from which one reads off the stress-tensor sourcing the linearized metric

$$\bar{h}_{\mu\nu}(x) = 4G \int d^4y\, \mathcal{G}_{\mathrm{ret}}(x-y) T_{\mu\nu}(y)$$

The worldline operators linear in R is then matched to the stress tensor

$$S_{int} = -\int d^4x\, \frac{1}{2} h_{\mu\nu}(x) T^{\mu\nu}(x)$$

$$T_{\mu\nu}(x) = m \int ds \left[u_\mu u_\nu \sum_{n=0}^{\infty} \frac{C_{\mathrm{ES}^{2n}}}{(2n)!} \left(-(a\cdot\partial)^2\right)^n \delta^4[x - x_{\mathrm{wl}}(s)] + u_{(\mu}\epsilon_{\nu)\alpha\beta\gamma} u^\alpha a^\beta \partial^\gamma \sum_{n=0}^{\infty} \frac{C_{\mathrm{BS}^{2n+1}}}{(2n+1)!} \left(-(a\cdot\partial)^2\right)^n \delta^4[x - x_{\mathrm{wl}}(s)] \right]$$

C#=1 for Kerr BHs !

Smoking gun!

Justin Vines 1709.06016

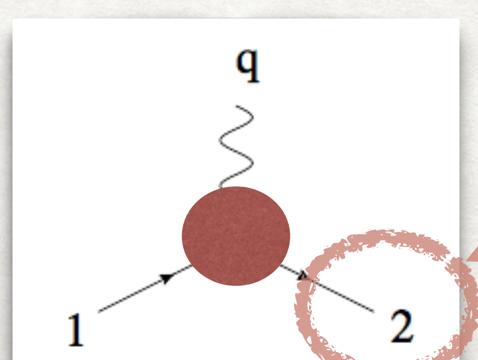
C#=1 for Kerr BHs !

- What are the physical principles that selects these coefficients ?
- How do they encode/differentiate distinct black hole (like) solutions?
- Let us approach this in a purely on-shell fashion: the operators yield the scattering amplitude of a massive object coupled to a graviton

$$\mathcal{O}^{phys_a}{}_{phys_b} \longrightarrow \epsilon_2^{(s),\,\mu\nu\cdots}\ \mathcal{O}\ \epsilon^{(s)}_{1,\,\mu\nu\cdots} \longrightarrow M_3(q^{+2},\mathbf{1}^s,\mathbf{2}^s)$$

Start with the worldliness operator

End with a scattering amplitude



These are objects with quantum number (M, s) and sources the non-trivial background via interactions!

GENERAL MASSIVE AMPLITUDES

N. Arkani-Hamed, Tzu-Chen Huang, Y-t H 1709.04891

Consider an amplitude for massive states. Since it is a scalar function that carries the quantum number of the physical state (Little group)

$$\langle in\ t \to +\infty | out\ t \to -\infty \rangle \to M_n^{\{I_1 I_2 \cdots I_{2s_1}\},\{J_1 J_2 \cdots J_{2s_2}\}\cdots}$$

I =1,2 are doublets of SU(2) Little group.

We introduce spinor-helicity formalism

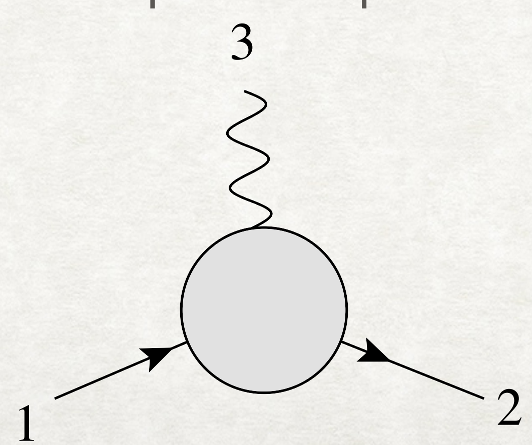
$$p^{\alpha\dot{\alpha}} = \begin{pmatrix} -p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^0-p^3 \end{pmatrix}$$

$$Det(p^{\alpha\dot{\alpha}}) = m^2 \qquad p^{\alpha\dot{\alpha}} = \lambda_1^{\alpha}\tilde{\lambda}_1^{\dot{\alpha}} + \lambda_2^{\alpha}\tilde{\lambda}_2^{\dot{\alpha}} = \lambda^{I\alpha}\tilde{\lambda}_I^{\dot{\alpha}}$$

This naturally introduces the requisite SU(2) indices

$$M_n^{\{I_1 I_2 \cdots I_{2s_1}\},\cdots} = \lambda_1^{I_1\alpha_1}\lambda_1^{I_2\alpha_2}\cdots\lambda_1^{I_{2s_1}\alpha_{2s_1}} M_{n,\ \{\alpha_1\alpha_2\cdots\alpha_{2s_1}\}\cdots}$$

Let us consider three-point amplitude with one massless and two equal mass



$$M^h{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S_1}\},\{\beta_1\beta_2\cdots\beta_{2S_2}\}}$$

GENERAL MASSIVE AMPLITUDES

We need two vectors to span the SL(2,C) space: we have the massless spinor of the massless leg

$$Det(p^{\alpha\dot{\alpha}}) = 0 \qquad p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$$

$$(u_\alpha, v_\alpha) = (\lambda_{3,\alpha}, \epsilon_{\alpha\beta}\lambda_3^\beta)$$

There is another variable that carry the opposite helicity weight of the massless leg

$$2p_2 \cdot p_3 = \langle 3|p_2|3] = 0 \longrightarrow x\lambda_3^\alpha = \frac{p_2^{\alpha\dot{\alpha}}\tilde{\lambda}_{3\dot{\alpha}}}{m}$$

This allows us to define the x factor which carries positive helicity

$$M^h{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S_1}\},\{\beta_1\beta_2\cdots\beta_{2S_2}\}}$$

Three point amplitude is constructed from (x, **λ**, **ε**)

We have a parameterization of the three-point coupling that is purely kinematic in nature. For example for photon (h=1)

g-2 for electron and W

$$s = \frac{1}{2}: \qquad x\left(\epsilon_{\alpha\beta} + g_1\, x\frac{\lambda_\alpha\lambda_\beta}{m}\right)$$

$$s = 1: \qquad x\left(\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2} + g_1\epsilon_{\alpha_2\beta_2}\, x\frac{\lambda_{\alpha_1}\lambda_{\beta_1}}{m} + g_2\, x^2\frac{\lambda_{\alpha_1}\lambda_{\beta_1}\lambda_{\alpha_2}\lambda_{\beta_2}}{m^2}\right)$$

$$\vdots$$

$$s = 2: \qquad x\left(\epsilon^4 + g_1\epsilon^3 x\frac{\lambda^2}{m} + g_2\,\epsilon^2 x^2\frac{\lambda^4}{m^2} + g_3\epsilon\frac{\lambda^6}{m^3} + g_4\frac{\lambda^8}{m^3}\right)$$

$$\vdots$$

THE SIMPLEST MASSIVE-AMPLITUDE

Let's consider simplest possible amplitude is given by a pure x term

$$M_3(q^{+1},\mathbf{1}^s,\mathbf{2}^s)=x\,m\epsilon^{2s},\quad M_3(q^{+2},\mathbf{1}^s,\mathbf{2}^s)=x^2\,m\epsilon^{2s}$$

