Generalizations of Reflected entropy and the holographic dual

Yang Zhou(周洋)

Fudan University

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NCTS

Quantum Entanglement in AdS/CFT

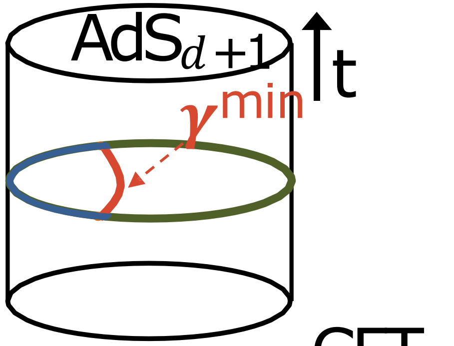
- Recently there are celebrated success:
  - Eternal black hole/TFD, Ryu-Takayanagi, HRT, entanglement dynamics…

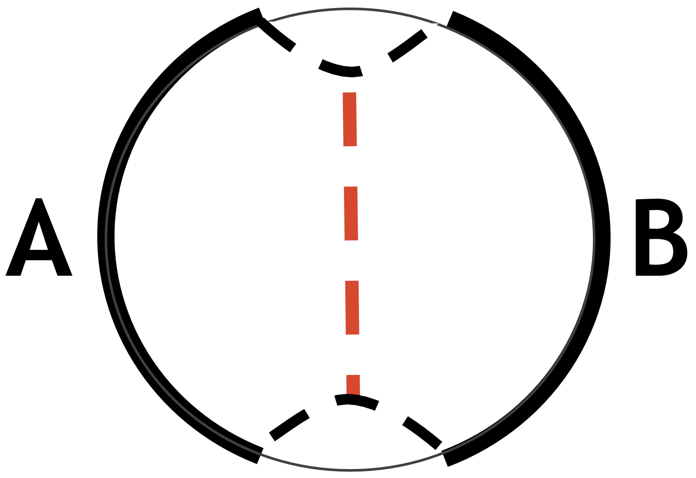
- Mixed state entanglement measure (and AdS dual) is less known

- Multipartite entanglement measure (and AdS dual) is less known

- General quantum information dictionary of AdS/CFT is unknown

Entanglement wedge cross-section /mixed state correlation measure

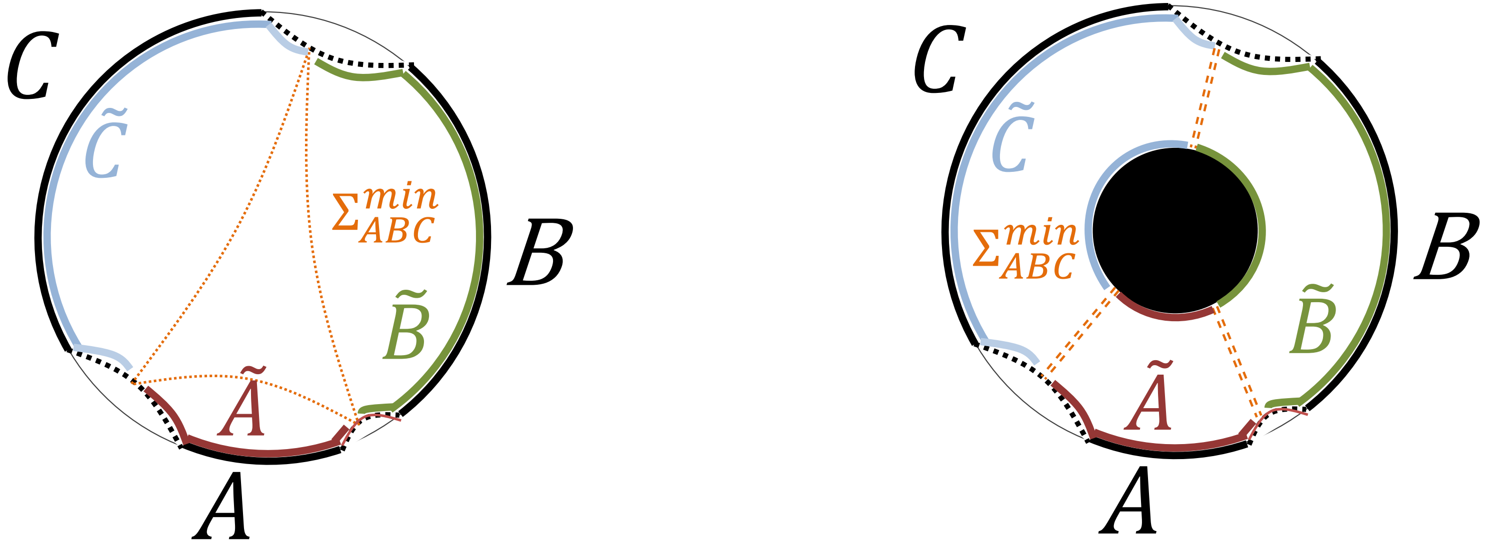




$$E_P(\rho_{AB}) = \frac{1}{2} \min_{|\psi\rangle_{AA'BB'}} [S_{AA'} + S_{BB'}]$$

- Entanglement of purification [Umemoto-Takayanagi, NDHZS]
- Logarithmic negativity [Kudler-Flam and Ryu]
- Odd entropy [Tamaoka]
- Reflected entropy [Dutta-Faulkner]

Multipartite EWCS



[Umemoto-YZ]

$$\Delta_P(A:B:C) := \min_{|\psi\rangle_{AA'BB'CC'}} [S_{AA'} + S_{BB'} + S_{CC'}]$$

Canonical purification

- Consider a mixed state on a bipartite Hilbert space

$$\rho_{AB}$$

- Flipping Bras to Kets for the basis

$$|i\rangle\langle j| \Longrightarrow |i\rangle \otimes |j\rangle$$

- A canonical purification

$$|\sqrt{\rho_{AB}}\rangle \in (\mathcal{H}_A \otimes \mathcal{H}_A^\star) \otimes (\mathcal{H}_B \otimes \mathcal{H}_B^\star) \equiv \mathcal{H}_{AA^\star BB^\star}$$

$$\mathrm{Tr}_{\mathcal{H}_A^\star \otimes \mathcal{H}_B^\star} |\sqrt{\rho_{AB}}\rangle\langle\sqrt{\rho_{AB}}| = \rho_{AB}$$

$$\rho_{AB} = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow|_{AB} + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|_{AB})$$

$$|\sqrt{\rho_{AB}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\uparrow\rangle_{AA'BB'} + |\downarrow\downarrow\downarrow\downarrow\rangle_{AA'BB'})$$

- Reflected entropy

$$S_R(A:B) \equiv S(AA^\star)_{\sqrt{\rho_{AB}}}$$

Reflected entropy

- Properties

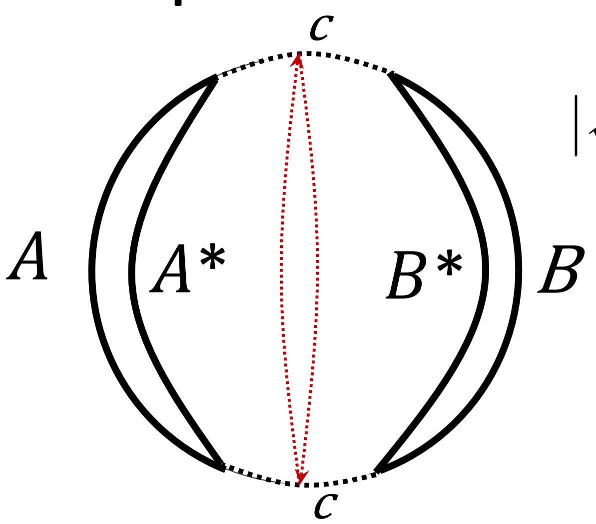
$$\text{pure state}: \ S_R(A:B) = 2S(A) \ ,$$

