WEAK COSMIC CENSORSHIP CONJECTURE— PROOF & APPLICATION

Feng-Li Lin (NTNU)

Based on 1902.00949 (PRD 2019) & on-going with
Bo Ning (Sichuan U) & Baoyi Chen (Caltech)

MOTIVATION

- The black hole contains a curvature singularity, which will not be naked if the mass M, charge Q and angular momentum J=aM satisfying

$$M^2 \geq a^2 + Q^2.$$

- When the equality holds, it is called extremal black hole.
- Penrose proposed the so-called weak cosmic censorship conjecture (WCCC) that a gravitational singularity should be hidden inside a black hole horizon.
- This implies non-existence of super-extremal black hole.

GEDENKEN EXPERIMENT

- Can we overspin or overcharge to destroy a black hole by throwing matter of large spin or charge into it?

- Consider throwing a charged particle of mass m and charge e into an extremal Reissner-Nordstrom (RN) black hole with M=Q. The energy of the particle is given by

$$E = -(mu_\mu + eA_\mu)\xi^\mu \geq e\Phi_H = e \qquad \text{with } \Phi_H = (-A_\mu \xi^\mu)|_H = 1 \text{ for extermal RN black hole.}$$

Thus, M+E>=Q+e, impossible to overcharge.

- Dynamically, this is a very complicated problem due to the self-force and self-energy. E.g., self-force of electric charged particle:

$$ma^\mu = f^\mu_{\text{ext}} + e^2(\delta^\mu{}_\nu + u^\mu u_\nu)\left(\frac{2}{3m}\frac{Df^\nu_{\text{ext}}}{d\tau} + \frac{1}{3}R^\nu{}_\lambda u^\lambda\right) + 2e^2 u_\nu \int_{-\infty}^{\tau^-} \nabla^{[\mu} G_+{}^{\nu]}{}_{\lambda'}(z(\tau), z(\tau'))u^{\lambda'}\, d\tau',$$

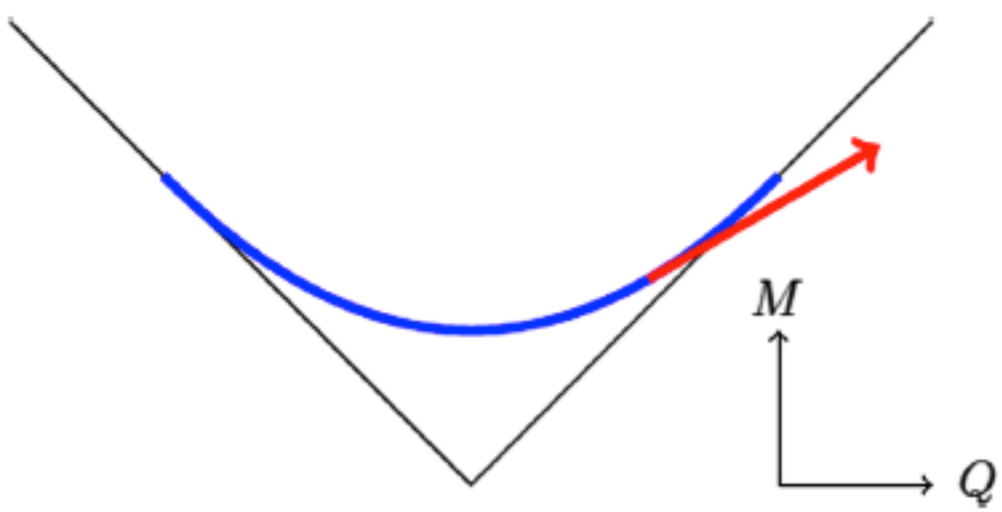
HUBENY'S ARGUMENT

- Hubeny (1999) argued that it is possible to overcharge a near-extremal black hole. Parametrizing the near-extremality: $\epsilon = \sqrt{1-q^2}$ with $q = Q/M$.

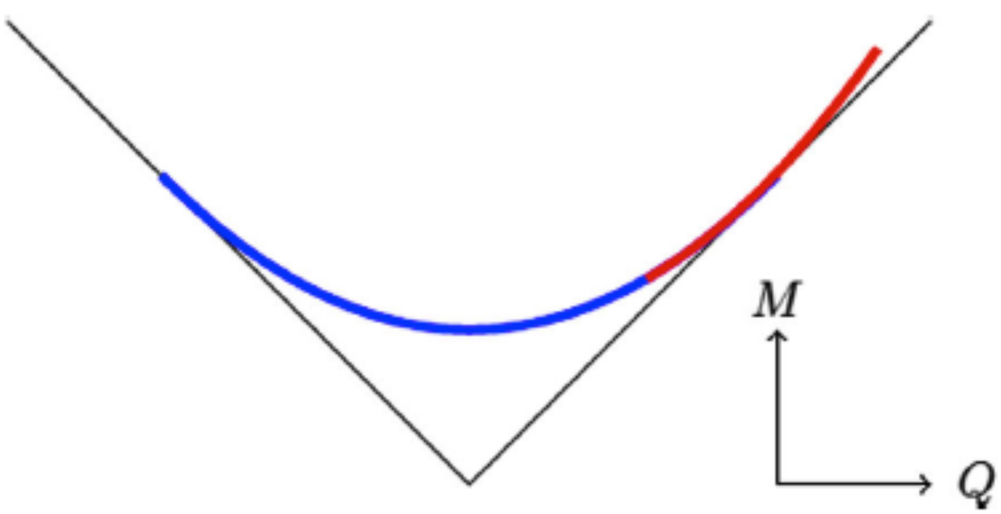
- The EM potential now is $\Phi_H = Q/r_+ \approx 1-\epsilon$, and the energy of the charged particle $E \geq e(1-\epsilon)$. Thus, we have

$$M + E - (Q + e) \approx -e\epsilon + M\epsilon^2/2$$

- It seems that we can overcharge to destroy a black hole if $e > \epsilon M/2$. However, this is not the whole story since the e^2 effect is involved for the argument without also including it in estimating E.



