Entanglement, free energy and C-theorem in defect CFT

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Dec 12, 2019

NCTS annual theory meeting 2019

based on 1810.06995 with N. Kobayashi, Y. Sato and K. Watanabe

Outline

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1. Defect conformal field theories

2. Entanglement entropy and sphere free energy in DCFT

3. Towards a C-theorem in DCFT

Defects in quantum field theory

Defects = Non-local objects in QFTs

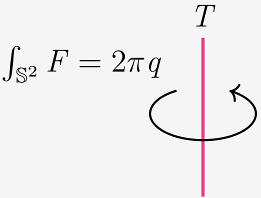
- Defined by boundary conditions around them
- Many examples:

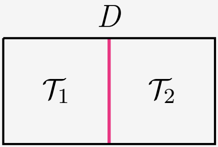
1-dim : Line operators (Wilson-'t Hooft loops)

2-dim : Surface operators

Codim-1 : Domain walls, interfaces and boundaries

Codim-2 : Entangling surface for entanglement entropy





Conformal defects

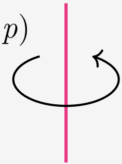
- In Euclidean CFT$_d$, the conformal group is $\mathrm{SO}(d+1,1)$
- p-dimensional conformal defects $\mathcal{D}^{(p)}$ are either flat or spherical, preserving

$\mathrm{SO}(p+1,1)$: conformal symmetry on defects

$\mathrm{SO}(d-p)$: rotation in the transverse direction

$\mathbb{R}^p$

$\mathrm{SO}(d-p)$



$\mathcal{D}^{(p)}$

- Defects allow for defect local operators $\hat{\mathcal{O}}_n(\hat{x})$
  $\hat{x}^a$: parallel coordinates on $\mathcal{D}^{(p)}$ $(a = 1, \cdots, p)$

One-point function

The residual conformal symmetry constrains one-point functions

- Scalar primary:

$$\langle \mathcal{O}(x) \rangle^{(\mathrm{DCFT})} = \frac{a_{\mathcal{O}}}{|x_\perp|^\Delta} \,, \qquad x_\perp^i \text{ :transverse coordinates}$$

$$(i = p+1, \cdots, d)$$

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- Stress tensor:

$$\langle\, T^{ab}(x)\,\rangle^{(\mathrm{DCFT})} = \frac{d-p-1}{d}\,\frac{a_T}{|x_\perp|^d}\,\delta^{ab}$$

$$\langle\, T^{ij}(x)\,\rangle^{(\mathrm{DCFT})} = -\frac{a_T}{|x_\perp|^d}\left(\frac{p+1}{d}\delta^{ij} - \frac{x_\perp^i x_\perp^j}{|x_\perp|^2}\right)$$

$$\langle\, T^{ai}(x)\,\rangle^{(\mathrm{DCFT})} = 0$$

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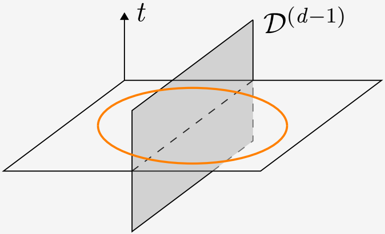
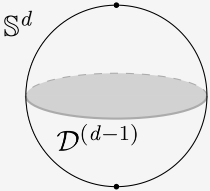
$$\langle\, T^{ai}(x) \,\rangle^{(\mathrm{DCFT})} = 0$$

N.B. $\langle\, T^{\mu\nu}(x) \,\rangle^{(\mathrm{DCFT})} = 0$ in BCFT $(p = d-1)$ [McAvity-Osborn 95]

Goal of this talk

- In DCFT we will study
  - Entanglement entropy across a sphere
  - Sphere free energy
- How do they depend on defect data?
- What is the measure of degrees of freedom associated to defects?





Outline

Definition of entanglement entropy

Divide a system to A and $B = \bar{A}$: $\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$

Entanglement entropy

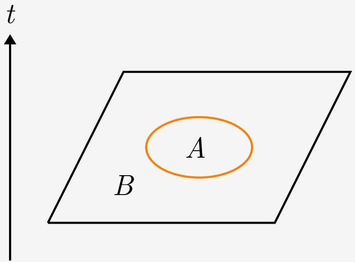
$$S_{\text{EE}} = -\text{tr}_A\, \rho_A \log \rho_A$$