$$\text{factorized state}: \ S_R(A:B) = 0 \ ,$$

$$\text{bounded from below}: \ S_R(A:B) \geq I(A:B) \ ,$$

$$\text{bounded from above}: \ S_R(A:B) \leq 2\text{min}\{S(A), S(B)\}$$

- Graph description



$$\psi_{ABc} \in \mathcal{H}_{ABc}$$

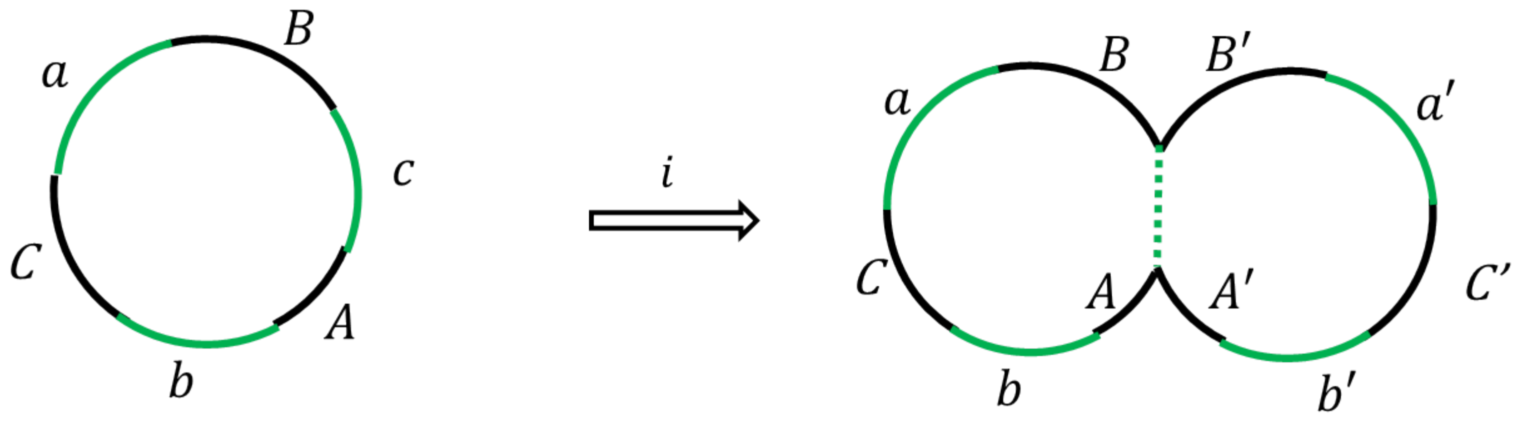
$$|\sqrt{\rho_{AB}}\rangle = |\sqrt{\text{Tr}_c|\psi\rangle\langle\psi|}\rangle \in (\mathcal{H}_A \otimes \mathcal{H}_{A^*}) \otimes (\mathcal{H}_B \otimes \mathcal{H}_{B^*})$$

$$S_R(A:B) = S(AA^* : BB^*)_{\sqrt{\rho_{AB}}}$$

$$= \text{Entanglement Entropy of Red Curve}$$

Generalizations

- Trace some of a,b,c first



$$\psi_1 = |\sqrt{\mathrm{Tr}_c|\psi_{ABCabc}\rangle\langle\psi_{ABCabc}|}\,\rangle$$

Generalizations

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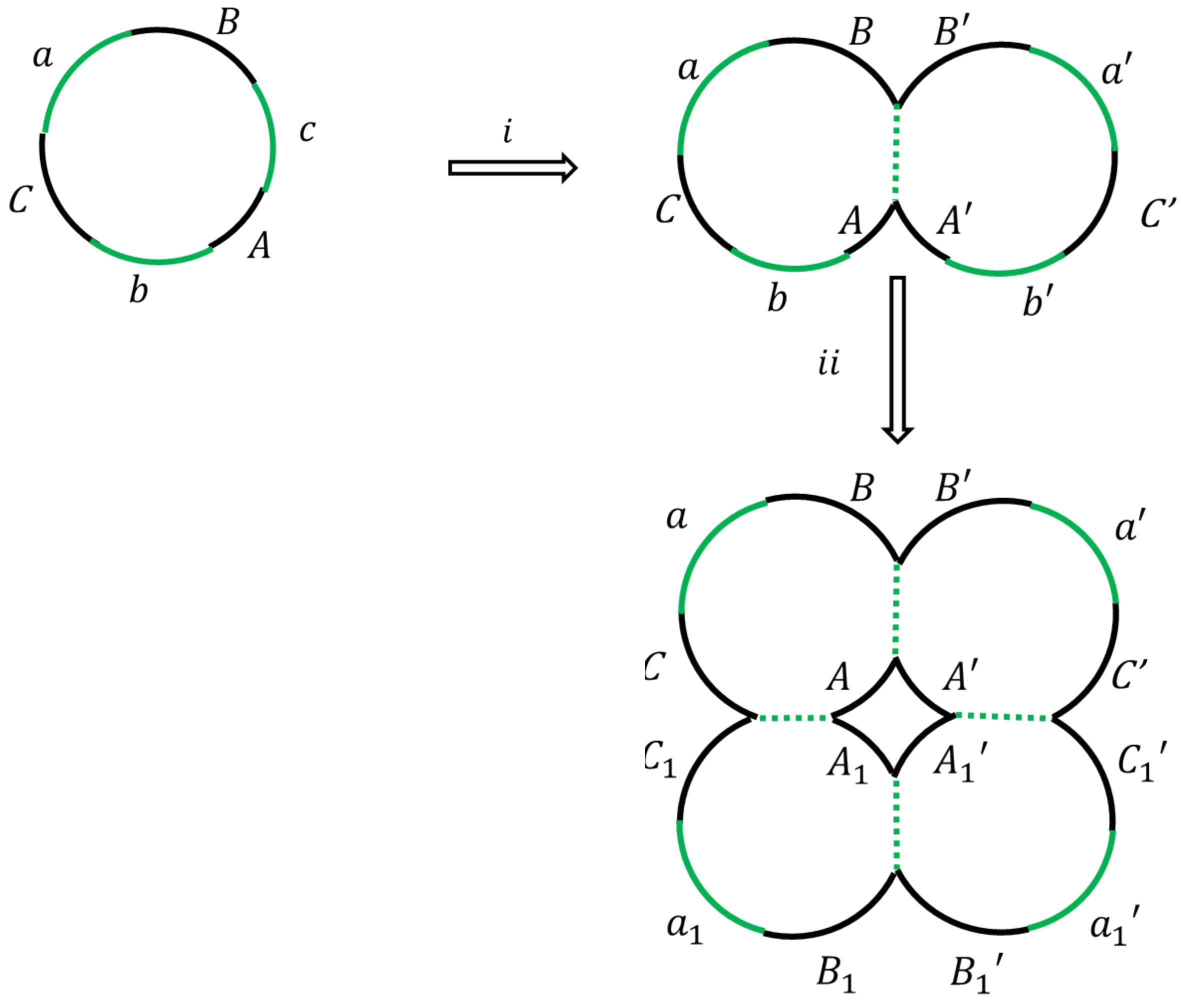
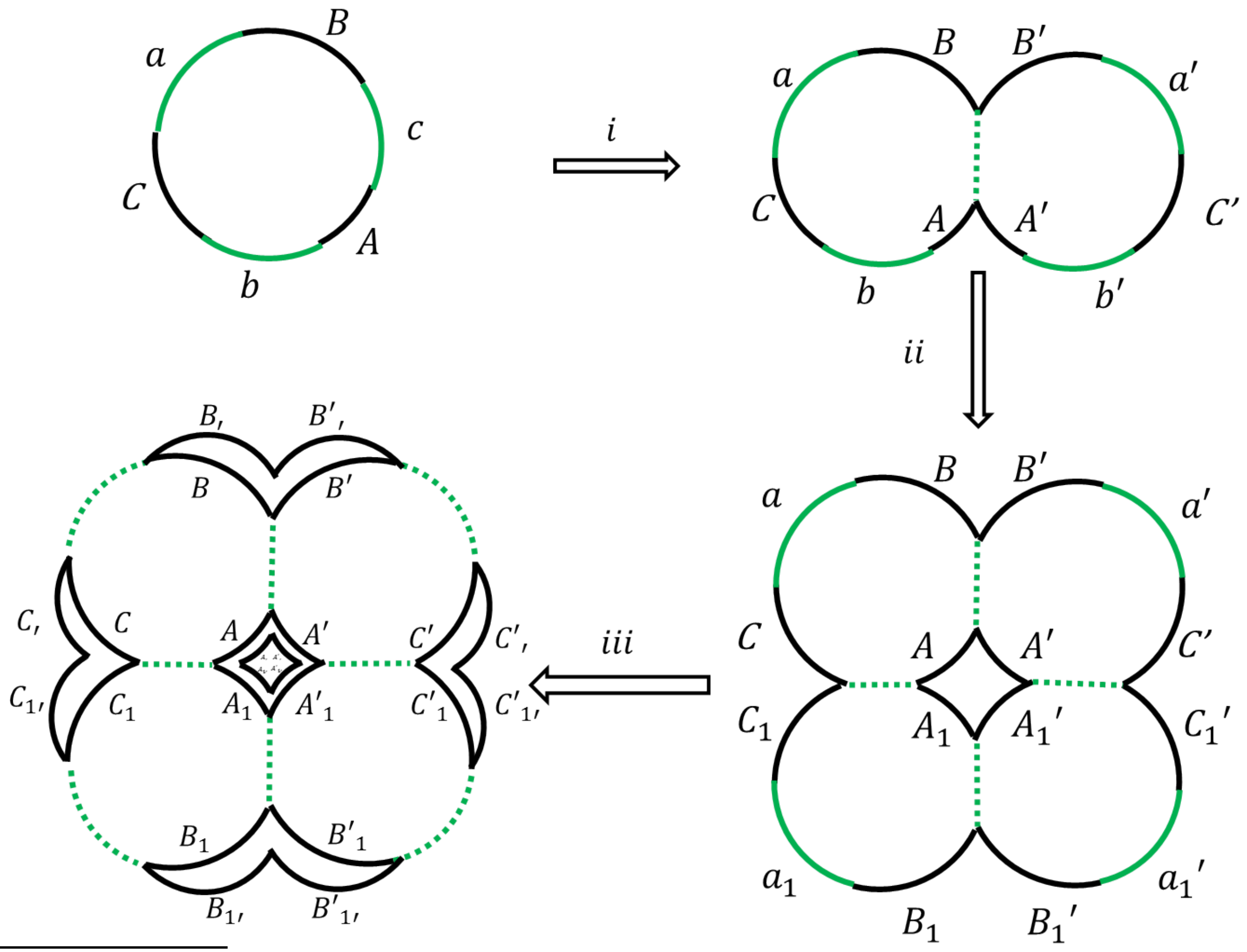


