The Gravity Dual of Lorentzian OPE Block

Heng-Yu Chen
National Taiwan University

(Based on 1912.04105 with Lung-Chuan Chen, Nozomu Kobayashi and Tatsuma Nishioka.)

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- The so-called "OPE block" in its simplest form is a bi-local operator arising from the OPE between two local primary operators: [Ferrara et al, Mack et al, 1970s]:

$$\mathcal{O}_i(x_1)\,\mathcal{O}_j(x_2) = \sum_k c_{ijk}\,\mathcal{B}_k(x_{12})$$

which is entirely kinematical and fixed by conformal symmetries.

- In Euclidean $\mathbb{R}^d$, the explicit form of $\mathcal{B}_{\Delta,J}(P_1, P_2)$ can be extracted from contraction with the shadow projector:

$$|\mathcal{O}_{\Delta,J}| = \frac{1}{\alpha_{\Delta,J}\,\alpha_{\bar{\Delta},J}}\,\frac{1}{J!(h-1)_J}\int [\mathrm{D}^d P]_{\mathrm{E}}\,|\,\tilde{\mathcal{O}}_{\bar{\Delta},J}(P, D_Z)\,\rangle\,\langle\,\mathcal{O}_{\Delta,J}(P,Z)\,|$$
$$\bar{\Delta} = d - \Delta\ , \qquad h = \frac{d}{2}\ ,$$

where D_Z is the spin Todorov operator, and $\tilde{\mathcal{O}}_{\Delta,J}(P, Z)$ is

$$\tilde{\mathcal{O}}_{\bar{\Delta},J}(P,Z) \equiv \frac{1}{J!(h-1)_J}\int [\mathrm{D}^d P']_{\mathrm{E}}\,E_{[\bar{\Delta},J]}(P,Z;P',D_{Z'})\,\mathcal{O}_{\Delta}(P',Z')$$
$$= \int [\mathrm{D}^d P']_{\mathrm{E}}\,\frac{1}{(-2P\cdot P')^{\bar{\Delta}}}\,\mathcal{O}_{\Delta,J}(P', Z\cdot\mathcal{I}(P',P))\ ,$$

- The generic form of OPE block can be expressed as:

$$\mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_1,P_2)=\frac{\gamma_{\Delta,J}}{\alpha_{\Delta,J}\,\alpha_{\bar{\Delta},J}}\frac{1}{J!(h-1)_J}\int[\mathrm{D}^dP_0]_{\mathrm{E}}\,E_{\Delta_1,\Delta_2,[\bar{\Delta},J]}(P_1,P_2,P_0;D_{Z_0})\,\mathcal{O}_{\Delta,J}(P_0,Z_0)$$

where the normalized Euclidean three point function is

$$E_{\Delta_1,\Delta_2,[\Delta,J]}(P_1,P_2,P_3;Z_3)=\frac{\left[-2P_1\cdot C_3\cdot P_2\right]^J}{P_{12}^{\frac{\Delta^+_{12}-\Delta+J}{2}}\,P_{13}^{\frac{\Delta+\Delta^-_{12}+J}{2}}\,P_{23}^{\frac{\Delta-\Delta^-_{12}+J}{2}}}\,,\qquad \Delta^{\pm}_{ij}=\Delta_i\pm\Delta_j$$

where $P_{ij}=-2P_i\cdot P_j$ and $C^{AB}=Z^AP^B-P^AZ^B$.

- Consider the Feynman parametrization for the P_0 dependences:

$$\begin{aligned}\frac{1}{P_{10}^{\frac{\bar{\Delta}+\Delta^-_{12}+J}{2}}\,P_{20}^{\frac{\bar{\Delta}-\Delta^-_{12}+J}{2}}}&=\mathrm{B}^{-1}\left(\bar{\delta}^+_{12}+J,\bar{\delta}^-_{12}+J\right)\int_0^1\frac{\mathrm{d}u}{u(1-u)}\frac{u^{\bar{\delta}^+_{12}+J}\,(1-u)^{\bar{\delta}^-_{12}+J}}{(u\,P_{10}+(1-u)\,P_{20})^{\bar{\Delta}+J}}\\&=2\,\mathrm{B}^{-1}\left(\bar{\delta}^+_{12}+J,\bar{\delta}^-_{12}+J\right)\frac{1}{(P_{12})^{\frac{\bar{\Delta}+J}{2}}}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,e^{\lambda\Delta^-_{12}}\frac{1}{(-2X(\lambda)\cdot P_0)^{\bar{\Delta}+J}}\end{aligned}$$

where $\mathrm{B}(x,y)$ is the beta function.