- The reduced density matrix

$$\rho_A \equiv \text{tr}_B\, \rho_{\text{tot}}$$

- For a pure ground state $|\Psi\rangle$

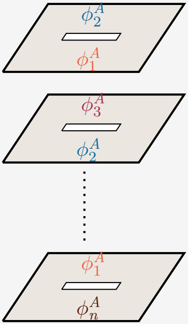
$$\rho_{\text{tot}} = |\Psi\rangle \langle \Psi|$$



Replica trick and Rényi entropy

Entanglement entropy

$$S_{\text{EE}} = \lim_{n\to 1} S_n$$



Replica trick and Rényi entropy

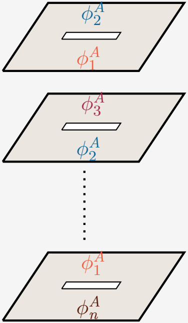
Entanglement entropy

$$S_{\text{EE}} = \lim_{n \to 1} S_n$$

n^{th} Rényi entropy

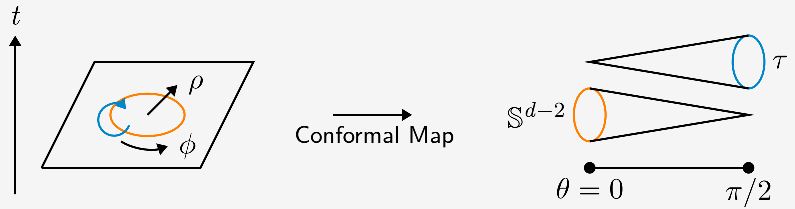
$$S_n = \frac{1}{1-n} \log \frac{Z_n}{(Z_1)^n}$$

Z_n: partition function on the n-fold cover branched over A



Conformal map for a spherical region

- Exact calculations limited to a few simple cases (free fields, planar ∂A in CFT, etc)
- For a spherical region in CFT, however, there exists a conformal map to the n-fold cover of $\mathbb{S}^d$ (CHM map) [Casini-Huerta-Myers 11]



$\mathbb{R}^d$ $\qquad\qquad$ $\mathbb{S}^d$

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\Omega_{d-2}^2 \qquad \mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\tau^2 + \cos^2\theta\,\mathrm{d}\Omega_{d-2}^2$$

Entanglement entropy across a sphere

- CFT partition function is invariant under the CHM map

$$Z_n^{(\mathrm{CFT})} = Z^{(\mathrm{CFT})}[\mathbb{S}_n^d] \,, \qquad \mathbb{S}_n^d\text{: } n\text{-fold cover of } \mathbb{S}^d \qquad (\tau \sim \tau + 2\pi n)$$

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- Hence the Rényi entropy is given by

The Rényi entropy across a sphere in CFT

$$S_n = \frac{1}{1-n} \log \frac{Z^{(\mathrm{CFT})}[\mathbb{S}_n^d]}{\left(Z^{(\mathrm{CFT})}[\mathbb{S}^d]\right)^n}$$

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$$S_n = \frac{1}{1-n} \log \frac{Z^{(\mathrm{CFT})}[\mathbb{S}^d_n]}{(Z^{(\mathrm{CFT})}[\mathbb{S}^d])^n}$$

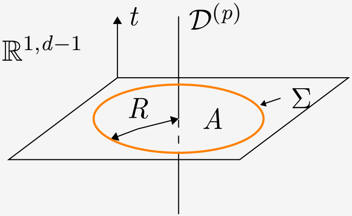
- Taking $n \to 1$ limit

Entanglement entropy across a sphere in CFT [Casini-Huerta-Myers 11]

$$S_{\mathrm{EE}} = \log Z^{(\mathrm{CFT})}[\mathbb{S}^d]$$

up to UV divergence

Entanglement entropy across a sphere with defects



- A: a ball centered at the origin
- $\mathcal{D}^{(p)}$: a p-dim flat defect
- After a conformal transformation (CHM map [Casini-Huerta-Myers 11, Jensen-O'Bannon 13])

The n-th Rényi entropy

$$S_n^{(\mathrm{DCFT})} = \frac{1}{1-n} \log \frac{Z^{(\mathrm{DCFT})}[\mathbb{S}_n^d]}{(Z^{(\mathrm{DCFT})}[\mathbb{S}^d])^n}$$

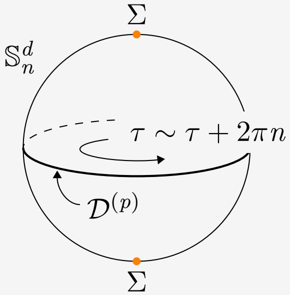
$\mathbb{S}_n^d$: n-fold cover of $\mathbb{S}^d$