$$\psi_2 = |\sqrt{\mathrm{Tr}_{bb'}|\psi_1\rangle\langle\psi_1|}\,\rangle$$

Generalizations

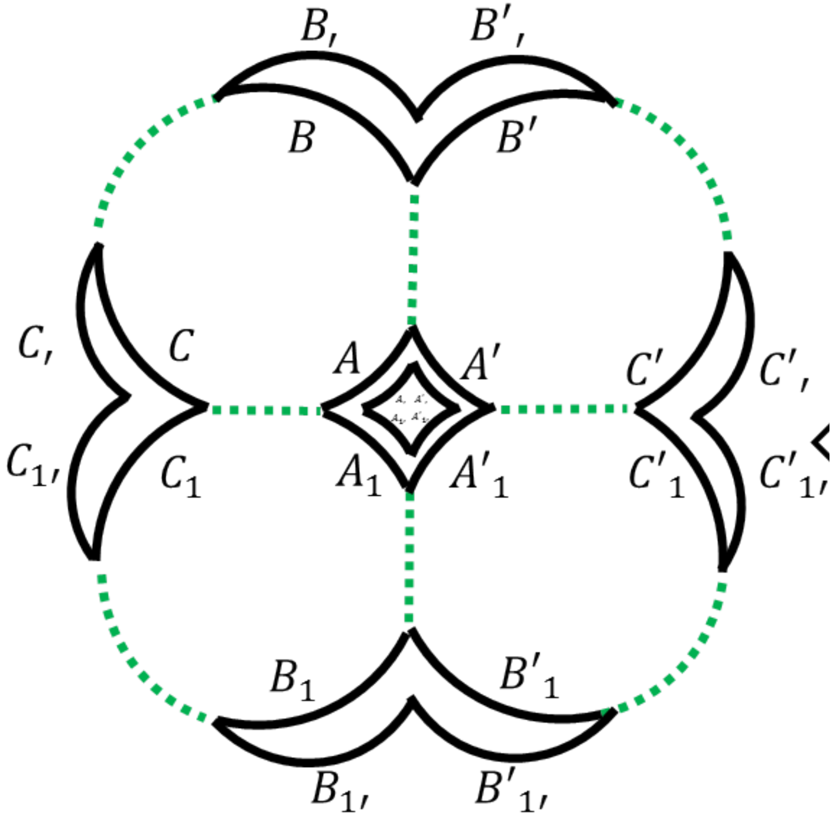
$$\psi_1 = \left|\sqrt{\mathrm{Tr}_c |\psi_{ABCabc}\rangle\langle\psi_{ABCabc}|}\,\right\rangle$$



$$\psi_2 = \left|\sqrt{\mathrm{Tr}_{bb'} |\psi_1\rangle\langle\psi_1|}\,\right\rangle$$

$$\psi_3 = \left|\sqrt{\mathrm{Tr}_{aa'a''a'''} |\psi_2\rangle\langle\psi_2|}\,\right\rangle = \psi_{AA'A_1A'_1A_{\prime}A'_{\prime}A_{1'}A'_{1'}BB'B_{\prime}B'_{\prime}B_1B'_1B_{1'}B'_{1'}CC_{\prime}C_1C_{1'}C'C'_{\prime}C'_1C'_{1'}}$$