- It is interesting to note that the combined coordinate $X^A(\lambda)$:

$$\gamma_{12} \ : \ X^A(\lambda) \equiv \frac{e^{\lambda} P_1^A + e^{-\lambda} P_2^A}{P_{12}^{\frac{1}{2}}} \ , \qquad X(\lambda)^2 = -1$$

naturally belongs to AdS$_{d+1}$ space and is interpreted as a geodesic.

- We can now express the euclidean OPE block as:

$$\mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_1, P_2) = 2\,\kappa_{\Delta,J}\,\mathcal{N}_{12,[\Delta,J]}\,\Gamma(\bar{\Delta}+J)\,\frac{1}{J!(h-1)_J}\int [\mathrm{D}^d P_0]_{\mathrm{E}}$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \frac{1}{(-2P_1\cdot X(\lambda))^{\Delta_1}(-2P_2\cdot X(\lambda))^{\Delta_2}}\, \frac{\left[V_{0,12}\right]^J |_{Z_0 \to D_{Z_0}}}{(-2P_0\cdot X(\lambda))^{\bar{\Delta}+J}}\, \mathcal{O}_{\Delta,J}(P_0, Z_0)$$

where the unique three point scalar-scalar-tensor structure is:

$$V_{0,12} \equiv \frac{P_1\cdot C_0\cdot P_2}{P_1\cdot P_2} = \frac{-2P_1\cdot C_0\cdot P_2}{P_{12}}$$

- For the scalar case, we can express the scalar OPE block into:

$$\mathcal{B}^{(\mathrm{E})}_{\Delta,0}(P_1,P_2) = 2\,\kappa_{\Delta,0}\,\mathcal{N}_{12,[\Delta,0]}\,\Gamma(\bar{\Delta})\,\frac{1}{P_{12}^{\frac{\Delta^+_{12}}{2}}}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,e^{\lambda\Delta^-_{12}}\,\Phi^{(\mathrm{E})}_{\Delta,0}(X(\lambda))$$

  where we have introduced so-called HKLL scalar field $\Phi^{(\mathrm{E})}_{\Delta,0}(X)$ [Hamilton et. al. 2006] such that:

$$\Phi^{(\mathrm{E})}_{\Delta,0}(X) = \int [\mathrm{D}^d P_0]_{\mathrm{E}}\,K_{[\bar{\Delta},0]}(X,P_0)\,\mathcal{O}_{\Delta,0}(P_0)$$

  whose form was first derived from solving the AdS scalar equation of motion for arbitrary bulk point X.

- The natural interpretation of OPE block is that we integrate $\Phi^{(\mathrm{E})}_{\Delta,0}(X)$ along the geodesic γ_{12} with measure $e^{\Delta^-_{12}\lambda}$. [Czech et. al. 2016], [de Boer et. al. 2016], [da Cunha et. al. 2016]

- For $J \neq 0$, notice that we have the identity along γ_{12}:

$$V_{0,12} = -\frac{\mathrm{d}X(\lambda)}{\mathrm{d}\lambda} \cdot C_0 \cdot X(\lambda)$$

and rewrite the spinning OPE block as

$$\begin{aligned}\mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_1, P_2) &= 2\,\kappa_{\Delta,J}\,\mathcal{N}_{12,[\Delta,J]}\,\Gamma(\bar{\Delta}+J)\,\frac{1}{J!(h-1)_J}\int_{-\infty}^{\infty}\mathrm{d}\lambda\, e^{\lambda\Delta_{12}^{-}}\left[\frac{\mathrm{d}X(\lambda)}{\mathrm{d}\lambda}\cdot K\right]^J \Phi^{(\mathrm{E})}_{\Delta,J}(X(\lambda), \\ &= 2\,\kappa_{\Delta,J}\,\mathcal{N}_{12,[\Delta,J]}\,\Gamma(\bar{\Delta}+J)\,\frac{1}{P_{12}^{\frac{\Delta_{12}^{+}}{2}}}\int_{-\infty}^{\infty}\mathrm{d}\lambda\, e^{\lambda\Delta_{12}^{-}}\,\Phi^{(\mathrm{E})}_{\Delta,J}\left(X(\lambda), \frac{\mathrm{d}X(\lambda)}{\mathrm{d}\lambda}\right)\ .\end{aligned}$$