Or after putting back the external polarization vectors:

$$M_3(q^{+1},\mathbf{1}^s,\mathbf{2}^s)=x\,m\left(\frac{\langle\mathbf{12}\rangle}{m}\right)^{2s},\quad M_3(q^{+2},\mathbf{1}^s,\mathbf{2}^s)=x^2\,m\left(\frac{\langle\mathbf{12}\rangle}{m}\right)^{2s}$$

Note that beyond spin-2 there are no Lagrangian for consistent fundamental higher-spin particles. Could it be strings ?

The couplings of leading trajectory spin-s state coupled to massless field

$$M_3^{+1,s,s}=x\sum_{n=0}^{s}\sum_{k=0}^{n}(\alpha')^{2s-n+\frac{k-1}{2}}\begin{pmatrix}s\\n\end{pmatrix}\begin{pmatrix}n\\k\end{pmatrix}\frac{s-n-k+1}{2^{s-n}\,(s-n+1)!}\langle\mathbf{12}\rangle^{2n-k}\left(x\langle\mathbf{23}\rangle\langle\mathbf{31}\rangle\right)^{2s-2n+k}$$

maximally complicated !

THE SIMPLEST MASSIVE-AMPLITUDE

To see what kind of interaction this describes, we compare this with amplitude from the 1-particle EFT

$$M_s^2 = \sum_{a+b\leq s} \frac{\kappa m x^2}{2} C_{\mathrm{S}^{a+b}} n_{a,b}^s \langle \mathbf{21}\rangle^{s-a} \left(-\frac{x\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle}{2m}\right)^a [\mathbf{21}]^{s-b} \left(\frac{[\mathbf{2}q][q\mathbf{1}]}{2mx}\right)^b,$$
$$n_{a,b}^s \equiv \frac{1}{m^{2s}} \binom{s}{a} \binom{s}{b},$$

Our minimal coupling is given as:

$$M_{s,min}^{+2} = \frac{\kappa m x^2}{2} \frac{\langle \mathbf{21}\rangle^{2s}}{m^{2s}}$$

They are not in the same basis:

$$\langle \mathbf{21}\rangle^{2s} = (\langle \mathbf{21}\rangle^2)^s = \sum_{a,b=0}^{s} c_{a,b}^s \langle \mathbf{21}\rangle^{s-a} \left(-\frac{x\langle \mathbf{23}\rangle\langle \mathbf{31}\rangle}{2m}\right)^a [\mathbf{21}]^{s-b} \left(\frac{[\mathbf{23}][\mathbf{31}]}{2mx}\right)^b$$

we find that

$$C_{\mathrm{S}^n} = 1 + \frac{n(n-1)}{4s} + \frac{n(n-1)(n^2-5n+10)}{32s^2} + \frac{n(n-1)(n^4-14n^3+71n^2-154n+144)}{384s^3} + \mathcal{O}(s^{-4})$$

It matches to C=1 in the large s limit!

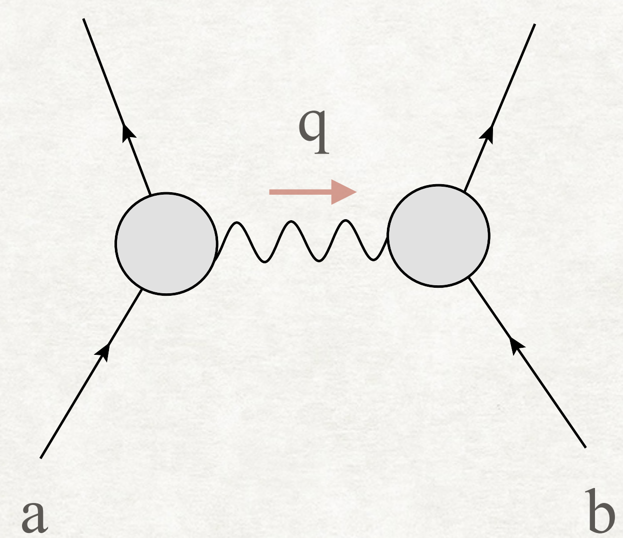
Isolated higher-spin elementary particles do exists, they are Kerr black holes

THE X-AMPLITUDE

Minimal coupling in the large s limit is identical to Kerr BH:

$$M^{+2}_{s,min} = \frac{\kappa m x^2}{2}\frac{\langle \mathbf{21}\rangle^{2s}}{m^{2s}}$$

Observables: the classical gravitational potential: using minimal coupling at tree-level



$$V(p,q) = \left.\frac{M_4(s,t)}{4E_aE_b}\right|_{t\to 0} = \frac{\mathrm{Res}_t}{4E_aE_b\,q^2}$$

$$\mathrm{Res}_t = M^+_{3a}M^-_{3b} + M^-_{3a}M^+_{3b}$$

we find that

$$V^{\mathrm{BBN}}_{cl} = \left(-\cosh\left[\left(\frac{\vec{S}_a}{m_a}+\frac{\vec{S}_b}{m_b}\right)\times\vec{\nabla}\right] - 2\left(\frac{\vec{p}_a}{m_a}-\frac{\vec{p}_b}{m_b}\right)\cdot\sinh\left[\left(\frac{\vec{S}_a}{m_a}+\frac{\vec{S}_b}{m_b}\right)\times\vec{\nabla}\right]\right)\frac{Gm_am_b}{r}$$
$$+\frac{1}{2}\left(\left[\frac{\vec{p}_a}{m_a}\times\frac{\vec{S}_a}{m_a}-\frac{\vec{p}_b}{m_b}\times\frac{\vec{S}_b}{m_b}\right]\cdot\vec{\nabla}\right)\cosh\left[\left(\frac{\vec{S}_a}{m_a}+\frac{\vec{S}_b}{m_b}\right)\times\vec{\nabla}\right]\frac{Gm_am_b}{r}.$$

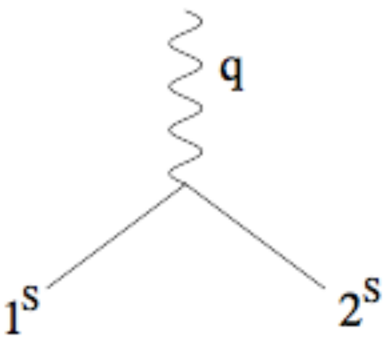
which matches to the classical GR result

Justin Vines 1709.06016

- What is minimal coupling telling us about Kerr?