Multipartite entropy

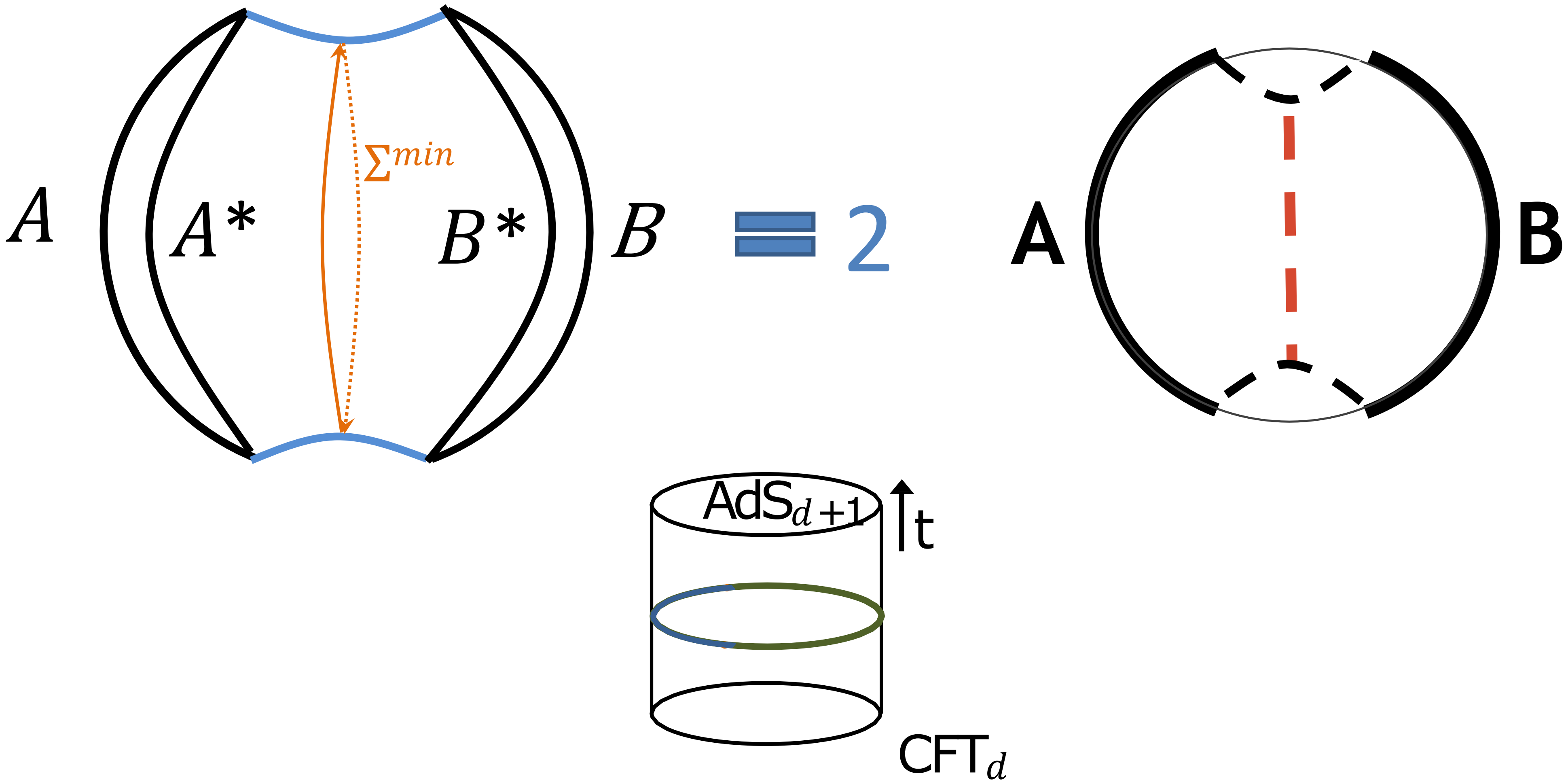


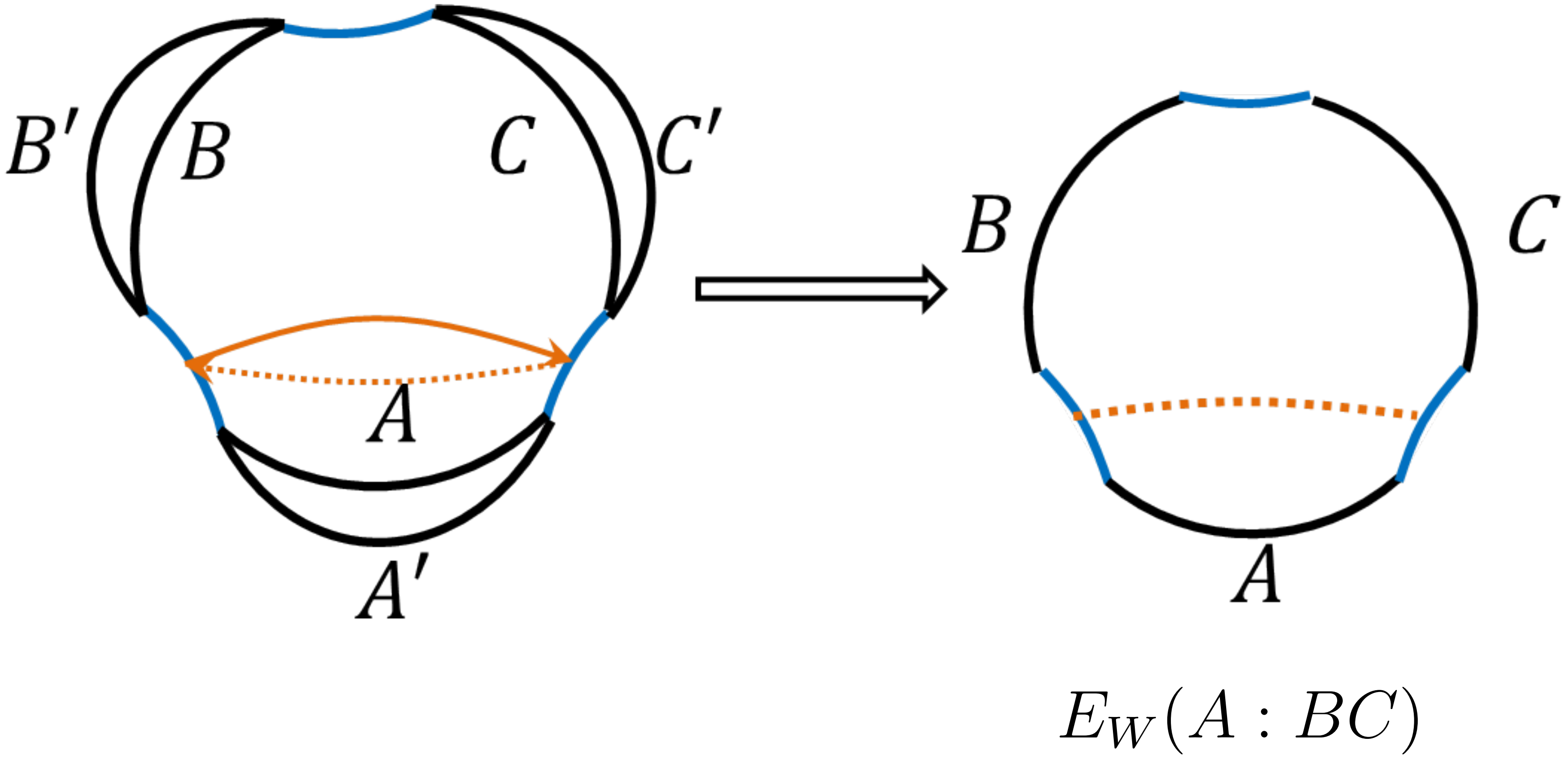
$$\rho_3 = |\psi_3\rangle\langle\psi_3|$$

$$\Delta_R(A:B:C) \equiv S(AA'A_1A_1'B_1B_1'B_{1'}B_{1'}'CC_{\prime}C_1C_{1'} : A_{\prime}A_{\prime}'A_{1'}A_{1'}'BB'B_{\prime}B_{\prime}'C'C_{\prime}'C_1'C_{1'}')_{\psi_3}$$

$$\Delta_R(A:B:C) = \lim_{\boldsymbol{n}\to 1} S_{\boldsymbol{n}}, \quad S_{\boldsymbol{n}} = \frac{1}{1-\boldsymbol{n}} \ln \mathrm{Tr}_R(\mathrm{Tr}_L \rho_3)^{\boldsymbol{n}}$$