Hubeny's argument (1999)



Including 2nd order effect by

Sorce & Wald (2017)

SORCE & WALD

- Thus, to fix the WCCC violation of Hubeny's argument, one needs to take into account the 2nd order effect.
- Also, a general proof is needed to go beyond particle-matter.
- In 2017, Sorce & Wald carried out this task and gave a general proof of WCCC based on the variational identities, which is the generalization of black hole's first law when considering the falling-in of the generic matters up to 2nd order variation.

DESTROY A BTZ BLACK HOLE

- The gedanken experiment to overspin or overcharge a BH by throwing chargers or spinning matters is an operational statement toward the third law of black hole mechanics/ thermodynamics (Israel, 1986).
- By AdS/CFT correspondence, this third law statement corresponds to the third law of thermodynamics for the dual CFTs.
- This also motivates us to check if one can violate WCCC for BTZ black holes.
- To enlarge the scope, we incorporate torsion to go beyond the gravity of Riemannian geometry.

CONSTRAIN HIGHER GRAVITIES BY WCCC

- Promote WCCC to a principle, we may constrain the higher gravities from UV corrections.
- Quartic corrections to Einstein-Maxwell: (Katz, Motl & Padi, 2007)

$$\Delta L = c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + c_4 R F_{\mu\nu}F^{\mu\nu} + c_5 R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho} + c_6 R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + c_7 F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + c_8 F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} + c_9(\nabla^\mu F_{\mu\rho})(\nabla_\nu F^{\nu\rho})\,.$$

$$\begin{aligned} g_{rr}^{-1} = 1 - \frac{\kappa m}{r} + \frac{\kappa q^2}{2r^2} &+ c_2\left(\frac{3\kappa^3 m q^2}{r^5} - \frac{6\kappa^3 q^4}{5r^6} - \frac{4\kappa^2 q^2}{r^4}\right) \\ &+ c_3\left(\frac{12\kappa^3 m q^2}{r^5} - \frac{24\kappa^3 q^4}{5r^6} - \frac{16\kappa^2 q^2}{r^4}\right) + c_4\left(\frac{14\kappa^2 m q^2}{r^5} - \frac{6\kappa^2 q^4}{r^6} - \frac{16\kappa q^2}{r^4}\right) \\ &+ c_5\left(\frac{5\kappa^2 m q^2}{r^5} - \frac{11\kappa^2 q^4}{5r^6} - \frac{6\kappa q^2}{r^4}\right) + c_6\left(\frac{7\kappa^2 m q^2}{r^5} - \frac{16\kappa^2 q^4}{5r^6} - \frac{8\kappa q^2}{r^4}\right) \\ &+ c_7\left(-\frac{4\kappa q^4}{5r^6}\right) + c_8\left(-\frac{2\kappa q^4}{5r^6}\right)\,, \end{aligned}$$

- The condition for extremal black hole:

$$m = \sqrt{\frac{2}{\kappa}}|q| \times \left(1 - \frac{4}{5q^2}c_0\right) \qquad c_0 \equiv c_2 + 4c_3 + \frac{c_5}{\kappa} + \frac{c_6}{\kappa} + \frac{4c_7}{\kappa^2} + \frac{2c_8}{\kappa^2}\,.$$

- WCCC will lead to the condition: $\delta m - \sqrt{\frac{2}{\kappa}}\delta q - \frac{4c_0}{5q^2}\delta q \geq 0\,.$
- Check if consistent with linear perturbation around extremal BH.

OUTLINE

- 1. Review of Source & Wald: proof of WCCC in Einstein gravity
  - a. Variational Identities & Canonical Energy
  - b. Gedanken Experiments to destroy a Kerr-Newman Black holes
- 2. Our proof for WCCC in Mielke-Baekler (MB) model of Topological Massive Gravity by adopting Source & Wald
- 3. Conclusion

WALD'S NOETHER METHOD

- To construct the energy and its variation in GR, we can follow Wald's Noether method. Introduce Lagrangian n-form $L(\phi)$ with $\phi = (g_{\mu\nu}, \psi)$, its variation yields $\delta L = E(\phi)\delta\phi + d\Theta(\phi, \delta\phi)$.

- One can then define the symplectic and Noether current (n-1)-forms:

$$\omega(\phi, \delta_1\phi, \delta\phi_2) = \delta_1\Theta(\phi, \delta_2\phi) - \delta_2\Theta(\phi, \delta_1\phi) \Leftrightarrow \delta p \wedge \delta x \quad (\text{with } \Theta \Leftrightarrow p\delta x)$$

$$J_\xi = \Theta(\phi, \mathcal{L}_\xi\phi) - i_\xi L$$

- Easy to show $dJ_\xi = -E(\phi)\mathcal{L}_\xi\phi$ so that $J_\xi = dQ_\xi + C_\xi$ with $C_\xi \propto E(\phi)$

- Use the identity $\delta J_\xi = \omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) + d[i_\xi\Theta(\phi, \delta\phi)]$, one can derive

$$\delta h_\xi := \int_\Sigma \omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) = \int_\Sigma [\delta J_\xi - d i_\xi \Theta(\delta\phi)] = \int_{\partial\Sigma} [\delta Q_\xi - i_\xi\Theta(\phi, \delta\phi)] + \int_\Sigma \delta C_\xi$$

$$\Leftrightarrow \delta H = \delta p\, \dot{x} - \delta x\, \dot{p} = \delta(p\dot{x} - L) + e.o.m.$$

- Note that $\omega(\phi, \delta\phi, \mathcal{L}_\xi\phi = 0) = 0$ if ξ is a Killing vector field. Thus,

$$\int_{\partial\Sigma} [\delta_\xi Q_\xi - i_\xi\Theta(\phi, \delta\phi)] = -\int_\Sigma \delta C_\xi$$

2ND ORDER VARIATIONAL & CANONICAL ENERGY

- The contribution of the boundary integral from infinity yields variation of ADM quantities: for $\xi = t + \Omega_H \varphi$,

$$\int_\infty [\delta Q_\xi - i_\xi \Theta(\phi, \delta\phi)] = \delta M - \Omega_H \delta J$$

- For 2nd order variation, we define canonical energy to characterize the effect of gravitational & EM fluxes

$$\mathcal{E}_\Sigma(\phi, \delta\phi) = \int_\Sigma \omega(\phi, \delta\phi, \mathcal{L}_\xi \delta\phi)$$