- Here we have introduced the spinning generalization of HKLL field:

$$\begin{aligned}\Phi^{(\mathrm{E})}_{\Delta,J}(X, W) &\equiv \frac{1}{2^J J!(h-1)_J}\int [\mathrm{D}^d P_0]_{\mathrm{E}}\, K_{[\bar{\Delta},J]}(X, P_0; W, D_{Z_0})\,\mathcal{O}_{\Delta,J}(P_0, Z_0) \\ &= \frac{1}{2^J}\int [\mathrm{D}^d P_0]_{\mathrm{E}}\,\frac{1}{(-2P_0\cdot X)^{\bar{\Delta}}}\,\mathcal{O}_{\Delta,J}(P_0, W\cdot\mathcal{J}(P_0, X))\end{aligned}$$

[HYC + Chen, Kobayashi, Nishioka 1912.04105]

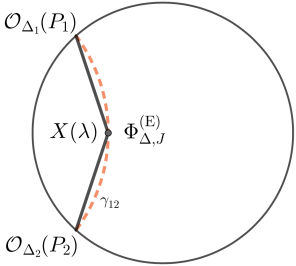
- It is interesting to note that the boundary polarization vector:

$$(W \cdot \mathcal{J}(P_0, X))_A = W_A - \frac{(W \cdot P_0)}{(X \cdot P_0)} X_A$$

  is necessary for preserving the bulk symmetry: $W^A \to W^A + \alpha X^A$. However the explicit X^A dependence can be packaged into total derivative and vanishes when imposing conservation law.

- We therefore have the Euclidean Spinning OPE block as integrating the pull-back of $\Phi^{(\mathrm{E})}_{\Delta,J}(X, W)$ along γ_{12}:

$$\mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_1, P_2) \quad =$$

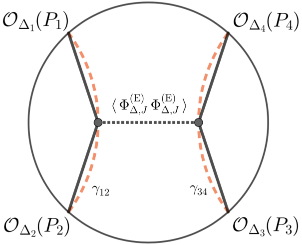


We can also consider the correlation function between two OPE blocks, which also has the natural holographic interpretation:

$$
\begin{aligned}
&\langle \mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_1,P_2)\,\mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_3,P_4)\rangle \\
&= \left(2\,\kappa_{\Delta,J}\,\Gamma(\bar{\Delta}+J)\right)^2 \mathcal{N}_{12,[\Delta,J]}\,\mathcal{N}_{34,[\Delta,J]} \\
&\quad\times \int_{-\infty}^{\infty}\mathrm{d}\lambda \int_{-\infty}^{\infty}\mathrm{d}\lambda' \frac{\left\langle \Phi^{(\mathrm{E})}_{\Delta,J}\left(X(\lambda),\frac{\mathrm{d}X(\lambda)}{\mathrm{d}\lambda}\right)\,\Phi^{(\mathrm{E})}_{\Delta,J}\left(\tilde{X}(\lambda'),\frac{\mathrm{d}\tilde{X}(\lambda')}{\mathrm{d}\lambda'}\right)\right\rangle_{\mathrm{E}}}{(-2P_1\cdot X(\lambda))^{\Delta_1}(-2P_2\cdot X(\lambda))^{\Delta_2}(-2P_3\cdot \tilde{X}(\lambda'))^{\Delta_3}(-2P_4\cdot \tilde{X}(\lambda'))^{\Delta_4}}\,.
\end{aligned}
$$