THE ON-SHELL VIEW POINT (DOUBLE COPY)

First, we see that gravitational minimally coupling, is simply a double copy of electric-magnetic minimal coupling !



$$= g(xm)^h \frac{\langle \mathbf{12} \rangle^{2S}}{m^{2S}}$$

where $h = (1, 2)$ and $g = (\frac{\kappa}{2}, \frac{e}{\sqrt{2}})$

This imply that black holes are a double copy of some electrically charged object. Indeed such a relation was shown for the Kerr-Schild form of the metric:

Monteiro, O'Connell, White 1410.0239

$$g_{\mu\nu} = g^0_{\mu\nu} + k_\mu k_\nu \phi(r), \quad A^{\mu a} = c^a k^\mu \phi(r)$$

Schwarzschild sol -> Coloumb potential

Kerr sol -> rotating charged disk with radius

in the one body EFT approach in terms of computations involving x^2. A tantalising example would be the fact for charged black holes, one also has $g = 2$ [42], and one can conjecture that x gives the correct Wilson coefficient for the electricmagnetic couplings. This would be the simplest example of double copy for classical objects.

THE ON-SHELL VIEW POINT(EXPONENTIALS)

Minimal coupling is really a reflection that the spin is completely "intrinsic"

$$h=+1:\quad \frac{e_2}{\sqrt{2}}x\frac{\langle\mathbf{2}\mathbf{2}'\rangle^S}{m^{S-1}},\quad h=-1:\quad \frac{e_2}{\sqrt{2}}\frac{1}{x}\frac{[\mathbf{2}\mathbf{2}']^S}{m^{S-1}}$$

The only spin-pieces is contained in the external wave functions (the λs). Indeed 2′ is related to 2 by a boost

.

$$|\mathbf{2}'\rangle=|\mathbf{2}\rangle+\frac{1}{4}\omega_{\mu\nu}(\sigma^\mu\bar\sigma^\nu-\sigma^\nu\bar\sigma^\mu)|\mathbf{2}\rangle$$

$$\omega_{\mu\nu}=-\frac{\hbar}{m_2}p_{2[\mu}\bar q_{\nu]}$$

Each spinor bracket can be written in terms of spin operators

$$\frac{1}{m_2}\langle\mathbf{2}\mathbf{2}'\rangle=\mathbb{I}+\frac{1}{m_2^2}\hbar\langle\mathbf{2}|\bar{\not q}|\mathbf{2}]=\mathbb{I}+\frac{1}{Sm_2}\bar q\cdot s\,.$$

$$s^\mu=-\frac{1}{2m_2}\epsilon^{\mu\nu\rho\sigma}p_{2\nu}J_{\rho\sigma}$$

Thus the spinor brackets becomes operators on the Hilbert-space with

$$\langle\mathbf{2}'\mathbf{2}\rangle^S=\left(1+\frac{\bar q\cdot s}{Sm_2}\right)^S$$

Taking S-> infinity

$$\lim_{S\to\infty}\frac{e_2}{\sqrt{2}}m\,x\left(\mathbb{I}\pm\frac{1}{Sm}\bar q\cdot s\right)^S=\frac{e_2}{\sqrt{2}}mxe^{\pm\bar q\cdot a}$$

$$a=\frac{s}{m}$$

The spin factor exponentiates!!

THE ON-SHELL VIEW POINT (COMPLEX SHIFT)

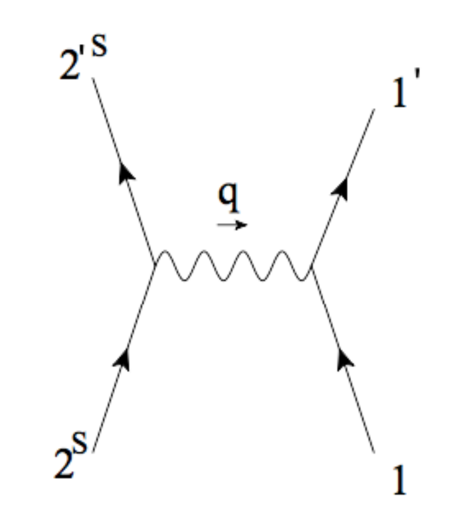
For minimal coupling the spin-dependence exponentiates in the large spin-limit

$$h=+1:\quad \frac{e_2}{\sqrt{2}}x\frac{\langle\mathbf{22'}\rangle^S}{m^{S-1}},\qquad h=-1:\quad \frac{e_2}{\sqrt{2}}\frac{1}{x}\frac{[\mathbf{22'}]^S}{m^{S-1}}$$

$$\lim_{S\to\infty}\frac{e_2}{\sqrt{2}}m\,x\left(\mathbb{I}\pm\frac{1}{Sm}\bar{q}\cdot s\right)^S=\frac{e_2}{\sqrt{2}}mxe^{\pm\bar{q}\cdot a}$$

Note that q is the transverse momenta, and hence after Fourier transform, relates to impact parameter. So the difference between s=0 and spinning case

This implies a complex-shift relating Kerr to Schwarzschild in the context of physical observables. For exp the impulse imparted on a probe



$$\Delta p_1^\mu=\frac{1}{4m_1m_2}\int\hat{d}^4q\,\hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)e^{-iq\cdot b}\,iq^\mu\,M_4\left(1,2\to1',2'\right)|_{q^2\to0}$$

Kosower, Maybee, O'Connell 1811.10950

Once again in the q^2=0 limit the amplitude factorizes to our minimal coupling!

$$M_4\left(1,2\to1',2'\right)|_{q^2\to0}=-\frac{e_1e_2}{2\bar{q}^2}\left(\frac{x_{11'}}{x_{22'}}e^{-\bar{q}\cdot a}+\frac{x_{22'}}{x_{11'}}e^{\bar{q}\cdot a}\right)$$

THE ON-SHELL VIEW POINT (COMPLEX SHIFT)

For minimal coupling the spin-dependence exponentiates in the large spin-limit

$$h=+1: \quad \frac{e_2}{\sqrt{2}} x \frac{\langle \mathbf{22'}\rangle^S}{m^{S-1}}, \qquad h=-1: \quad \frac{e_2}{\sqrt{2}} \frac{1}{x} \frac{[\mathbf{22'}]^S}{m^{S-1}}$$

$$\lim_{S\to\infty} \frac{e_2}{\sqrt{2}} m\, x \left(\mathbb{I} \pm \frac{1}{Sm}\bar{q}\cdot s\right)^S = \frac{e_2}{\sqrt{2}} m x e^{\pm \bar{q}\cdot a}$$