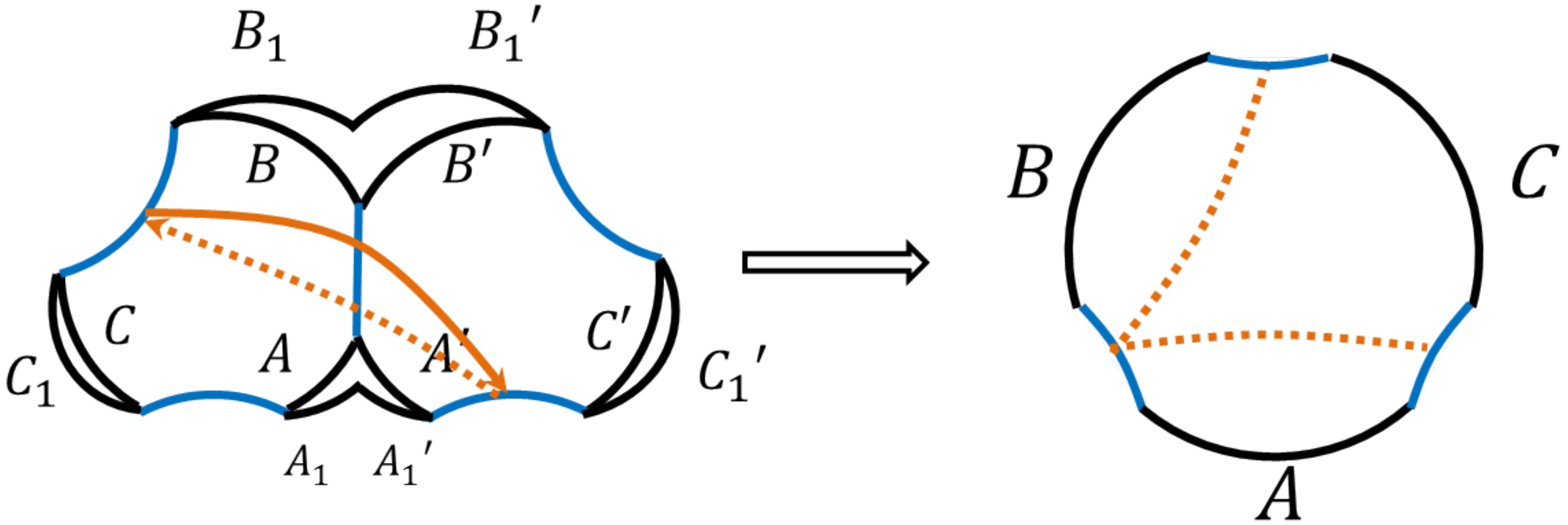
Holographic reflected entropy





B'
B
C
C'
A
A'
B
C
A
$E_W(A:BC)$

Holographic generalized entropy (4 copy)



$$E_W(A:BC) + E_W(B:CA) \leq \Sigma_{(1)}^{min}(C:A:B)$$

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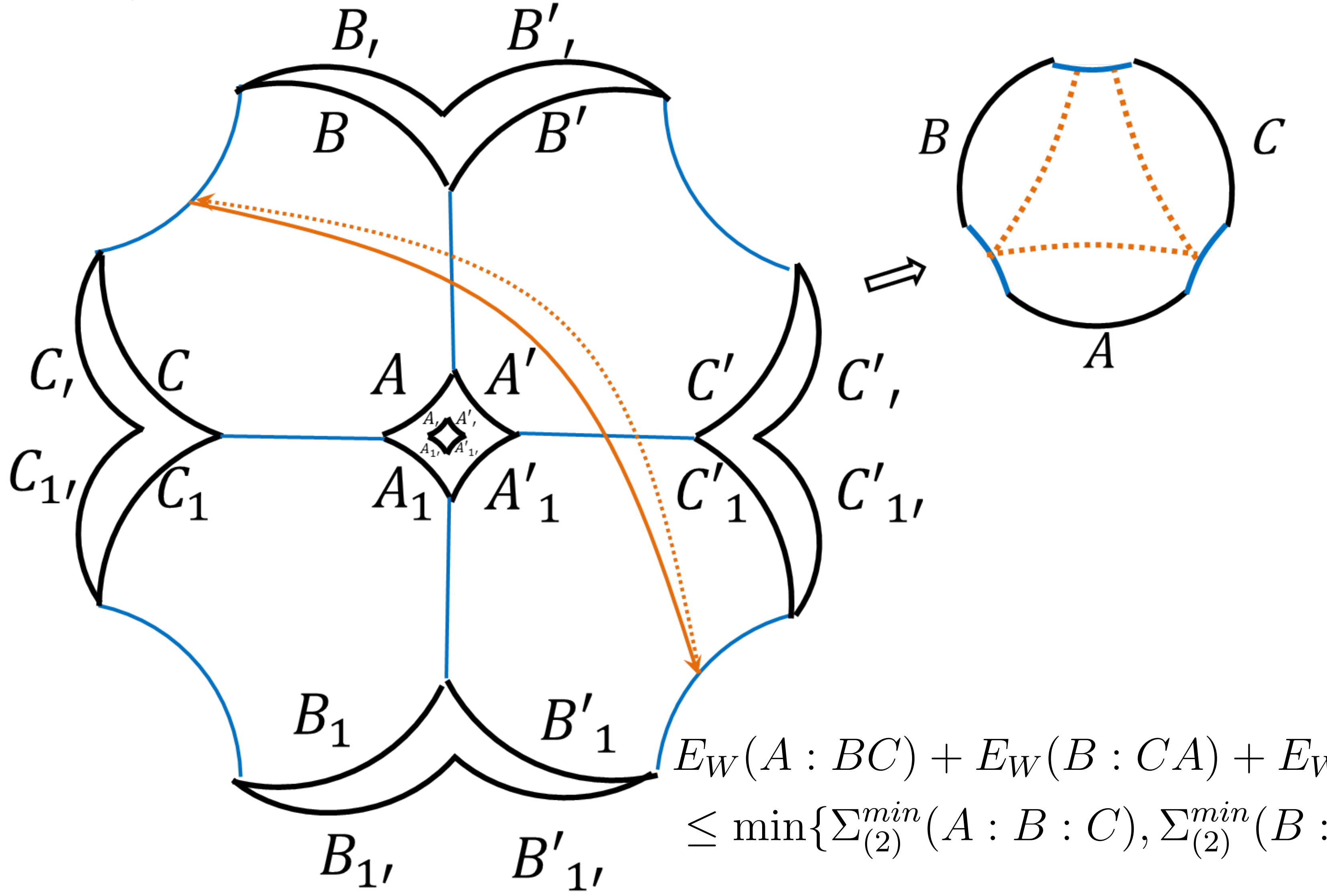
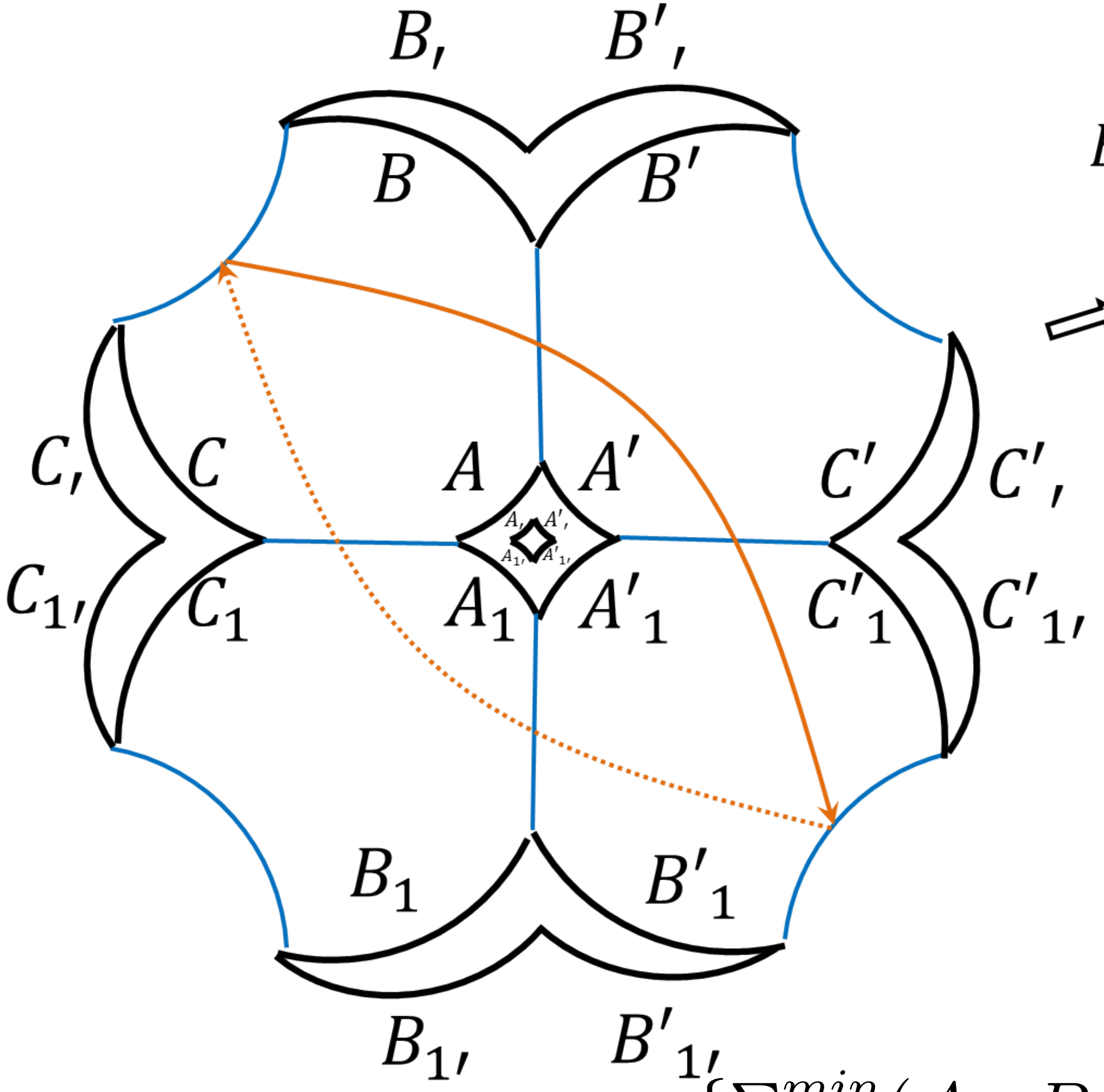
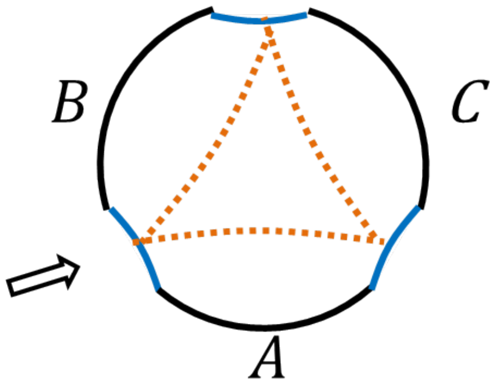