- Vary the 1st order variation Id, we can arrive

$$\mathcal{E}_\Sigma(\phi, \delta\phi) = \int_{\partial\Sigma} [\delta^2 Q_\xi - i_\xi \delta\Theta(\phi, \delta\phi)] + \int_\Sigma \delta^2 C_\xi + \int_\Sigma i_\xi (\delta E \cdot \delta\phi)$$

- Similarly, we can define the 2nd order variation of ADM:

$$\int_\infty [\delta^2 Q_\xi - i_\xi \delta\Theta(\phi, \delta\phi)] = \delta^2 M - \Omega_H \delta^2 J$$

EINSTEIN-MAXWELL

- For black hole, we need to evaluate the boundary term at horizon B. This needs explicit form, for Einstein-Maxwell:

$$\Theta_{abc}(\phi, \delta\phi) = \frac{1}{16\pi}\epsilon_{dabc}(\nabla^b \delta g_{db} - g^{ce}\nabla_d \delta g_{ce} - 4F_d^b \delta A_b) := \Theta^{GR} + \Theta^{EM}$$

$$Q_{ab} = -\frac{1}{16\pi}\epsilon_{abcd}(\nabla^c \xi^d + 2F^{cd}A_e \xi^e) := Q^{GR} + Q^{EM}$$

$$C_{bcda} = \epsilon_{ebcd}(T_a^e + J^e A_a), \text{ where } 8\pi T^{de} := G^{de} - 8\pi T_{EM}^{de}, \quad 4\pi J^a = \nabla_b F^{ab}$$

$$\delta C_{bcda} = \epsilon_{ebcd}(\delta T_a^e + A_a \delta J^e)$$

- Note that $\xi = 0$ on B of non-extremal BH, thus $i_\xi \Theta|_B = 0$. Thus, $\int_B \delta Q_\xi^{GR} = \frac{\kappa}{8\pi}\delta A_B, \quad \int_B \delta Q_\xi^{EM} = \Phi_H \delta Q_B$, we then have the 1st variational Id:

$$\delta M - \Omega_H \delta J - \frac{\kappa}{8\pi}\delta A_B - \Phi_H \delta Q_B = -\int_\Sigma \epsilon_{ebcd}[\delta T_a^e + A_a \delta J^e]\xi^a$$

- Similarly, the 2nd order variational Id:

$$\delta^2 M - \Omega_H \delta^2 J - \frac{\kappa}{8\pi}\delta^2 A_B - \Phi_H \delta^2 Q_B = \mathcal{E}(\phi, \delta\phi) - \int_\Sigma \epsilon_{ebcd}[\delta^2 T_a^e + A_a \delta^2 J^e]\xi^a$$

where we use the fact: $i_\xi(\delta E \cdot \delta\phi)_{abc} = -\xi^d \epsilon_{dabc}[\frac{1}{2}\delta T^{ef}\delta_{ef} + \delta j^e \delta\delta A_e] = 0$ when pulling back to B as $\xi|_B = 0$.

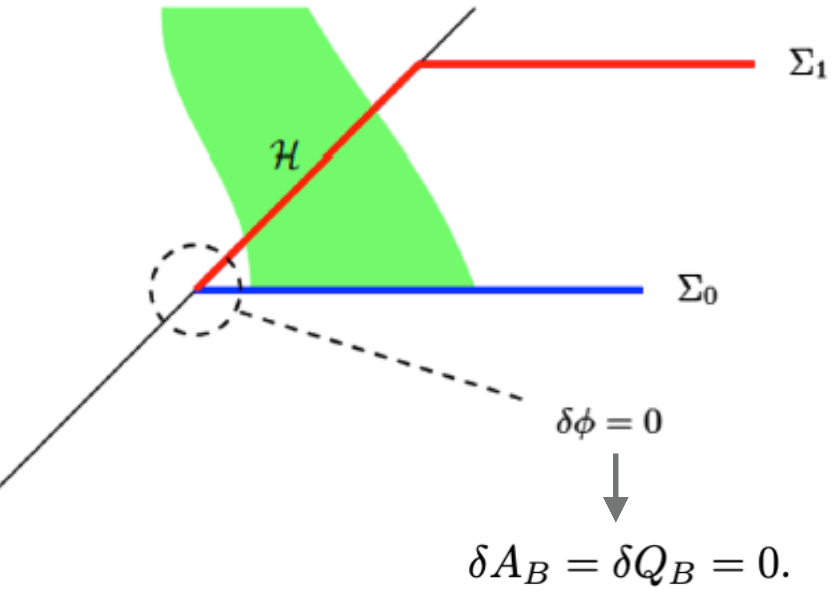
EXAMPLE HIGHER GR

- Take c4 R F ^ 2 as an example:

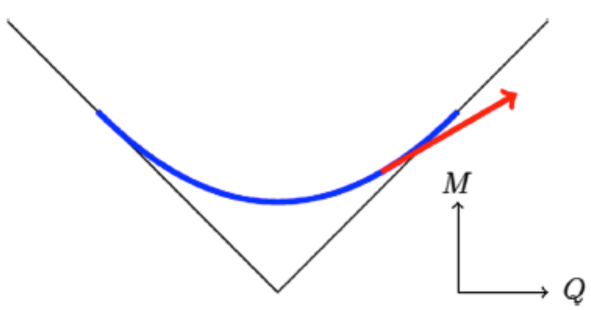
$$Q^{c_4}_{ab} = c_4\ \epsilon_{abcd}(F^2\nabla^c\xi^d + 2\xi^c\nabla^d F^2 - 2RF^{cd}A_e\xi^e)$$
$$C^{c_4}_{abc} = c_4\ \epsilon_{dabc}\xi_e(T^{de}_{c_4} + J^d_{c_4}A^e)$$
$$T^{ab}_{c_4} := 2(R^{ab} - \frac{1}{2}g^{ab}R + g^{ab}\Box - \nabla^{(a}\nabla^{b)})F^2 + 4RF^a_e F^{eb}$$
$$J^a_{c_4} := 4\nabla_b(RF^{ba})$$