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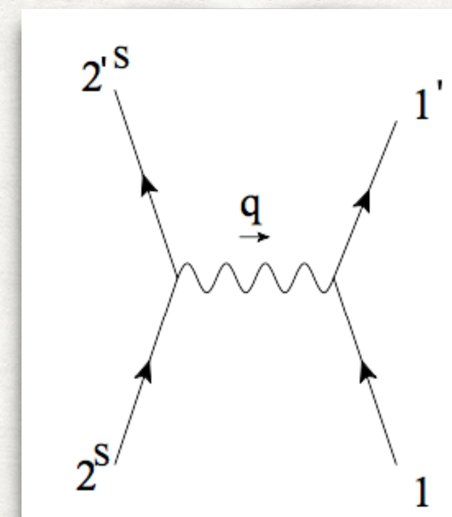


$$\Delta p_1^\mu = \frac{1}{4m_1 m_2}\int \hat{d}^4 q\, \hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2) e^{-iq\cdot b} i q^\mu M_4\left(1,2 \to 1',2'\right)|_{q^2\to 0}$$

Kosower, Maybee, O'Connell 1811.10950

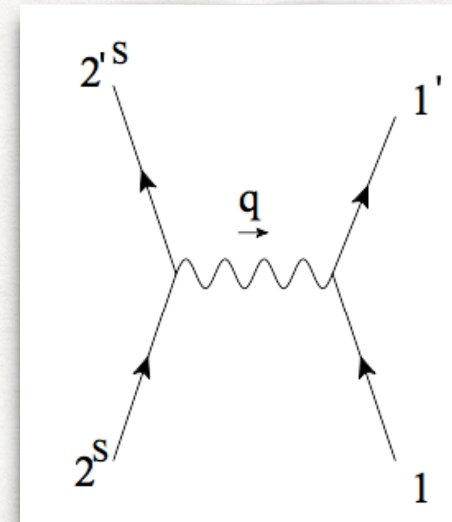
Once again in the q^2=0 limit the amplitude factorizes to our minimal coupling!

$$M_4\left(1,2\to 1',2'\right)|_{q^2\to 0} = -\frac{e_1 e_2}{2\bar{q}^2}\left(\frac{x_{11'}}{x_{22'}} e^{-\bar{q}\cdot a} + \frac{x_{22'}}{x_{11'}} e^{\bar{q}\cdot a}\right)$$

The Kerr sol is a complex shift of the Schwarzschild sol!

KERR FROM SCHWARZSCHILD

Convert
$$\frac{x_{11'}}{x_{22'}} = e^{w}, \quad \frac{x_{22'}}{x_{11'}} = e^{-w},$$



where w is the rapidity, we find that the EM impulse is

$$\Delta p_1^\mu = i\frac{e_1 e_2}{2}\int d^4\bar{q}\,\delta(\bar{q}\cdot u_1)\delta(\bar{q}\cdot u_2)e^{-i\bar{q}\cdot b}\frac{\bar{q}^\mu}{\bar{q}^2}$$
$$\left((\cosh w + \sinh w)e^{\bar{q}\cdot\Pi a} + (\cosh w - \sinh w)e^{-\bar{q}\cdot\Pi a}\right)$$

For gravity we just square x!

$$\Delta p_1^\mu = -i2\pi G m_1 m_2\int d^4\bar{q}\,\delta(\bar{q}\cdot u_1)\delta(\bar{q}\cdot u_2)e^{-i\bar{q}\cdot b}\frac{\bar{q}^\mu}{\bar{q}^2}$$
$$\left((\cosh 2w + \sinh 2w)e^{\bar{q}\cdot\Pi a} + (\cosh 2w - \sinh 2w)e^{-\bar{q}\cdot\Pi a}\right)$$

after Fourier transforming q we have

$$\Delta p_1^\mu = -\frac{2m_1 m_2 G}{\sinh w}\,\mathrm{Re}\left[\frac{\cosh 2w\, b_\perp^\mu + i2\cosh w\,\epsilon^{\mu\nu\alpha\beta}u_{1\alpha}u_{2\beta}b_{\perp\nu}}{b_\perp^2}\right]$$

where $b_\perp = \Pi(b + ia)$, in agreement with Justin Vines [1709.06016](1709.06016)

THE JANIS NEWMAN SHIFT

This is simply the mysterious Janis Newman shift!

Once again consider BH solutions in the Kerr-Schild form

$$g_{\mu\nu} = g^0_{\mu\nu} + k_\mu k_\nu \phi(r),$$

$$\text{Schwarzschild}: \quad \phi_{\text{Sch}}(r) = \frac{r_0}{r}, \quad k_\mu = (1, \hat{r})$$

$$\text{Kerr}: \quad \phi_{\text{Kerr}}(r) = \frac{r_0 r}{r^2 + a^2\cos^2\theta}, \quad k_\mu = (1, \hat{r})$$

The Kerr solution is simply a complex shift of Schwaz-Schild!

Newman, Janis
J. Math. Phys. 6, 915 (1965)

$$\begin{aligned}\phi_{\text{Sch}}(r)\big|_{r\to r+ia\cos\theta} &= \frac{r_0}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right)\Bigg|_{r\to r+ia\cos\theta} \\ &= \frac{r_0 r}{r^2 + a^2\cos^2\theta} = \phi_{\text{Kerr}}(r)\,.\end{aligned}$$

This is precisely the shift
induced by the exponentiation!

THE ON-SHELL VIEW POINT (DUALITIES)

Is there other ways of ``exponentiating" minimal coupling ?

$$x \rightarrow x f(q^\mu, s_\mu, p_\mu)$$

Since $p \cdot q = p \cdot s = 0$ already have the spin-shift $x \rightarrow x\, e^{\frac{q \cdot s}{m}}$ the only other possibility is a complex phase shift! $x \rightarrow x e^{i\theta}$

A complex charge describes the coupling of a dyon.
By double copy, there must be a corresponding gravitation solution:

$$x^2 \rightarrow x^2 e^{i2\theta}$$

As we will see, the double copy of a dyon is Taub-NUT!
The double copy of S-duality! Taub-NUT spacetime must be related to Schwarzschild by some kind of electric-magnetic duality transformation!