Defect entropy

- The excess of EE is measured by

Defect entropy

$$S_{\text{defect}} \equiv \lim_{n\to 1}\left(S_n^{(\text{DCFT})} - S_n^{(\text{CFT})}\right)$$



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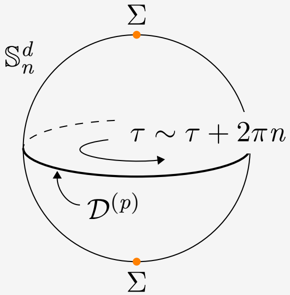
$$S_{\text{defect}} \equiv \lim_{n\to 1}\left(S_n^{(\text{DCFT})} - S_n^{(\text{CFT})}\right) = \lim_{n\to 1}\frac{1}{1-n}\log\frac{\langle\,\mathcal{D}^{(p)}\,\rangle_n}{\langle\,\mathcal{D}^{(p)}\,\rangle^n}$$



$$\langle\,\mathcal{D}^{(p)}\,\rangle_n \equiv \frac{Z^{(\text{DCFT})}[\mathbb{S}_n^d]}{Z^{(\text{CFT})}[\mathbb{S}_n^d]}$$

$$\langle\,\mathcal{D}^{(p)}\,\rangle \equiv \langle\,\mathcal{D}^{(p)}\,\rangle_1 \quad (\text{vev of } \mathcal{D}^{(p)})$$

$n \to 1$ limit

- Expansion around $n = 1$ ($\delta g_{\mu\nu} = O(n-1)$)

$$\log Z^{(\mathrm{DCFT})}[\mathbb{S}^d_n] = \log Z^{(\mathrm{DCFT})}[\mathbb{S}^d] - \frac{1}{2}\int_{\mathbb{S}^d} \delta g_{\mu\nu} \langle T^{\mu\nu} \rangle^{(\mathrm{DCFT})}_{\mathbb{S}^d} + O\left((n-1)^2\right)$$

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- In CFT [Casini-Huerta-Myers 11]

$$\langle\, T^{\mu\nu} \,\rangle^{(\mathrm{CFT})}_{\mathbb{S}^d} = 0$$

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- In DCFT

$$\langle\, T^{\mu\nu} \,\rangle^{(\mathrm{DCFT})}_{\mathbb{S}^d} \neq 0 \quad (\propto a_T)$$

Universal relation

Defect entropy and sphere free energy [Kobayashi-TN-Sato-Watanabe 18]

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(p)} \rangle - \frac{2(d-p-1)\, \pi^{d/2+1}}{\sin(\pi p/2)\; d\, \Gamma(p/2+1)\; \Gamma((d-p)/2)}\, a_T$$

- Dimensional regularization is assumed
- Equality holds up to UV divergences
- Reproduces a known result when $p = 1$ [Lewkowycz-Maldacena 13]
- The second term in rhs vanishes when $p = d-1$ (codimension-one)

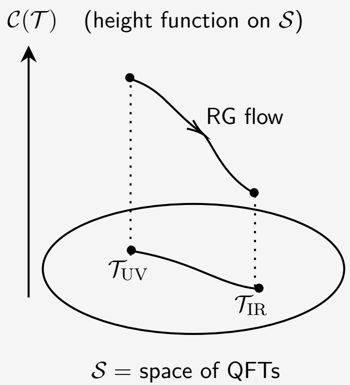
Outline

C-theorem

C-theorem (weak)

$\exists$ a function $\mathcal{C}(\mathcal{T})$ on a theory space s.t.

$$\mathcal{T}_{\mathrm{UV}} \xrightarrow[\text{RG flow}]{} \mathcal{T}_{\mathrm{IR}}$$
$$\Rightarrow \quad \mathcal{C}(\mathcal{T}_{\mathrm{UV}}) \geq \mathcal{C}(\mathcal{T}_{\mathrm{IR}})$$



- $\mathcal{C}(\mathcal{T})$ called a $\mathcal{C}$-function ($\approx$ resource measure)
- Regarded as a measure of degrees of freedom in QFT
- Constrains the dynamics under RG if holds

Examples and conjectures

- $2d$: Zamolodchikov's c-theorem [Zamolodchikov 86]
- even d: A-theorem ($\langle T^{\mu}_{\mu} \rangle_{\mathbb{S}^d} \propto A$) [Cardy 88, Komargodski-Schwimmer 11]