- Even the extremal BH condition remain the same as M=Q, we see that the first term in $Q^{c_4}_{ab}$ is nonzero at horizon, and can contribute to the 1st order variational Id.
- Moreover, the new positive energy condition might be required.
- Work in progress to see if WCCC will not hold for some higher GR …

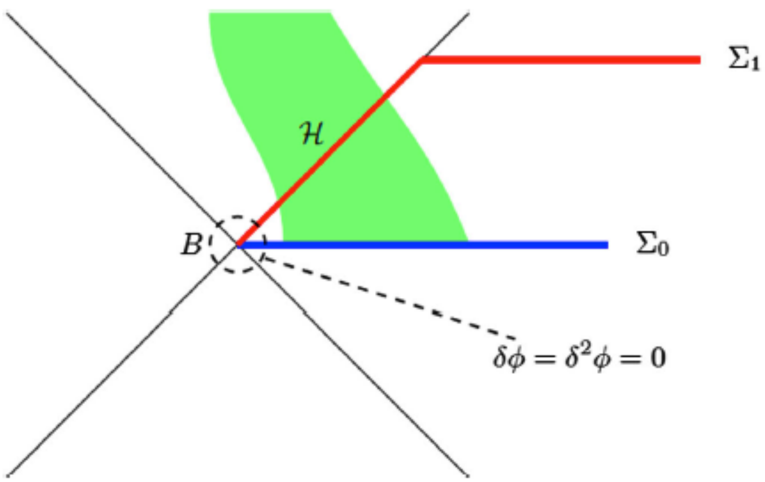
DESTROY AN EXTREMAL BH?



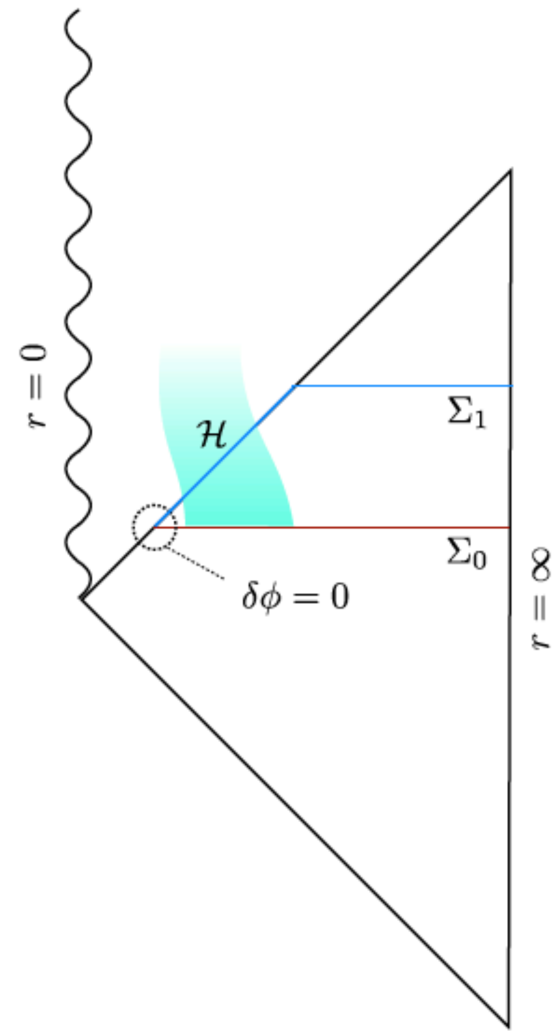
- Choose a hypersurface $\Sigma = \mathcal{H} \cup \Sigma_1$ for the falling matters as shown so that at earlier time $\delta\phi = 0.$
- For the chosen Σ, $\delta A_B = \delta Q_B = 0.$ Use 1st order variational ID and define $\delta Q = \int_{\mathcal{H}} \delta(\epsilon \cdot J)$, we have
$$\delta M - \Omega_H \delta J - \Phi_H \delta Q = -\int_{\mathcal{H}} \delta T^{ab} \xi_a k_b \geq 0 \ \text{ if } \delta T^{ab} \xi_a \xi_b \geq 0.$$
- Note $\Omega_H = \frac{a}{r_+^2 + a^2}, \quad \Phi_H = \frac{Q r_+}{r_+^2 + a^2}, \quad r_+ = M + \sqrt{M^2 - a^2 - Q^2}$
for extremal BH, $r_+ = M$, we have
$$\delta M \geq \frac{a}{M^2 + a^2} \delta J + \frac{QM}{M^2 + a^2} \delta Q$$
- This is exactly the condition we cannot violate WCCC: $\delta M^2 = 2M\delta M \geq \delta[(J/M)^2 + Q^2] = 2(J/M)(M\delta J - J\delta M)/M^2 + 2Q\delta Q$
- This inequality implies the perturbation moves upward along the light-cone of the M-(Q & J) space.