DYON FROM PHASE ROTATIONS

To see that the phase rotation indeed leads to a dyon, once again we consider the impulse

$$M_4\left(1,2\to 1',2'\right)|_{q^2\to 0} = 2\frac{m_1 m_2 e_1 e_2}{q^2}\left(\frac{x_1}{x_2}e^{-i\theta}+\frac{x_2}{x_1}e^{i\theta}\right) = 2e^2\frac{m_1 m_2 n_1 n_2}{q^2}\left(\frac{x_1}{x_2}e^{-i\theta}+\frac{x_2}{x_1}e^{i\theta}\right)$$

leading to

$$\Delta p_1^\mu = ie_1\int \hat{d}^4q\,\hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)e^{-iq\cdot b}\frac{1}{q^2}\left[Q\,q^\mu\,\cosh w - \tilde{Q}\,\epsilon^\mu(u_1,u_2,q)\right]$$

This can be compared with the Lorentz force. The field strength can be determined from the electric and magnetic fields

$$\frac{dp_1^\mu}{d\tau} = e_1 F_2^{\mu\nu}u_{1\nu},$$

$$F_2^{0i} = E^i = \frac{Q}{4\pi|r|^3}r^i,\quad F_2^{ij} = -\epsilon^{ijk}B_k = -\epsilon^{ijk}\frac{\tilde{Q}}{4\pi|r|^3}r_k$$

From this one can read off the relativistic form of the field strength

$$\widetilde{F}_2^{\mu\nu} = i\delta(q\cdot u_2)e^{-iq\cdot b}\frac{1}{q^2}(Qq^{[\mu}u_2^{\nu]} - \tilde{Q}\epsilon^{\mu\nu\rho\sigma}u_{2\rho}q_\sigma).$$

The impulse then follows

$$\begin{aligned}\Delta p_1^\mu &= \int d\tau\frac{dp_1^\mu}{d\tau} = e_1\int d\tau\int d^4q\,e^{-iq\cdot x}\widetilde{F}_2^{\mu\nu}u_{1\nu}\\ &= ie_1\int d^4q\,\hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)\,e^{-iq\cdot b}\frac{1}{q^2}[Qq^\mu(u_1\cdot u_2) - \tilde{Q}\epsilon^\mu(u_1,u_2,q)]\end{aligned}$$

TAUB-NUT FROM DYON

The dyon impulse

$$\Delta p_1^\mu = -i\frac{q_e|C|}{2}\int d^4\bar{q}\,\delta(\bar{q}\cdot u_1)\delta(\bar{q}\cdot u_2)e^{-i\bar{q}\cdot\Pi b}\frac{\bar{q}^\mu}{\bar{q}^2}$$
$$\left((\cosh w + \sinh w)e^{-i\theta} + (\cosh w - \sinh w)e^{+i\theta}\right)$$
$$= -i\int d^4\bar{q}\,\delta(\bar{q}\cdot u_1)\delta(\bar{q}\cdot u_2)e^{-i\bar{q}\cdot\Pi b}\frac{\bar{q}^\mu}{\bar{q}^2}q_e\left(Q\cosh w - iG\sinh w\right)$$

Double copying then yields

$$i\int d^4\bar{q}\,\delta(\bar{q}\cdot u_1)\delta(\bar{q}\cdot u_2)e^{-i\bar{q}\cdot\Pi b}\frac{\bar{q}^\mu}{\bar{q}^2}\left(Q_G\cosh 2w - iG_G\sinh 2w\right)$$

Indeed this matches with the impulse computed in Taub-Nut space-time at 1 PM

$$ds^2 = -\frac{A}{B}(dt + 2n\cos\theta d\phi)^2 + B(d^2\theta + \sin^2\theta d^2\phi) + \frac{B}{A}d^2r, \qquad A = r^2 - 2mr - \ell^2,\ B = r^2 + \ell^2$$

At 1 PM, we are considering the linear approximation,

$$h_{00} = \frac{2m}{r},$$
$$h_{0i} = 4\ell\cos\theta d\varphi = 4\ell\frac{\cos\theta}{x^2+y^2}\left(-ydx + xdy\right),$$
$$h_{ij} = \frac{2mx_ix_j}{r^3}.$$

$$\longrightarrow$$

$$\Gamma^i_{00} = -\frac{1}{2}\partial^i h_{00},$$
$$\Gamma^i_{0j} = \frac{1}{2}\eta^{ik}\left(\partial_k h_{0j} - \partial_j h_{0k}\right),$$
$$\Gamma^i_{jk} = \frac{2mx^i}{r^3}\delta_{jk} - \frac{3mx^ix_jx_k}{r^5}.$$

TAUB-NUT FROM DYON

The dyon impulse

$$\Delta p_1^\mu = -i\frac{q_e|C|}{2}\int d^4\bar{q}\,\delta(\bar{q}\cdot u_1)\delta(\bar{q}\cdot u_2)e^{-i\bar{q}\cdot\Pi b}\frac{\bar{q}^\mu}{\bar{q}^2}$$
$$((\cosh w+\sinh w)e^{-i\theta}+(\cosh w-\sinh w)e^{+i\theta})$$
$$= -i\int d^4\bar{q}\,\delta(\bar{q}\cdot u_1)\delta(\bar{q}\cdot u_2)e^{-i\bar{q}\cdot\Pi b}\frac{\bar{q}^\mu}{\bar{q}^2}q_e\,(Q\cosh w - iG\sinh w)$$

Double copy then yields

$$i\int d^4\bar{q}\,\delta(\bar{q}\cdot u_1)\delta(\bar{q}\cdot u_2)e^{-i\bar{q}\cdot\Pi b}\frac{\bar{q}^\mu}{\bar{q}^2}\,(Q_G\cosh 2w - iG_G\sinh 2w)$$

Indeed this matches with the impulse computed in Taub-Nut space-time at 1 PM

At 1 PM, we are considering the linear approximation,

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}.$$

$$-\frac{d\vec{v}}{d\tau} = \left(\frac{2\gamma^2-1}{\gamma}\right)\vec{\nabla}\phi - 4\gamma\,\vec{v}\times(\vec{\nabla}\times\vec{A})$$
$$= \left(\frac{2\gamma^2-1}{\gamma}\right)\vec{E} - 4\gamma\,\vec{v}\times\vec{B}.$$

$$\phi \equiv -\frac{1}{2}h_{00}$$
$$A_i = -\frac{1}{4}h_{0i}$$

$$\frac{d\vec{v}}{d\tau} = -\left(\frac{\cosh 2w}{\cosh w}\right)\vec{E} + 2\left(\frac{\sinh 2w}{\sinh w}\right)\vec{v}\times\vec{B}.$$

TAUB-NUT FROM SCHWARZSCHILD

A complex charge describes the coupling of a dyon.

$$x \to x e^{i\theta}$$

By double copy, there must be a corresponding gravitation solution:

$$x^2 \to x^2 e^{i2\theta}$$

The double copy of a dyon is Taub-Nut!

Also shown directly for the classical solution
A. Luna, R. Monteiro, D. O'Connell, C. D. White

The phase transformation indicates that

1. The Taub-Nut metric is again some complex shift acting on the Schwarzschild metric
2. The shift has an interpretation as a electric-magnetic duality transformation.