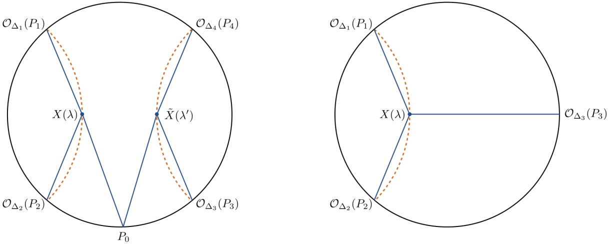
We can directly identify the integrand as the pull-back of bulk to bulk propagator for spin-J tensor field along γ_{12} and γ_{34}:

$$\langle \mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_1,P_2)\,\mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_3,P_4)\rangle \quad =$$



Connection with Geodesic Witten Diagram

For $J \in \mathbb{Z}_{\geq 0}$ in $\mathbb{R}^d$, the holographic dual of conformal block (plus its shadow) is "Geodesic Witten Diagram" (GWD) [Hijano et. al.]:



Here we have diagrammatically introduced the "split representation", and their building block: three point GWD [Chen, Kyono, Kuo.].
This naturally leads the holographic interpretation of OPE block as "half" of four point GWD.

Generalization to Lorentzian CFT/AdS?

- It is interesting to ask if we can extend the construction of Euclidean Holographic OPE block to Lorentzian case?

$$\mathcal{B}^{(\mathrm{L})}_{\Delta,J}(x_1,x_2) \underset{?}{=} \int [\mathrm{d}^d y]_{\mathrm{L}} \, \langle \, \mathcal{O}_1(x_1) \, \mathcal{O}_2(x_2) \, \tilde{\mathcal{O}}_{\tilde{\Delta},J}(y,d_z) \, \rangle \, \mathcal{O}_{\Delta,J}(y,z)$$

  Basically three issues:

  1. Integration Region for Space-like and Time-like separation?
  2. Which Three pt function?
  3. Which bulk field and geodesic?

- This belongs to part of the systematic program of relating boundary primary fields to bulk ones (i.e. Lorentzian HKLL?), their correlation functions to the corresponding Witten diagrams in Lorentzian setting.

We need to introduce instead momentum space shadow operator:

$$\mathbf{1} = |0\rangle\langle 0| + \sum_{\Delta,J} \frac{1}{C_{\Delta,J}\, C_{\bar{\Delta},J}} \int [\mathrm{D}^d p]_{\mathrm{L}}\, |\, \tilde{\mathcal{O}}_{\bar{\Delta},J}(-p)\,\rangle\, \langle\, \mathcal{O}_{\Delta,J}(p)\,|$$

Here we have introduced $\mathcal{O}_{\Delta,J}(p) \equiv \mathcal{O}_{\Delta,J}(p,z)$ as:

$$|\, \mathcal{O}_{\Delta,J}(p,z)\,\rangle \equiv \int [\mathrm{d}^d x]_{\mathrm{L}}\, e^{\mathrm{i}\,p\cdot x}\, \mathcal{O}_{\Delta,J}(x^0 + \mathrm{i}\,\epsilon, x^i, z)\, |0\rangle$$

and its shadow operator:

$$|\, \tilde{\mathcal{O}}_{\bar{\Delta},J}(p,z)\,\rangle \equiv \frac{1}{J!\,(h-1)_J}\, W_{[\bar{\Delta},J]}(p; z, d_{z'})\, |\, \mathcal{O}_{\Delta,J}(p,z')\,\rangle$$

where d_z is the spin Todorov operator and the two point Wightman function $W_{\Delta,J}(p; z_1, z_2)$ is obtained from conformal Ward identity:

$$W_{[\Delta,J]}(p; z_1, z_2) = C_{\Delta,J}\, \Theta(p^0)\, \Theta(-p^2)\, (-p^2)^{\Delta-h} \sum_{r=0}^{J} 2^r \binom{J}{r} \frac{(h-\Delta)_r}{(2-\Delta-J)_r}\, (z_1\cdot z_2)^{J-r} \left(\frac{(p\cdot z_1)(p\cdot z_2)}{-p^2}\right)^r$$