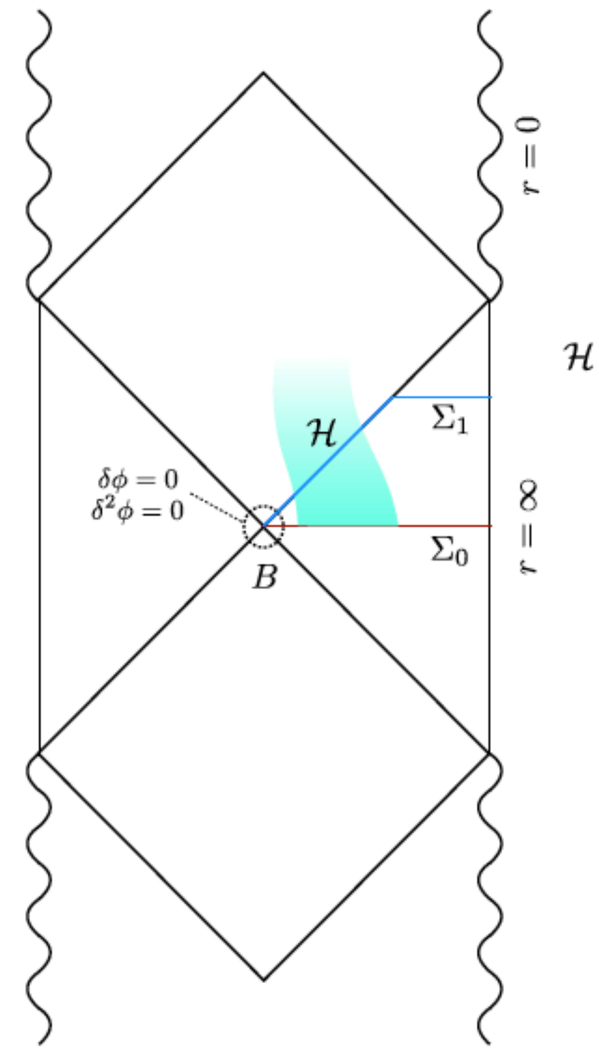
from Source & Wald



Σ_1
$\mathcal{H}$
B
Σ_0
$\delta\phi = \delta^2\phi = 0$



$r = 0$
$\mathcal{H}$
Σ_1
Σ_0
$\delta\phi = 0$
$r = \infty$



$r = 0$
$\mathcal{H}$
$\mathcal{H}$
Σ_1
$\delta\phi = 0$
$\delta^2\phi = 0$
$r = \infty$
Σ_0
B

DESTROY A NEAR-EXTREMAL BH?

- We first assume the first order variation is done optimally:

$$\delta M = \Omega_H \delta J + \Omega_H \delta Q$$

- For chosen Cauchy surface, we have $\delta^2 Q_B = \delta^2 A_B = 0.$
- Define $\delta^2 Q = \int_{\mathcal{H}} \delta^2(\epsilon \cdot J)$, then the 2nd order variation Id becomes

$$\delta^2 M - \Omega_H \delta^2 J - \Phi_H \delta^2 Q = \mathcal{E}(\phi, \delta\phi) - \int_\Sigma \epsilon_{ebcd} \delta^2 T^e_a \xi^a \geq \mathcal{E}(\phi, \delta\phi) \ \text{ if } \delta^2 T^{ab} \xi_a \xi_b \geq 0$$

- The evaluation of canonical energy on horizon is tedious, just summarize the results:

$$\mathcal{E}_{\mathcal{H}}(\phi, \delta\phi) = \frac{1}{4\pi} \int_{\mathcal{H}} [(\xi^a \nabla_a u) \delta\sigma_{bc} \delta\sigma^{bc} + 2k^d \xi^e \delta F^f_d \delta F_{ef}] = \text{ infalling fluxes of gravitational and EM waves} \geq 0$$

- To evaluate the $\mathcal{E}_{\Sigma_1}(\phi, \delta\phi)$, we need to introduce the linear stability assumption (Sorce & Wald):

Any source free solution to linearly Einstein-Maxwell equations approaches a perturbation towards another Kerr-Newman BH at sufficient late time.

- With this assumption, we have
$\mathcal{E}_{\Sigma_1}(\phi,\delta\phi)=\mathcal{E}_{\Sigma_1}(\phi,\delta\phi^{KN})=\mathcal{E}_{\Sigma}(\phi,\delta\phi^{KN})|_{\text{no fluxes}}=\mathcal{E}_{\Sigma}(\phi,\delta\phi^{KN})|_{\delta^2M=\delta^2J=\delta^2Q_B=\delta E=\delta^2C=0}=-\frac{\kappa}{8\pi}\delta^2A_B^{KN}$
- For simplicity, below we consider the RN BH only. Start with $A_B=4\pi r_+^2$, then $\delta^2A_B^{KN}|_{\delta^2M=\delta^2Q=0}=\frac{8\pi}{\epsilon^3}\left[(1+\epsilon)^2(2\epsilon-1)(\delta M)^2-[(Q/M)^2+(1+\epsilon)\epsilon^2](\delta Q)^2-2(Q/M)(\epsilon^2-1)\delta M\delta Q\right]$
- Use $\kappa=\frac{\epsilon}{M(1+\epsilon)^2}$, the 2nd order variations obey

$$\delta^2M-\Omega_H\delta^2J-\Phi_H\delta^2Q\geq(\delta Q)^2/M+\mathcal{O}(\epsilon)$$

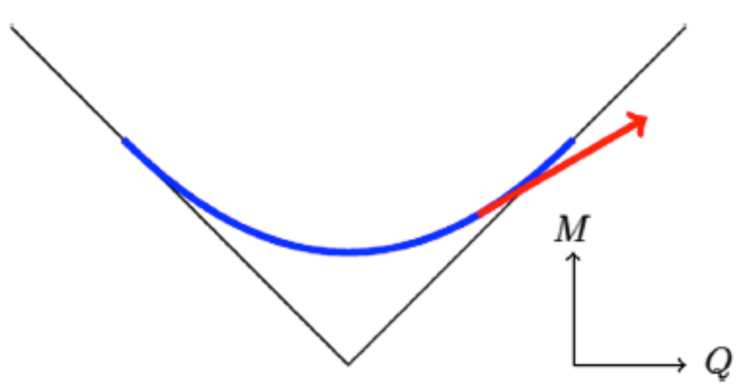
- Now, we check the WCCC condition for one-parameter family of RN solution up to 2nd order: $f(\lambda)=(M+\lambda\delta M+\lambda^2\delta^2M/2)^2-(Q+\lambda\delta Q+\lambda^2\delta^2Q/2)^2$
- Using 2nd order variation but up to first order in λ, we obtain Hubeny's result: $f(\lambda)\geq M^2\epsilon^2-2Q\delta Q\lambda\epsilon+\mathcal{O}(\lambda^2,\epsilon^3,\epsilon^2\lambda)$
- However, up to full 2nd order in both ϵ,λ, we have