TAUB-NUT FROM E&M DUALITY

not long after Janis and Newman's realization of Kerr/Schwarzschild correspondence, Talbot generalized the complex shift to obtain Taub-NUT The shift has an interpretation as a electric-magnetic duality transformation.

$$u = u' - ia\cos\theta + 2i\ell\log\sin\theta, \qquad r = r' + ia\cos\theta - i\ell,$$

$$m = m' - i\ell.$$

The complex shift introduced by Talbot can be generated by a BMS super translation:

The expansion of asymptotically flat metrics around future null infinity

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &\quad + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 - 2U_z dudz - 2U_{\bar{z}}dud\bar{z} \\ &\quad + \dots, \end{aligned}$$

$$\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$$

$$U_z = -\frac{1}{2}D^z C_{zz}$$

The metric is invariant under BMS translation

$$T(f) = \frac{1}{4\pi G}\int_{\mathcal{I}^+_-} d^2z\gamma_{z\bar{z}}f(z,\bar{z})m_B.$$

v $$\xi_f = f\partial_u - \frac{1}{r}\left(D^z f\partial_z + D^{\bar{z}} f\partial_{\bar{z}}\right) + D^z D_z f\partial_r + \mathcal{O}\left(r^{-2}\right), \qquad f = f(z,\bar{z}).$$

For real f BMS translation is a symmetry of the metric

TAUB-NUT FROM E&M DUALITY

2. The shift has an interpretation as a electric-magnetic duality transformation.

$$u = u' - ia\cos\theta + 2i\ell\log\sin\theta, \qquad r = r' + ia\cos\theta - i\ell,$$

$$m = m' - i\ell.$$

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$$\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$$

$$U_z = -\frac{1}{2}D^z C_{zz}$$

The Taub-NUT metric is

Uri Kol, Massimo Porrati 1907.00990

$$C_{zz} = i\ell\gamma_{z\bar{z}}\frac{1+|z|^4}{z^2}$$

This is generated from an imaginary super translation

$$f(z,\bar{z}) = -2i\ell\log\frac{2(1+|z|^2)}{|z|} = 2i\ell\log\sin\theta. \longrightarrow (2i\ell\log\sin\theta)\partial_u - (i\ell)\partial_r - \frac{4i\ell}{r}\cot\theta d\theta + \mathcal{O}\left(r^{-2}\right)$$

TAUB-NUT FROM E&M DUALITY

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$$\longrightarrow \quad (2i\ell \log \sin\theta)\partial_u - (i\ell)\partial_r - \frac{4i\ell}{r}\cot\theta d\theta + \mathcal{O}\left(r^{-2}\right)$$

Let us see what happens to the charges. It was shown by Kol and Porrati that the super translation charge is accompanied by a dual charge

$$\mathcal{M}(\varepsilon) = \frac{i}{16\pi G}\int_{\mathcal{I}^+_-} d^2z\, \varepsilon(z,\bar{z})\gamma^{z\bar{z}}\left(D_{\bar{z}}^2 C_{zz} - D_z^2 C_{\bar{z}\bar{z}}\right)$$

Under the super translation, the T charge deforms as

$$T(\varepsilon) \to T(\varepsilon) + \frac{1}{4\pi G}\int_{\mathcal{I}^+_-} d^2z\, \varepsilon(z,\bar{z})\gamma^{z\bar{z}} D_z^2 D_{\bar{z}}^2 f(z,\bar{z}),$$

using $D_z^2 D_{\bar{z}}^2 f = -i\ell\gamma_{z\bar{z}}^2$ we find $T(\varepsilon) \to T(\varepsilon) - i\frac{\ell}{4\pi G}\int_{\mathcal{I}^+_-} d^2z\, \gamma_{z\bar{z}}\varepsilon(z,\bar{z}).$

or

$$T(\varepsilon) \to T(\varepsilon) - i\mathcal{M}_{\rm NUT}(\varepsilon)$$

Thus we see that the imaginary shift rotates the super-translation charge with the dual supertranslation

CHARGED BH

Schwarzschild, Kerr, (Kerr) Taub-Nut at 1 PM is described by minimal coupling. When there are other massless d.o.f. are they given by minimal coupling? Kerr-Newman!

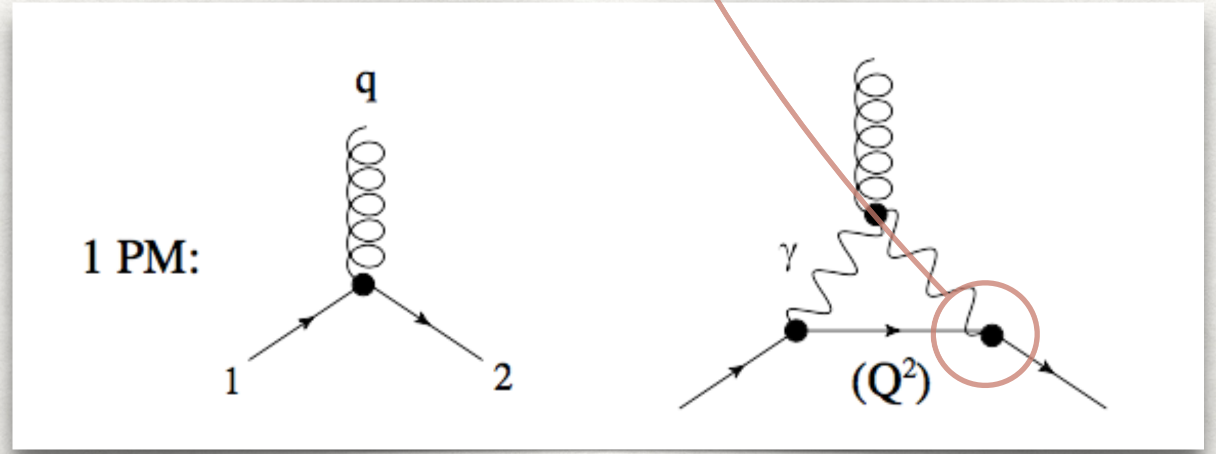
In the Kerr-Schild form the Kerr-Newman solution is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + f k_\mu k_\nu$$
$$f = \frac{Gr^2}{r^4 + a^2 z^2}[2Mr - Q^2]$$
$$k^\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right)$$
$$A_\mu = \frac{Qr^3}{r^4 + a^2 z^2} k_\mu$$

Let's consider the world line action again, where the couplings are derived from the stress tensor

$$\langle p_2 | T_{\mu\nu}(q) | p_1 \rangle = \frac{1}{\sqrt{4E_1E_2}} \left[2P_\mu P_\nu F_1(q^2) + (q_\mu q_\nu - \eta_{\mu\nu} q^2) F_2(q^2)\right]$$

Charge induces new photon contributions entering at one loop, but still 1 PM (we have GQ^2)



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$$A_\mu = \frac{Q r^3}{r^4 + a^2 z^2} k_\mu$$