- We can contract with the OPE block to obtain Lorentzian OPE block:

$$\mathcal{B}^{(\mathrm{L})}_{\Delta,J}(x_1,x_2) = \frac{1}{C_{\Delta,J}\,C_{\bar{\Delta},J}} \int [\mathrm{D}^d p]_{\mathrm{L}}\, W_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,-p)\, W_{[\bar{\Delta},J]}(-p)\, \mathcal{O}_{\Delta,J}(p)$$

- To extract $W_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,p;z)$, we can start from Euclidean momentum space three point function $E_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,p;z)$:

$$E_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,-p) = \frac{\pi^{h+1}\mathcal{N}_{12,[\Delta,J]}}{2^J \sin(\pi(h-\tau))}\Big[Q_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,-p) - \kappa_{\Delta,J}\, E_{[\Delta,J]}(-p)\, Q_{\Delta_1,\Delta_2,[\bar{\Delta},J]}(x_1,x_2,-p)\Big]$$

where we have introduced the Q-kernel:

$$\begin{aligned} Q_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,-p;z_3) = &-\frac{1}{(x_{12}^2)^{\frac{\Delta_{12}^+-\tau}{2}}}\,\mathcal{D}_J\left(\delta_{12}^+,\, z_3\cdot\partial_1;\, \delta_{12}^-,\, z_3\cdot\partial_2\right)\left(\frac{p^2}{4x_{12}^2}\right)^{\frac{\tau-h}{2}} \\ &\times \int_0^1 \mathrm{d}u\, u^{\frac{\Delta_{12}^-+h}{2}-1}(1-u)^{\frac{h-\Delta_{12}^-}{2}-1} e^{\mathrm{i}\,p\cdot(ux_1+(1-u)x_2)}\, I_{h-\tau}\left(\sqrt{u(1-u)\,p^2\,x_{12}^2}\right) \end{aligned}$$

To obtain $W_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,p;z)$, we perform the analytic continuation $x^d = ix^0 + \epsilon$ on:

$$W_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,x;z) = \lim_{\epsilon_i\to 0}\int \frac{\mathrm{d}^{d-1}\mathbf{p}}{(2\pi)^{d-1}}\, e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^d}{2\pi}\, e^{-p^d(x^0-\mathrm{i}\,\epsilon_3)}\, E_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,p;z)$$

By deforming the contour along the branch cut, we recover:

$$\begin{aligned} W_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,x) &= \int \frac{[\mathrm{d}^d p]_{\mathrm{L}}}{(2\pi)^d}\, e^{\mathrm{i}\,p\cdot x}\, W_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,p) \\ &= \int \frac{[\mathrm{d}^d p]_{\mathrm{L}}}{(2\pi)^d}\, e^{\mathrm{i}\,p\cdot x}\left[-\frac{\pi^{h+1}\,\kappa_{\Delta,J}\,\mathcal{N}_{12,[\Delta,J]}}{2^J\sin\left(\pi(h-\tau)\right)}\, Q_{\Delta_1,\Delta_2,[\bar{\Delta},J]}(x_1,x_2,p)\, W_{\Delta,J}(p)\right]\Bigg|_{p^d\to\mathrm{i}\,p^0} \end{aligned}$$

A useful representation of Lorentzian OPE block is thus given by:

$$\mathcal{B}^{(\mathrm{L})}_{\Delta,J}(x_1,x_2) = -\frac{\pi^{h+1}\,\kappa_{\Delta,J}\,\mathcal{N}_{12,[\Delta,J]}}{2^J\sin\left(\pi(h-\tau)\right)}\int [\mathrm{D}^d p]_{\mathrm{L}}\, Q_{\Delta_1,\Delta_2,[\bar{\Delta},J]}(x_1,x_2,-p)\,\mathcal{O}_{\Delta,J}(p)\big|_{p^d\to\mathrm{i}\,p^0}$$