$$E_W(A:BC)+E_W(B:CA)+E_W(C:AB)$$
$$\leq \min\{\Sigma_{(2)}^{min}(A:B:C),\Sigma_{(2)}^{min}(B:C:A),\Sigma_{(2)}^{min}(C:A:B)\}$$

Holographic multipartite entropy





$$\Delta_W(A:B:C) \geq$$
$$\max\{\Sigma_{(2)}^{min}(A:B:C), \Sigma_{(2)}^{min}(B:C:A), \Sigma_{(2)}^{min}(C:A:B)\}$$

Replica trick

- Replica trick in canonical purifications

$$\psi_1 = |\sqrt{\mathrm{Tr}_c|\psi_{ABCabc}\rangle\langle\psi_{ABCabc}|}\,\rangle \qquad \psi_1^{(m)} = |(\mathrm{Tr}_c\rho_0)^{\frac{m}{2}}\rangle\ ,$$

$$\psi_2 = |\sqrt{\mathrm{Tr}_{bb'}|\psi_1\rangle\langle\psi_1|}\,\rangle \qquad \psi_2^{(m)} = |(\mathrm{Tr}_{bb'}\rho_1^{(m)})^{\frac{m}{2}}\rangle\ ,$$

$$\psi_3 = |\sqrt{\mathrm{Tr}_{aa'a''a'''}|\psi_2\rangle\langle\psi_2|}\,\rangle \qquad \psi_3^{(m)} = |(\mathrm{Tr}_{aa'a''a'''}\rho_2^{(m)})^{\frac{m}{2}}\rangle$$

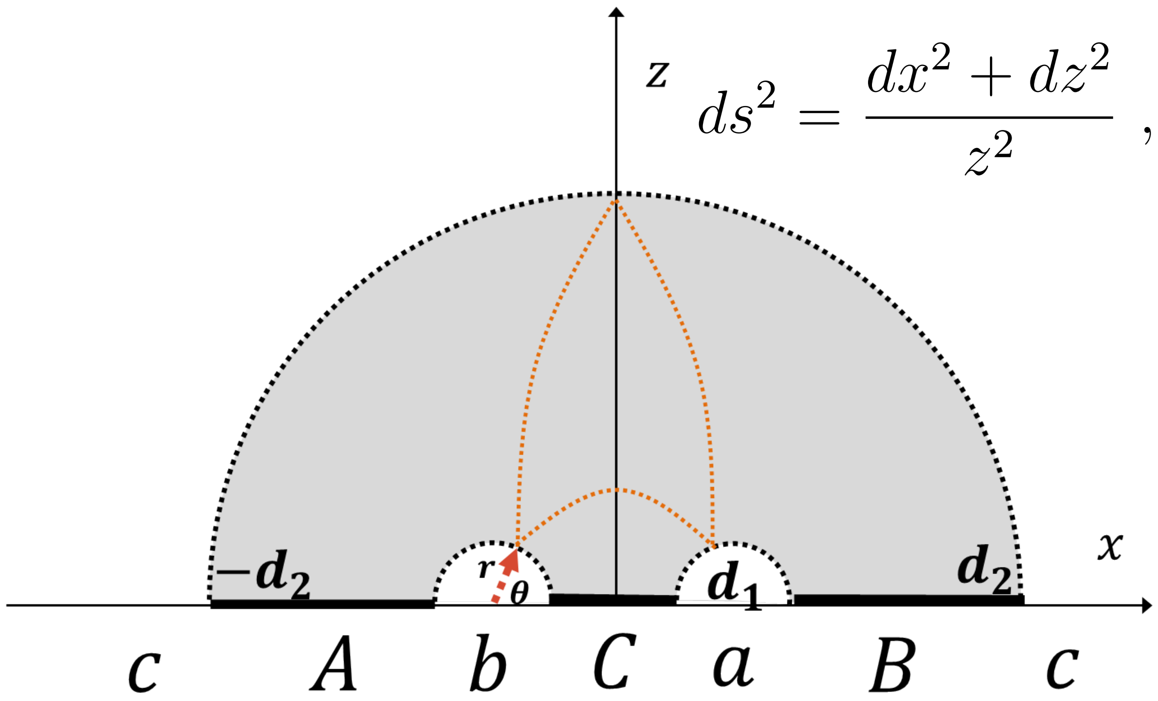
- Replica trick in Renyi index

$$\Delta_R(A:B:C) = \lim_{\substack{\boldsymbol{n}\to 1 \\ m\to 1}} S_{\boldsymbol{n}}, \qquad S_{\boldsymbol{n}} = \frac{1}{1-\boldsymbol{n}} \ln \frac{\mathrm{Tr}_R(\mathrm{Tr}_L\rho_3^{(m)})^{\boldsymbol{n}}}{(\mathrm{Tr}\rho_3^{(m)})^{\boldsymbol{n}}}$$