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Generalized F-theorem conjecture [Giombi-Klebanov 14]

$$\tilde{F} \equiv \sin\left(\frac{\pi d}{2}\right) \log Z[\mathbb{S}^d], \qquad \tilde{F}_{\mathrm{UV}} \ge \tilde{F}_{\mathrm{IR}}$$

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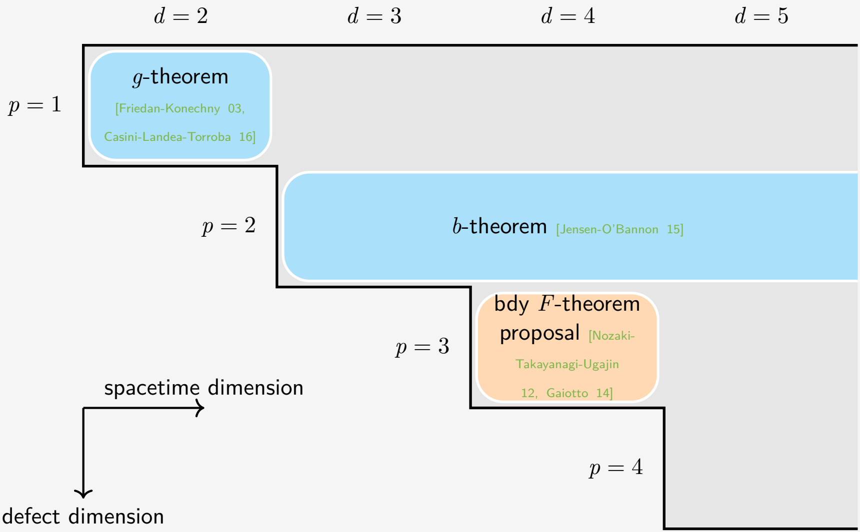
- Reduces to the F- and A-theorems

$$\tilde{F} = \begin{cases} F & d : \text{odd} \\ \frac{\pi}{2} A \quad (\text{conformal anomaly}) & d : \text{even} \end{cases}$$

Status of C-theorems (+ conjectures) in DCFT



$\mathcal{C}$-function in DCFT?

- Candidate of $\mathcal{C}$-functions
  - Entanglement entropy: holographic model $(p = d - 1)$
    [Estes-Jensen-O'Bannon-Tsatis-Wrase 14]
  - Sphere free energy: bdy F-thm, b-thm $(p = 2)$,
    Wilson loop RG flow $(d = 4, p = 1)$ [Beccaria-Giombi-Tseytlin 17]
- These two agree when $p = d - 1$ due to the universal relation

Are both $\mathcal{C}$-functions for any d and p?

$\mathcal{C}$-theorem in DCFT: conjecture

- Defect RG flow triggered by a relevant defect operator:

$$I = I_{\text{DCFT}} + \hat{\lambda} \int \mathrm{d}^p \hat{x} \sqrt{\hat{g}}\, \hat{\mathcal{O}}(\hat{x})$$

Conjecture [Kobayashi-TN-Sato-Watanabe 18]

The universal part of the sphere free energy

$$\tilde{D} \equiv \sin\left(\frac{\pi p}{2}\right) \log |\langle \mathcal{D}^{(p)} \rangle|$$

does not increase along any defect RG flow

$$\tilde{D}_{\text{UV}} \geq \tilde{D}_{\text{IR}}$$

- Same form as the generalized F-thm: $\tilde{F} = \sin\left(\frac{\pi d}{2}\right) \log Z[\mathbb{S}^d]$

Checks

- Sphere free energy always decreases under defect RG flows in
  - Conformal perturbation theory
  - Wilson loops $(p = 1)$ in $3d$ and $4d$
  - Holographic models (a proof assuming null energy condition)
- Entanglement entropy DOES NOT decrease in
  - Wilson loop RG flows [Kobayashi-TN-Sato-Watanabe 18], surface operators $(p = 2)$ [Jensen-O'Bannon-Robinson-Rodgers 18]
  - Holographic Wilson loops [Kumar-Silvani 16, 17] and surface operators [Rodgers 18]

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Summary and future work

- Summary:
  - Find the universal relation between EE and sphere free energy
  - Derive the interface entropy as Calabi's diastasis
  - Propose a $\mathcal{C}$-theorem in DCFT
- Future work:
  - Proof in SUSY theories? (cf. F- and a-maximizations [Jafferis 10, Closset-Dumitrescu-Festuccia-Komargodskia-Seiberg 12, Intriligator-Wecht 03])
  - Proof using entropic inequalities as in g-thm [Casini-Landea-Torroba 16]?
  - Constrains on the dynamics of defect RG flows?