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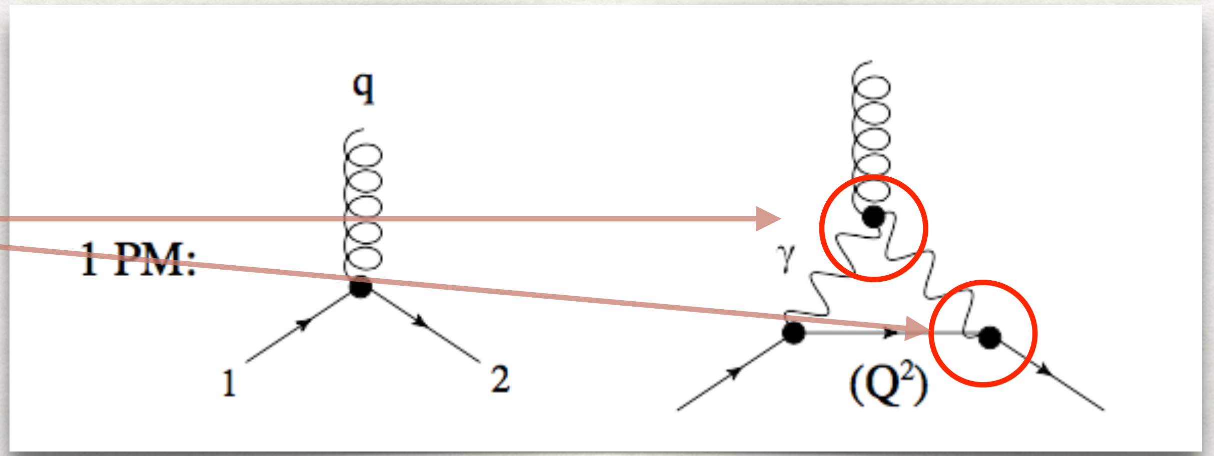
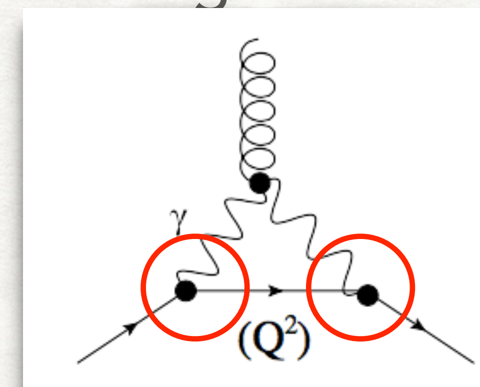
What are these new couplings kinematically?



CHARGED BH

For the photon coupling, lets work out the EFT coupled to the back ground gauge field

$$A'^{\mu} = \left(u^{\mu} \cos(a \cdot \partial) - \epsilon^{\mu\nu\alpha\beta} u_{\nu} a_{\alpha} \partial_{\beta} \frac{\sin(a \cdot \partial)}{a \cdot \partial} \right) \frac{Q}{R}$$



From which we can deduce the curent given by

$$\begin{aligned} j^{\mu} &= Q \int ds \left[u^{\mu} \cos(a \cdot \partial) - \epsilon^{\mu\nu\alpha\beta} u_{\nu} a_{\alpha} \partial_{\beta} \frac{\sin(a \cdot \partial)}{a \cdot \partial} \right] \delta^4 \left[x - x_{\mathrm{wl}}(s) \right] \\ &= Q \int ds \sum_{n=0}^{\infty} \left[u^{\mu} \frac{\left(-(a \cdot \partial)^2\right)^n}{(2n)!} - \epsilon^{\mu\nu\alpha\beta} u_{\nu} a_{\alpha} \partial_{\beta} \frac{\left(-(a \cdot \partial)^2\right)^n}{(2n+1)!} \right] \delta^4 \left[x - x_{\mathrm{wl}}(s) \right] \end{aligned}$$

this should appear in the interaction term of the EFT as

$$S_{int} = - \int d^4x A_{\mu} j^{\mu}$$

leading to

$$\begin{aligned} S_{int} = - \int d^4x \, Q \int ds \, \delta^4 \left[x - x_{\mathrm{wl}}(s) \right] \\ \times \sum_{n=0}^{\infty} \left[u^{\mu} \frac{\left(-(a \cdot \partial)^2\right)^n}{(2n)!} + \epsilon^{\mu\nu\alpha\beta} u_{\nu} a_{\alpha} \partial_{\beta} \frac{\left(-(a \cdot \partial)^2\right)^n}{(2n+1)!} \right] A_{\mu}(x) \end{aligned}$$

CHARGED BH

We again convert this to a three-point amplitude for charged spin-s particle interacting with a photon:

$$S_{int} = -\int d^4x\, Q \int ds\, \delta^4\left[x - x_{\mathrm{wl}}(s)\right] \times \sum_{n=0}^{\infty}\left[u^\mu \frac{\left(-(a\cdot\partial)^2\right)^n}{(2n)!} + \epsilon^{\mu\nu\alpha\beta} u_\nu a_\alpha \partial_\beta \frac{\left(-(a\cdot\partial)^2\right)^n}{(2n+1)!}\right] A_\mu(x)$$

converting to on-shell variables:

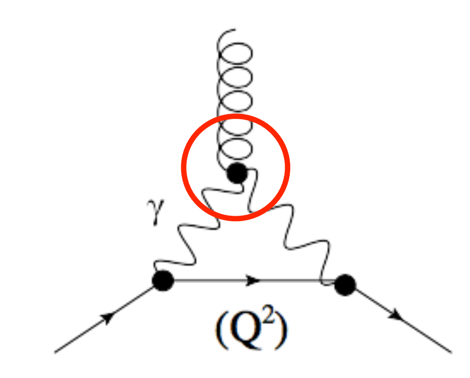
$$u^\mu A_\mu \to \frac{x^\eta}{\sqrt{2}}$$
$$\epsilon^{\mu\nu\alpha\beta} u_\nu a_\alpha \partial_\beta A_\mu \to \frac{x^\eta}{\sqrt{2}}\left(-\eta \frac{p_3\cdot S}{m}\right)$$
$$-(a\cdot\partial)^2 \to \left(\frac{p_3\cdot S}{m}\right)^2 = \left(-\eta\frac{p_3\cdot S}{m}\right)^2$$

we find that

$$M_s^\eta = \epsilon^*(\mathbf{2})\left[\frac{Qx^\eta}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\eta\frac{q\cdot S}{m}\right)^n\right]\epsilon(\mathbf{1}) \rightarrow M_3(q^{+1}, \mathbf{1}^s, \mathbf{2}^s) = x\, m\left(\frac{\langle \mathbf{12}\rangle}{m}\right)^{2s}$$

Again match with minimal coupling!