Example in AdS3/CFT2

- Example: ground state of CFT2 in infinite line

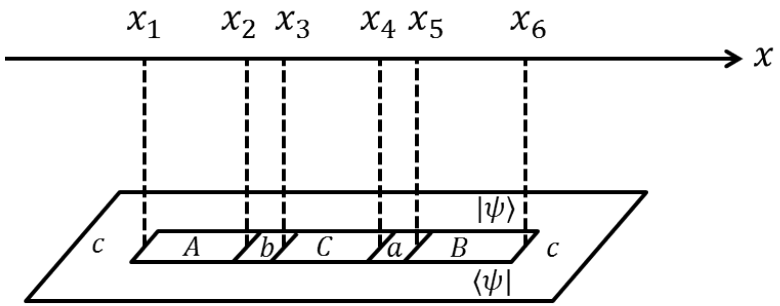
$$ds^2 = \frac{dx^2 + dz^2}{z^2} \,, \quad x \in (-\infty, +\infty), z \in [0, +\infty)$$



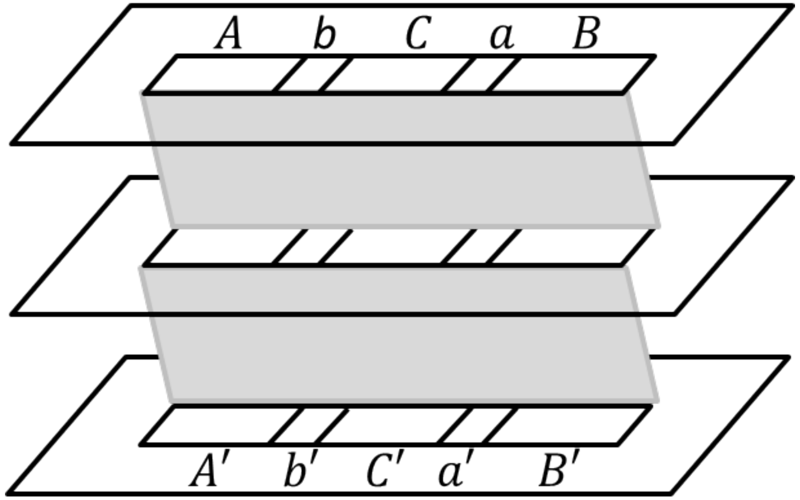
$$\Delta_W(A:B:C) = \min_{\theta} \left[\frac{L(\theta)}{4G_N} \right]$$

Path integral with replica trick

- Replica in purification



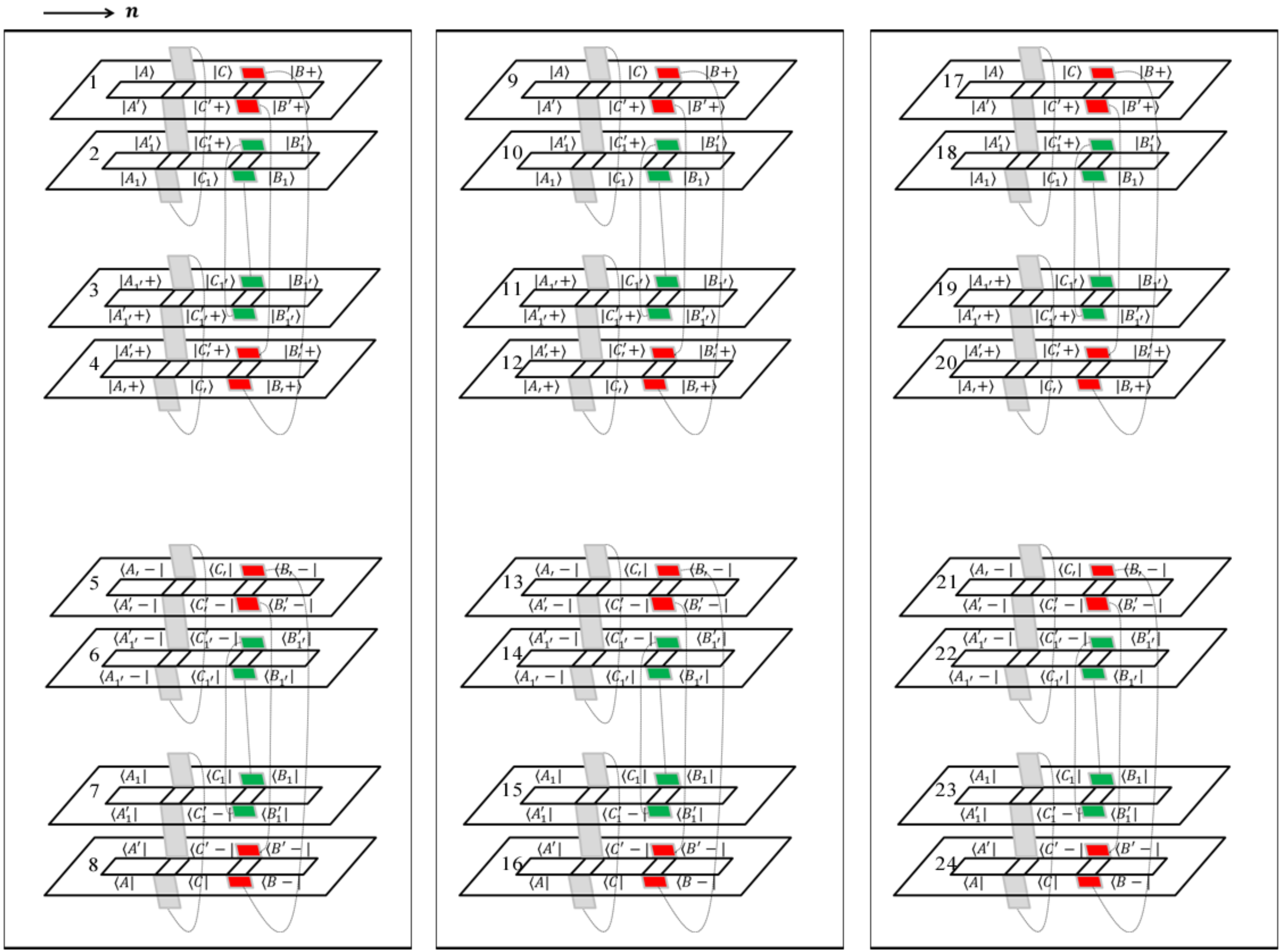
(a)



(b)

Use replica trick to represent $\psi_1^{(m)}$. (a): $\mathrm{Tr}_c \rho_0$ and (b): $\psi_1^{(m=6)}$.

$$m = 2, \boldsymbol{n} = 3$$



Twist operators

- Counting conformal weights for $\sigma_i(x_i)$

$$h_1 = h_6 = \frac{c}{24}(m^3 - m)\boldsymbol{n}\ , \quad h_2 = h_3 = \frac{c}{12}(m^2 - 1)\boldsymbol{n}\ , \quad h_4 = h_5 = \frac{c}{6}(m - \frac{1}{m})\boldsymbol{n}\ .$$

- OPE contraction

$$\sigma_i(x_i)\sigma_j(x_j) \to \sigma_f(x_f) \qquad h_{16} = h_{23} = h_{45} = \frac{c}{6}(\boldsymbol{n} - \frac{1}{\boldsymbol{n}})$$

- Partition function

$$\mathrm{Tr}_R(\mathrm{Tr}_L \rho_3^{(m)})^{\boldsymbol{n}} = \langle \sigma_1(x_1)\sigma_2(x_2)\sigma_3(x_3)\sigma_4(x_4)\sigma_5(x_5)\sigma_6(x_6)\rangle_{CFT^{\otimes m^3 \boldsymbol{n}}}$$

Large c simplification

- Large c 6-point block with fixed hf/c, hi/c

$$\mathcal{F} \approx \exp\left[-\frac{c}{6} f\left(\frac{h_f}{c}, \frac{h_i}{c}, x_i\right)\right]$$