$$f(\lambda)\geq(\epsilon M-\lambda Q\delta Q/M)^2\geq0$$

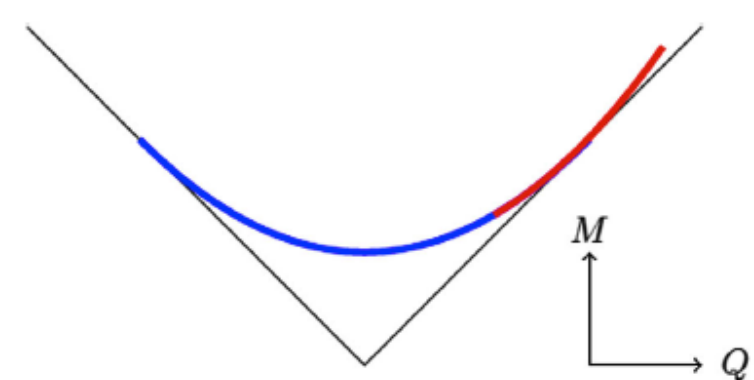
REMARKS

- For some special case, i.e., dropping a charged particle (no spin: $\delta J = 0$) from infinity into a Kerr BH (Q=0, $\Phi_H = 0$) along the BH's symmetry axis ($\delta^2 J = 0$), the result of the 2nd order variation yields

$$E := \delta^2 M \geq -\frac{\kappa}{8\pi}\delta^2 A_B^{KN} = \frac{1}{2M}(\delta Q)^2 = \text{work done by self-force + self-energy}$$
$$= \int_{r_+}^{\infty} \frac{Mr}{(r^2+a^2)^2}(\delta Q)^2 dr + \frac{1}{2r_p}(\delta Q)^2 \times (\kappa r_p)$$
$$= \frac{M}{2(r_+^2+a^2)}(\delta Q)^2 + \frac{\kappa}{2}(\delta Q)^2$$



Hubeny's argument (1999)



Including 2nd order effect by

Sorce & Wald (2017)

MB MODEL OF 3D GRAVITY

- MB model:

$$L = L_{\mathrm{EC}} + L_{\Lambda} + L_{\mathrm{CS}} + L_{\mathrm{T}} + L_{\mathrm{M}} ,$$

$$\begin{aligned} L_{\mathrm{EC}} &= \frac{1}{\pi} e^a \wedge R_a , \\ L_{\Lambda} &= -\frac{\Lambda}{6\pi} \epsilon_{abc} e^a \wedge e^b \wedge e^c , \\ L_{\mathrm{CS}} &= -\theta_{\mathrm{L}} \left(\omega^a \wedge \mathrm{d}\omega_a + \frac{1}{3} \epsilon_{abc} \omega^a \wedge \omega^b \wedge \omega^c \right) , \\ L_{\mathrm{T}} &= \frac{\theta_{\mathrm{T}}}{2\pi^2} e^a \wedge T_a , \end{aligned}$$

$$\omega^a = \frac{1}{2} \epsilon^a{}_{bc} \omega^{bc} , \qquad R^a = \frac{1}{2} \epsilon^a{}_{bc} R^{bc} ,$$

- EOMs are solved by

$$T^a = \frac{\mathcal{T}}{\pi} \epsilon^a{}_{bc} e^b \wedge e^c ,$$

$$R^a = -\frac{\mathcal{R}}{2\pi^2} \epsilon^a{}_{bc} e^b \wedge e^c ,$$

in which

$$\mathcal{T} \equiv \frac{-\theta_{\mathrm{T}} + 2\pi^2 \Lambda \theta_{\mathrm{L}}}{2 + 4\theta_{\mathrm{T}}\theta_{\mathrm{L}}} , \qquad \mathcal{R} \equiv -\frac{\theta_{\mathrm{T}}^2 + \pi^2 \Lambda}{1 + 2\theta_{\mathrm{T}}\theta_{\mathrm{L}}} .$$

THREE LIMITS

- We are consider 3 limits of MB model:

1. Einstein gravity: $\theta_L \longrightarrow 0, \qquad \theta_T \longrightarrow 0$
2. Chiral gravity: setting $\mathcal{T} = 0$ (reduce to TMG) first, then take $\theta_L \longrightarrow -1/(2\pi\sqrt{-\Lambda})$
3. Torsional chiral gravity: take $\theta_L \longrightarrow -1/(2\pi\sqrt{-\Lambda})$ so that $\mathcal{T} \longrightarrow \pi\sqrt{-\Lambda}/2$

- All three limits are well-defined without ghost, and admit BTZ solutions but with an effective cosmological constant:

$$\Lambda_{\text{eff}} \equiv -\frac{\mathcal{T}^2 + \mathcal{R}}{\pi^2}.$$ *so that*

extremal BH: $M^2 + \Lambda_{\text{eff}} J^2 = 0$

horizons: $r_\pm^2 = \frac{1}{2\Lambda_{\text{eff}}}\left(-M \mp \sqrt{M^2 + \Lambda_{\text{eff}} J^2}\right)$

angular velocity: $\Omega_H = \frac{J}{2r_+^2} = \frac{r_-}{r_+}\sqrt{-\Lambda_{\text{eff}}}.$

Hawking T: $T_H = -\frac{\Lambda_{\text{eff}}(r_+^2 - r_-^2)}{2\pi r_+},$

surface gravity: $\kappa_H = 2\pi T_H.$

DESTROY AN EXTREMAL BTZ?

- Follow Wald's construction method (though in the first order formulation of gravity theory), we obtain the first order variational Id: $\delta\mathcal{M} - \Omega_H \delta\mathcal{J} - T_H \delta S = -\int_\Sigma \delta C_\xi.$
- However, the ADM quantities and entropy are not conventional:

$$\mathcal{M} = M - 2\theta_L(\mathcal{T}M + \pi\Lambda_{\text{eff}}J),$$

$$\mathcal{J} = J + 2\theta_L(\pi M - \mathcal{T}J). \qquad S = 4\pi r_+ - 8\pi\theta_L(\mathcal{T}r_+ - \pi\sqrt{-\Lambda_{\text{eff}}}r_-).$$

c.f. Ning & Wei , 2018.