Consider holographic interpretation of $\mathcal{B}^{(\mathrm{L})}_{\Delta,J}(x_1,x_2)$, starting with $x_{12}^2>0$. For scalar $J=0$ case, the Q-kernel simplifies into:

$$\mathcal{B}^{(\mathrm{L})}_{\Delta}(x_1,x_2)=\frac{\pi^{h+1}\,\kappa_{\Delta,0}\,\mathcal{N}_{12,[\Delta,0]}}{2^{h-\Delta}\sin\left(\pi(h-\Delta)\right)}\,\frac{1}{(x_{12}^2)^{\frac{\Delta_{12}^+}{2}}}\int_0^1 \mathrm{d}u\,u^{\frac{\Delta_{12}^-}{2}-1}(1-u)^{-\frac{\Delta_{12}^-}{2}-1}$$
$$\times\int[\mathrm{D}^d p]_{\mathrm{L}}\,e^{\mathrm{i}\,p\cdot x(u)}\left(\sqrt{-p^2}\right)^{h-\Delta}\eta(u)^h\,J_{\Delta-h}\left(\sqrt{-p^2}\,\eta(u)\right)\mathcal{O}_\Delta(p)$$

and we can consider the Fourier transform identity for $J_\nu(|p_E|x)$:

$$J_\nu\left(\sqrt{p_0^2-\boldsymbol{p}^2}\,x\right)=\frac{1}{2^\nu\pi^h\Gamma(\nu-h+1)}\left(\frac{\sqrt{p_0^2-\boldsymbol{p}^2}}{x}\right)^\nu\int_{|y|\le x}[\mathrm{d}^d y]_{\mathrm{E}}\,(x^2-|y|^2)^{\nu-h}\,e^{\mathrm{i}(p_0 t+\mathrm{i}\,\boldsymbol{p}\cdot\boldsymbol{y})}$$

We can recover the original proposal for scalar OPE block :

$$\mathcal{B}^{(\mathrm{L})}_{\Delta}(x_1,x_2)=\frac{\pi\,\kappa_{\Delta,0}\,\mathcal{N}_{12,[\Delta,0]}}{\sin\left(\pi(h-\Delta)\right)\,\Gamma(1-\bar{\Delta})}\,\frac{1}{(x_{12}^2)^{\frac{\Delta_{12}^+}{2}}}\int_0^1 \mathrm{d}u\,u^{\frac{\Delta_{12}^-}{2}-1}(1-u)^{-\frac{\Delta_{12}^-}{2}-1}\Phi^{(\mathrm{L})}_{\Delta}\left(t(u),\boldsymbol{x}(u),\eta(u)\right)$$

- Here we have introduced the HKLL representation of AdS scalar field $\Phi_{\Delta}^{(\mathrm{L})}(t, \mathbf{x}, \eta)$:

$$\Phi_{\Delta}^{(\mathrm{L})}(t, \boldsymbol{x}, \eta) = \int_{t'^2+\boldsymbol{y}'^2 \leq \eta^2} \mathrm{d}t' \mathrm{d}^{d-1}\boldsymbol{y}' \left(\frac{\eta}{\eta^2 - t'^2 - \boldsymbol{y}'^2} \right)^{\bar{\Delta}} \mathcal{O}_{\Delta}\left(t + t', \, \boldsymbol{x} + \mathrm{i}\, \boldsymbol{y}'\right)$$

the integration is restricted to ensure the integral is well-defined.

- We have also introduced the physical AdS space geodesic coordinates:

$$x^{\mu}(u) \equiv u\, x_1^{\mu} + (1-u)\, x_2^{\mu} \,, \qquad \eta(u) \equiv \sqrt{u(1-u)\, x_{12}^2}$$

which can be lifted to the embedding AdS space geodesic via:

$$u = \frac{1}{1 + e^{-2\lambda}}$$

For $J \neq 0$, we can first consider the conserved primary operator:

$$p^{\mu_1}\,\mathcal{O}_{\Delta,\mu_1\mu_2\cdots\mu_J}(p,z) = 0$$

The Lorentzian OPE block now takes the form:

$$\mathcal{B}^{(\mathrm{L})}_{\Delta,J}(x_1,x_2) = \frac{(-1)^J\,\pi\,(\bar{\Delta}-1)_J\,\kappa_{\Delta,J}\,\mathcal{N}_{12,[\Delta,J]}}{\sin\left(\pi(h-\Delta)\right)\,\Gamma(1-\bar{\Delta})}\,\frac{1}{(x_{12}^2)^{\frac{\Delta_{12}^+}{2}}}$$
$$\times\,\frac{1}{2^J}\int_0^1 \mathrm{d}u\,u^{\frac{\Delta_{12}^-}{2}-1}(1-u)^{-\frac{\Delta_{12}^-}{2}-1}\,\Phi^{(\mathrm{L})}_{\mathrm{con},\,\mu_1\cdots\mu_J}\left(x^\mu(u),\eta(u)\right)\,w^{\mu_1}(u)\cdots w^{\mu_J}(u)$$

with the spinning HKLL field now given by: [Sarkar-Xiao et. al. 2014]