CHARGED BH



This corresponds to the stress tensor form factor with photons

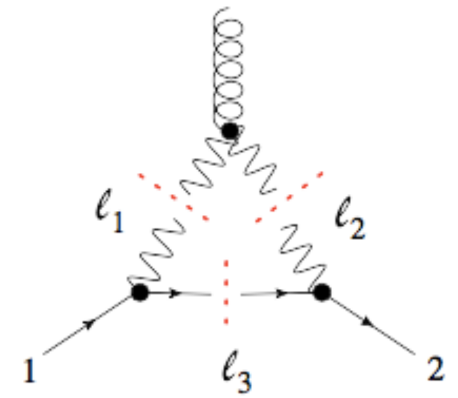
$$\langle \gamma_{\ell_1} | T_{\mu\nu}(q) | \gamma_{\ell_2} \rangle$$

The tree-level form factor for a stress tensor and two massless states can be identified as the three-point amplitude of massive spin-2 state and two massless states, it is again unique once the the helicities are fixed

$$\langle 1^{+1} | T^{\mu\nu}(q) | 2^{-1} \rangle \sim \lambda_1^{\{\alpha_1} \lambda_1^{\alpha_2} \lambda_1^{\alpha_3} \lambda_1^{\alpha_4\}} [12]^2$$

corresponds to $T_{\mu\nu} = F_{\mu\sigma} F^{\sigma}{}_{\nu} - \frac{1}{4} \eta_{\nu\nu} F^{\rho\sigma} F_{\rho\sigma}$

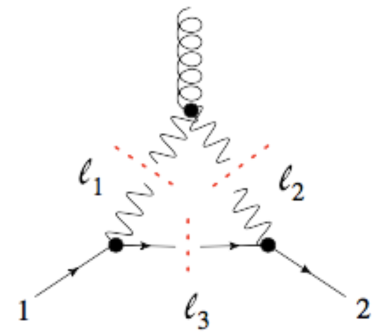
Thus all the vertices that determine the 1 PM photon contribution is uniquely determined



$$= \langle \gamma_{\ell_1} | T_{\mu\nu}(q) | \gamma_{\ell_2} \rangle \otimes A_3(1^s \ell_1 \ell_3^s) \otimes A_3(\ell_3^s \ell_2 2^s)$$

CHARGED BH

Applying unitarity methods we can extract the triangle coefficient from



$$= \langle \gamma_{\ell_1}|T_{\mu\nu}(q)|\gamma_{\ell_2}\rangle \otimes A_3(1^s\ell_1\ell_3^s)\otimes A_3(\ell_3^s\ell_2 2^s)$$

classical!

We are only interested in the triangle coefficients since

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k-q)^2((k-p)^2-m^2)} = \frac{-i}{32\pi^2 m^2}(L+S)$$

$$L=\log(\frac{-q^2}{m^2}) \quad \text{and} \quad S=\pi^2\sqrt{\frac{m^2}{-q^2}}.$$

We find:

$$T_{\mu\nu} = (-1)^{3s}\frac{Q^2\pi^2}{2}|\vec{q}|\left\{F_1u_\mu u_\nu + \frac{2F_2}{m^3}P_{(\mu}E_{\nu)} + F_3(q_\mu q_\nu - \eta_{\mu\nu}q^2)+\frac{F_5}{m^4}E_\mu E_\nu\right\}$$

$$E_\mu = \epsilon_{\mu\alpha\beta\gamma}P^\alpha q^\gamma S^\delta$$

Translated to non-relativistic frame

$$T_{00} = -\frac{Q^2\pi^2}{2}qJ_0(\vec{a}\times\vec{q})$$

$$T_{0i} = \frac{iQ^2\pi^2}{2}q\left[J_1(\vec{a}\times\vec{q})\right]^i = \frac{iQ^2\pi^2}{2}q(\vec{a}\times\vec{q})^i\left[\frac{J_1(\vec{a}\times\vec{q})}{\vec{a}\times\vec{q}}\right]$$

$$T_{ij} = \frac{Q^2\pi^2}{2}q\left[(\vec{a}\times\vec{q})^i(\vec{a}\times\vec{q})^j\right]\left[\frac{J_2(\vec{a}\times\vec{q})}{(\vec{a}\times\vec{q})^2}\right] + \frac{Q^2\pi^2}{2}\frac{q^iq^j-q^2\delta_{ij}}{q}\left[\frac{J_1(\vec{a}\times\vec{q})}{\vec{a}\times\vec{q}}\right]$$

Matches with the form factor obtained from

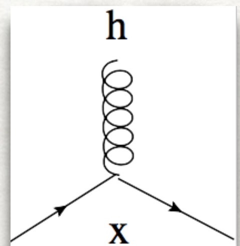
$$T_{\mu\nu} = F_{\mu\sigma}F^{\sigma}{}_{\nu} - \tfrac{1}{4}\eta_{\nu\nu}F^{\rho\sigma}F_{\rho\sigma}$$

minimal coupling captures all spin effects!

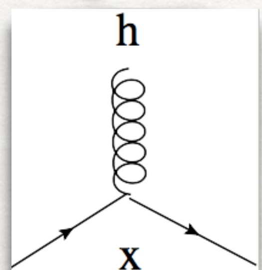
SUMMARY

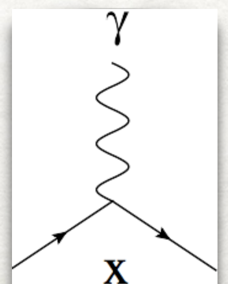
- **We have seen that properties of BH solutions can now be cleanly cast into on-shell elements providing convenient basis to manifest the simplicity of BHs.**

Schwarzschild

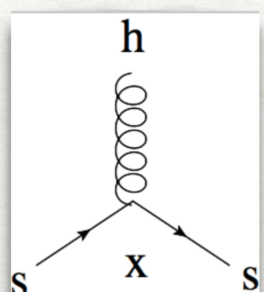


Reissner-Nordstorm

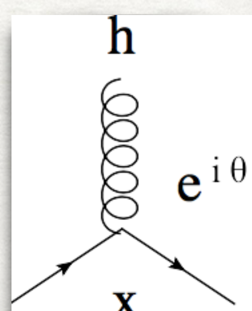




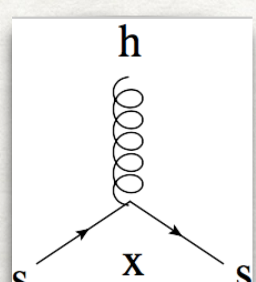
Kerr

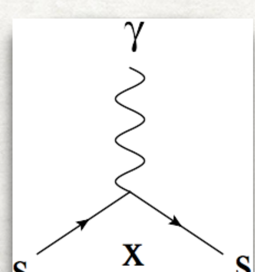


Taub-NUT



Kerr Newman





SUMMARY

- We have seen that in terms of on-shell basis, properties of BH solutions are cleanly captured
  → they are kinematically minimal
- The simplicity in the on-shell basis reflect hidden relations for the classical solutions: double copy, complex shifts, duality transformations
- These relations are in fact non-perturbative !
- More to understand at 2 PM, what selects the BH Compton amplitude ? (Causality ?)
- What is special about string amplitudes, and the difference between leading and subleading trajectories
- Can similar analysis applied to quasi-normal modes ?