- Also, $\delta C_\xi = (\mathrm{i}_\xi e^a) \wedge \delta\Sigma_a + (\mathrm{i}_\xi \omega^a) \wedge \delta\tau_a.$

$\delta\Sigma_a$ is related to the variartion of the canonical stress tensor obeying null energy condition.
$\delta\tau_a$ is related to the variation of the canonical spin angular momentum tensor, new in Einstein-Cartan theory

- As before, we consider the hypersurface with $\delta S = 0$, further simplification yields the 1st order variation Id

$$\delta\mathcal{M} - \Omega_H \delta\mathcal{J} = (1 - 2\theta_L \mathcal{T} - 2\pi\theta_L \Omega_H)(\delta M - \Omega_H \delta J) - 2\pi\theta_L \Lambda_{\text{eff}} \left(\frac{r_+^2 - r_-^2}{r_+^2}\right)\delta J$$
$$= \int_\Sigma d^2x \sqrt{-\gamma} \left\{ \xi_\mu k_\nu \delta\Sigma^{\mu\nu} - \left(\kappa_H n_{\mu\nu} + \frac{\mathcal{T}}{\pi} \epsilon_{\mu\nu}{}^\sigma \xi_\sigma\right) k_\lambda \delta\tau^{\mu\nu\lambda} \right\}.$$

- For extremal BTZ, it further reduces to

$$\delta\mathcal{M} - \Omega_H \delta\mathcal{J} = (1 - 2\theta_L \mathcal{T} - 2\pi\theta_L \sqrt{-\Lambda_{\text{eff}}})(\delta M - \sqrt{-\Lambda_{\text{eff}}}\delta J)$$
$$= \int_\Sigma d^2x \sqrt{-\gamma} \left\{ \xi_\mu k_\nu \delta\Sigma^{\mu\nu} - \frac{\mathcal{T}}{\pi} \epsilon_{\mu\nu}{}^\sigma \xi_\sigma k_\lambda \delta\tau^{\mu\nu\lambda} \right\} \geq 0$$

for both Einstein and Chiral gravity.

- Consider WCCC test for extremal BTZ:

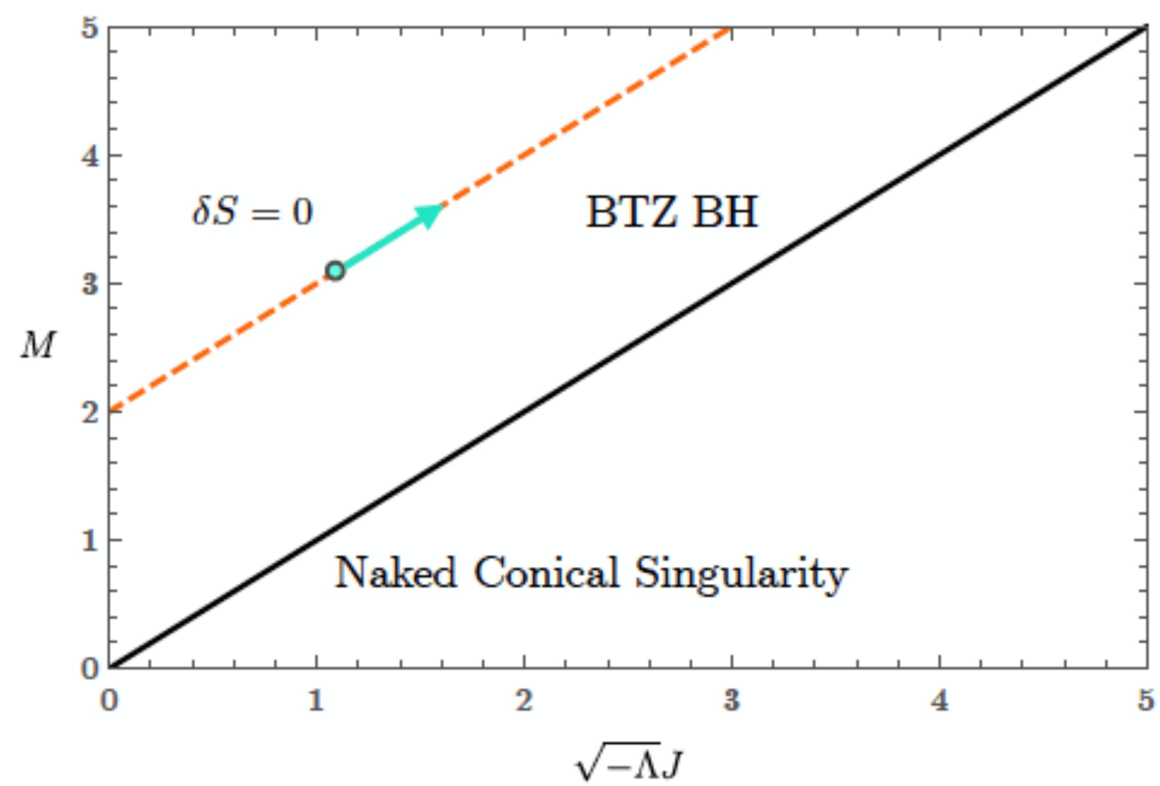
$$f(\lambda) = M^2(\lambda) + \Lambda_{eff} J^2(\lambda) = 2\lambda\sqrt{-\Lambda_{eff}}|J|(\delta M - \sqrt{\Lambda_{eff}}\delta J) + \mathcal{O}(\lambda^2)$$

- For WCCC to hold, it needs $1 - 2\theta_L \mathcal{T} - 2\pi\theta_L \sqrt{-\Lambda_{\text{eff}}} \geq 0.$

- Thus, WCCC holds for both Einstein and chiral gravity, but not clear for torsional chiral gravity due to the lack of positivity condition for canonical spin angular momentum tensor.

DESTROY A NEAR-EXTREMAL BTZ IN CHIRAL GRAVITY?

- For chiral gravity one can see that the following relation holds: $M = \left(\sqrt{-\Lambda}\right) J - \frac{\Lambda}{16\pi^2} S^2$.
- However, one requires the first order variation is done optimally, which just require keeping entropy constant. This then yields $\delta M = \sqrt{-\Lambda}\delta J$.
- Thus, the WCCC holds in chiral gravity trivially.