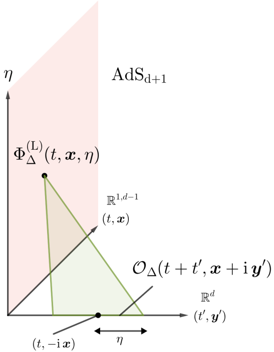
$$\Phi^{(\mathrm{L})}_{\mathrm{con},\,\mu_1\cdots\mu_J}(x^\mu,\eta) \equiv \frac{1}{\eta^J}\int_{t'^2+\boldsymbol{y}'^2\leq\eta^2}\mathrm{d}t'\,\mathrm{d}^{d-1}\boldsymbol{y}'\left(\frac{\eta}{\eta^2-t'^2-\boldsymbol{y}'^2}\right)^{\bar{\Delta}}\mathcal{O}_{\Delta,\mu_1\cdots\mu_J}\left(t+t',\boldsymbol{x}+\mathrm{i}\,\boldsymbol{y}'\right)$$

We can recover the same expression from Euclidean case by imposing conservation law.

For the non-conserved case, we have the following expression for $\Phi^{(\mathrm{L})}_{\Delta,J}$:

$$\begin{aligned}\Phi^{(\mathrm{L})}_{\Delta,J}(x^\mu(u),\eta(u)) = {} & \frac{1}{2^J\, J!(h-1)_J}\,(x_{12}^2)^{\bar\tau}\,\mathcal{D}_J\left(\bar\delta^+_{12},\, d_z\cdot\partial_1;\ \bar\delta^-_{12},\, d_z\cdot\partial_2\right)\,(x_{12}^2)^{-\bar\tau}\\ & \times \int_{t^2+\boldsymbol{y}^2\le\eta(u)^2} \mathrm{d}t\,\mathrm{d}^{d-1}\boldsymbol{y}\,\left(\frac{\eta(u)}{\eta(u)^2-t^2-\boldsymbol{y}^2}\right)^{\bar\tau}\mathcal{O}_{\Delta,J}\left(t(u)+t,\,\boldsymbol{x}(u)+\mathrm{i}\,\boldsymbol{y},z\right)\end{aligned}$$

The AdS bulk dual field is sourced by the boundary primary operator via shadow bulk-boundary propagator/HKLL kernel over sub-region:



Next let us consider time-like case $x_{12}^2 < 0$, we can again analytically continue from Euclidean case:

$$x_{\mathrm{E},12}^2 = (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + (\tau_{\mathrm{E},1} - \tau_{\mathrm{E},2})^2 \quad \rightarrow \quad x_{\mathrm{L},12}^2 = (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 - (t_1 - t_2)^2 + \mathrm{i}\,\epsilon\,(t_1 - t_2)$$

where $\epsilon = \epsilon_1 - \epsilon_2 > 0$, i. e. $\mathcal{O}_1$ is in the future light cone of $\mathcal{O}_2$. If we next take $\epsilon \to 0$, this amounts to replacing x_{12}^2 by $e^{i\pi}|x_{12}^2|$.

For scalar $J = 0$ case, we have:

$$\mathcal{B}_{\Delta}^{(\mathrm{L})}(x_1, x_2) = \frac{\pi^{h+1}\,\kappa_{\Delta,0}\,\mathcal{N}_{12,[\Delta,0]}\,e^{-\mathrm{i}\pi\frac{\Delta_{12}^{+}-h}{2}}}{\sin\left(\pi(h-\Delta)\right)} \int [\mathrm{D}^d p]_{\mathrm{L}} \frac{1}{\left(|x_{12}^2|\right)^{\frac{\Delta_{12}^{+}-\bar{\Delta}}{2}}} \left(\frac{-p^2}{4|x_{12}^2|}\right)^{\frac{h-\Delta}{2}}$$
$$\times \int_0^1 \mathrm{d}u\, u^{\frac{\Delta_{12}^{-}+h}{2}-1}(1-u)^{\frac{h-\Delta_{12}^{-}}{2}-1}\, e^{\mathrm{i}\,p\cdot(u x_1 + (1-u) x_2)} I_{\Delta-h}\left(\sqrt{-u(1-u)p^2|x_{12}^2|}\right) \mathcal{O}_{\Delta}(p)$$

where we have introduced the follow vectors (more about $\chi(u)$ later):

$$x^{\mu}(u) = u\,x_1^{\mu} + (1-u)\,x_2^{\mu}\,, \quad \chi(u) = \sqrt{u(1-u)\,|x_{12}^2|} = \sqrt{-u(1-u)\,x_{12}^2}$$

To obtain similar spacetime interpretation as in space-like case, consider:

$$I_{\Delta-h}\left(\sqrt{p_0^2-\boldsymbol{p}^2}\,\chi(u)\right)=\frac{1}{2^{\Delta-h}\pi^h\Gamma(1-\bar{\Delta})}\left(\frac{\sqrt{p_0^2-\boldsymbol{p}^2}}{\chi(u)}\right)^{\Delta-h}\int_{t^2+\boldsymbol{y}^2\leq\chi(u)^2}[\mathrm{d}^d y]_{\mathrm{E}}\frac{e^{-p_0 t+\mathrm{i}\,\boldsymbol{p}\cdot\boldsymbol{y}}}{(\chi(u)^2-t^2-\boldsymbol{y}^2)^{\bar{\Delta}}}$$

We can rewrite the time-like OPE block as:

$$\mathcal{B}_{\Delta}^{(\mathrm{L})}(x_1,x_2)=\frac{\pi\,\kappa_{\Delta,0}\,\mathcal{N}_{12,[\Delta,0]}\,e^{-\mathrm{i}\pi\frac{\Delta_{12}^{+}-h}{2}}}{\sin\left(\pi(h-\Delta)\right)\,\Gamma(1-\bar{\Delta})}\frac{1}{|x_{12}^2|^{\frac{\Delta_{12}^{+}}{2}}}\int_0^1\mathrm{d}u\,u^{\frac{\Delta_{12}^{-}}{2}-1}(1-u)^{-\frac{\Delta_{12}^{-}}{2}-1}$$
$$\times\int_{t^2+\boldsymbol{y}^2\leq\chi(u)^2}[\mathrm{d}^d y]_{\mathrm{E}}\left(\frac{\chi(u)}{\chi(u)^2-t^2-\boldsymbol{y}^2}\right)^{\bar{\Delta}}\mathcal{O}_{\Delta}\left(t(u)+\mathrm{i}\,t,\boldsymbol{x}(u)+\boldsymbol{y}\right)$$

where the corresponding HKLL field is

$$\Phi_{\Delta}^{(\mathrm{LT})}(t,\boldsymbol{x},\chi)\equiv\int_{t'^2+\boldsymbol{y}'^2\leq\chi^2}\mathrm{d}t'\mathrm{d}^{d-1}\boldsymbol{y}'\left(\frac{\chi}{\chi^2-t'^2-\boldsymbol{y}'^2}\right)^{\bar{\Delta}}\mathcal{O}_{\Delta}\left(t+\mathrm{i}\,t',\boldsymbol{x}+\boldsymbol{y}'\right)$$

In fact the geodesic $(x^\mu(u), \chi(u))$ now is in a hyperbloid in $\mathbb{R}^{2,d}$ instead:

$$(X^+, X^-, X^\mu) = \frac{1}{\chi}\left(1,\, x^2 - \chi^2,\, x^\mu\right) \ , \ X^2 = +1$$

This geometry can be obtained from the analytic continuation of de-Sitter space instead:

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\chi^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}{\chi^2}$$

Two ways of extending $\mathbb{R}^{1,d-1}$ in $\mathbb{R}^{2,d}$ correspond space- and time-like!