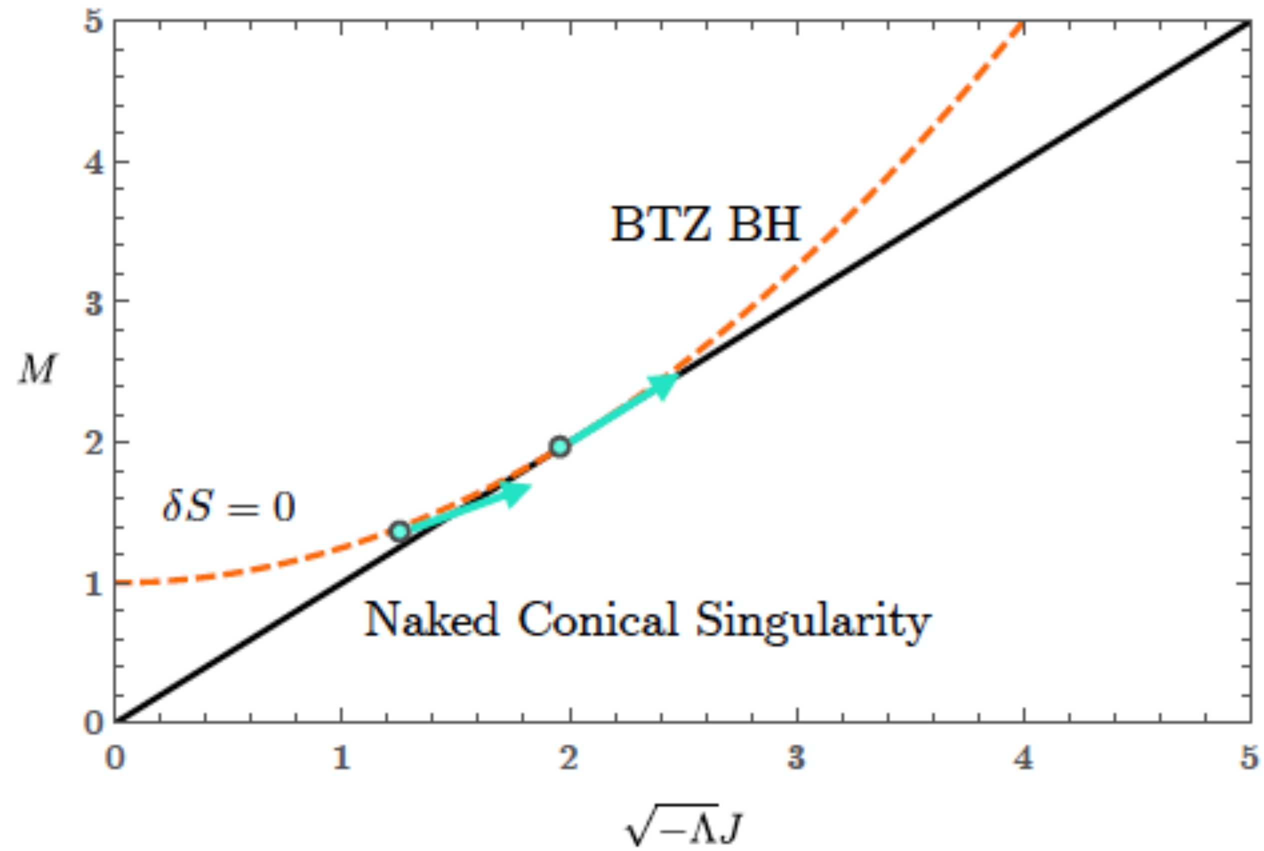
DESTROY A NEAR-EXTREMAL BTZ IN EINSTEIN GRAVITY?

- Without reciting the details, we write down the 2nd order variational Id: $$\delta^2\mathcal{M} - \Omega_{\mathrm{H}}\delta^2\mathcal{J} = \mathcal{E}_\Sigma(\phi;\delta\phi) + \delta^2\int_\Sigma d^2x\sqrt{-\gamma}\left\{\xi_\mu k_\nu \Sigma^{\mu\nu} - \left(\kappa_{\mathrm{H}} n_{\mu\nu} + \frac{\mathcal{T}}{\pi}\epsilon_{\mu\nu}{}^{\sigma}\xi_\sigma\right)k_\lambda \tau^{\mu\nu\lambda}\right\}.$$
- As there is no propagating d.o.f. in 3D gravity, thus $\mathcal{E}_{\mathcal{H}}(\phi,\delta\phi) = 0.$
- Thus, the 2nd order variation Id is reduced to $\delta^2 M - \Omega_{\mathrm{H}}\delta^2 J \geq -T_{\mathrm{H}}\delta^2 S^{\mathrm{BTZ}}.$
- Here $T_{\mathrm{H}} = -\frac{\Lambda(r_+^2 - r_-^2)}{2\pi r_+} = \frac{\alpha M\epsilon}{\pi\sqrt{2M(1+\epsilon)}}$ with $\alpha := \sqrt{-\Lambda}$, and

$$\delta^2 S^{\mathrm{BTZ}} = (\delta J)^2\left(-\frac{\pi\alpha M\left[\alpha^2 J^2(3\epsilon+2) + 2M^2\epsilon^2(\epsilon+1)\right]}{\sqrt{2}\epsilon^3\left[M^3(\epsilon+1)\right]^{3/2}}\right) + (\delta J\delta M)\left(\frac{\pi\sqrt{2}\alpha J(\epsilon+2)}{M\epsilon^3\sqrt{M^3(\epsilon+1)}}\right) + (\delta M)^2\left(\frac{\pi(\epsilon-2)(\epsilon+1)}{\sqrt{2}\alpha\epsilon^3\sqrt{M^3(\epsilon+1)}}\right)$$

- Check the WCCC test function to full 2nd order, we find

$$f(\lambda) \geq (M\epsilon - \lambda\frac{\alpha^2 J\delta J}{M})^2 \geq 0.$$


BTZ BH
$\delta S = 0$
Naked Conical Singularity
M
$\sqrt{-\Lambda}J$
0
1
2
3
4
5

CONCLUSION

- We have reviewed Sorce & Wald on the proof of WCCC.
- This proof based on variation of energy, charge and spin up to 2nd order, and bypass the difficulty of dynamical consideration with the complication of self-force and self-energy.
- We further extend this scheme of proof to the 3D TMG-like models. We find WCCC holds in most of cases including Einstein gravity & chiral gravity except for the cases with torsion.
- Our results implies the operational proof of third law of thermodynamics for the 2D dual CFTs.
- On-going progress to use WCCC to constrain higher GR.

THANKS!