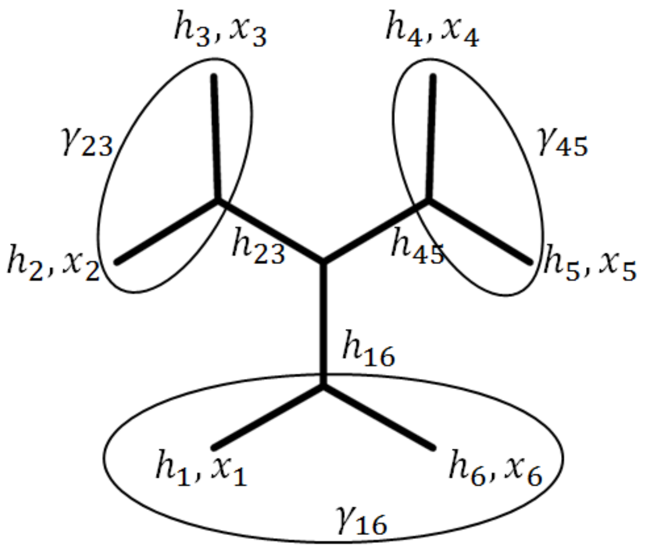
- Function f(x) can be solved from monodromy approach $\frac{\partial f}{\partial x_i} = c_i$

$$\psi''(z) + T(z)\psi(z) = 0 \qquad T(z) = \sum_{i=1}^{6}\left(\frac{6h_i/c}{(z-x_i)^2} - \frac{c_i}{z-x_i}\right)$$

$$\sum_{i=1}^{6} c_i = 0\,, \quad \sum_{i=1}^{6}(c_i x_i - \frac{6h_i}{c}) = 0\,, \quad \sum_{i=1}^{6}(c_i x_i^2 - \frac{12h_i}{c} x_i) = 0$$

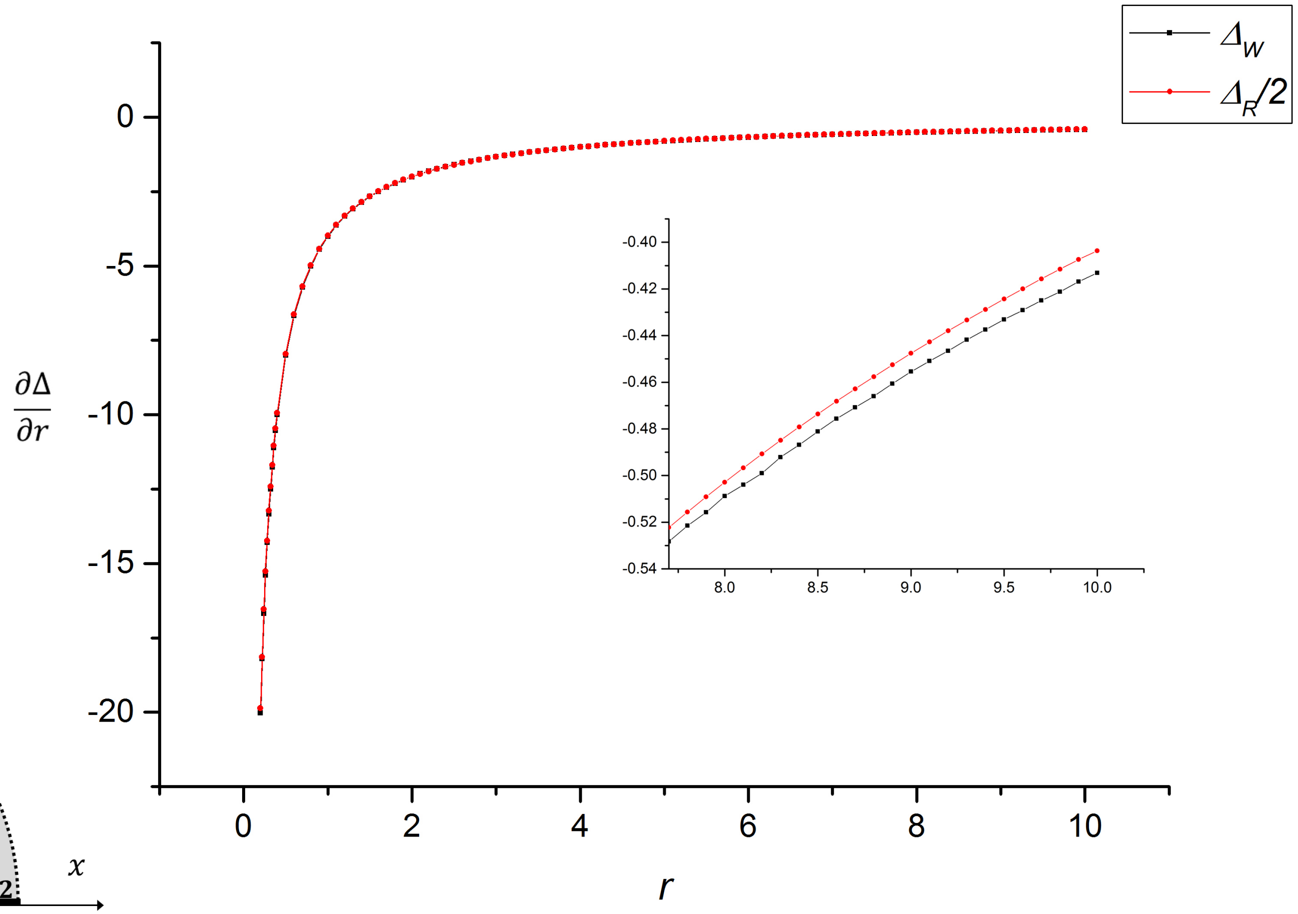
$$\mathrm{Tr} M_f = -2\cos\left(\pi\sqrt{1 - \frac{24}{c} h_f}\right)$$

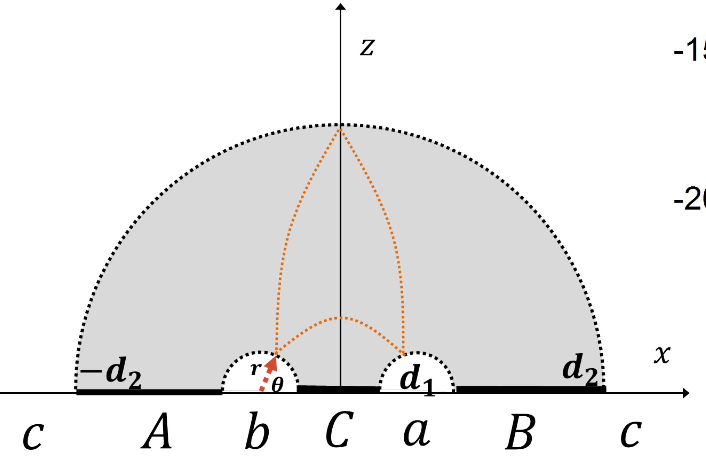
- Contours for monodromy matrices



Comparison

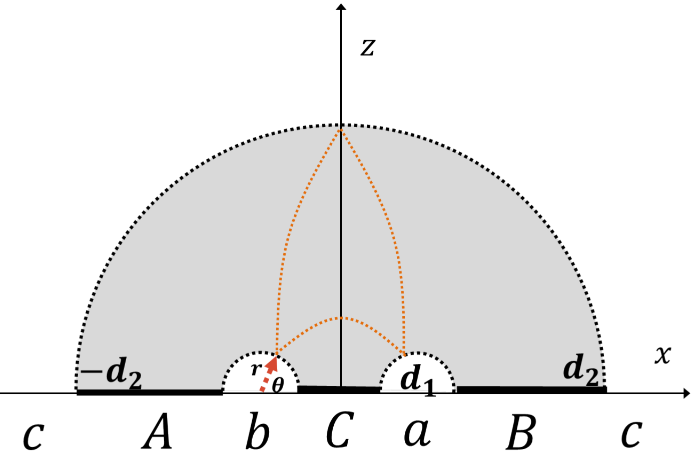
(a): to r, with $d_1 = 20, d_2 = 100$

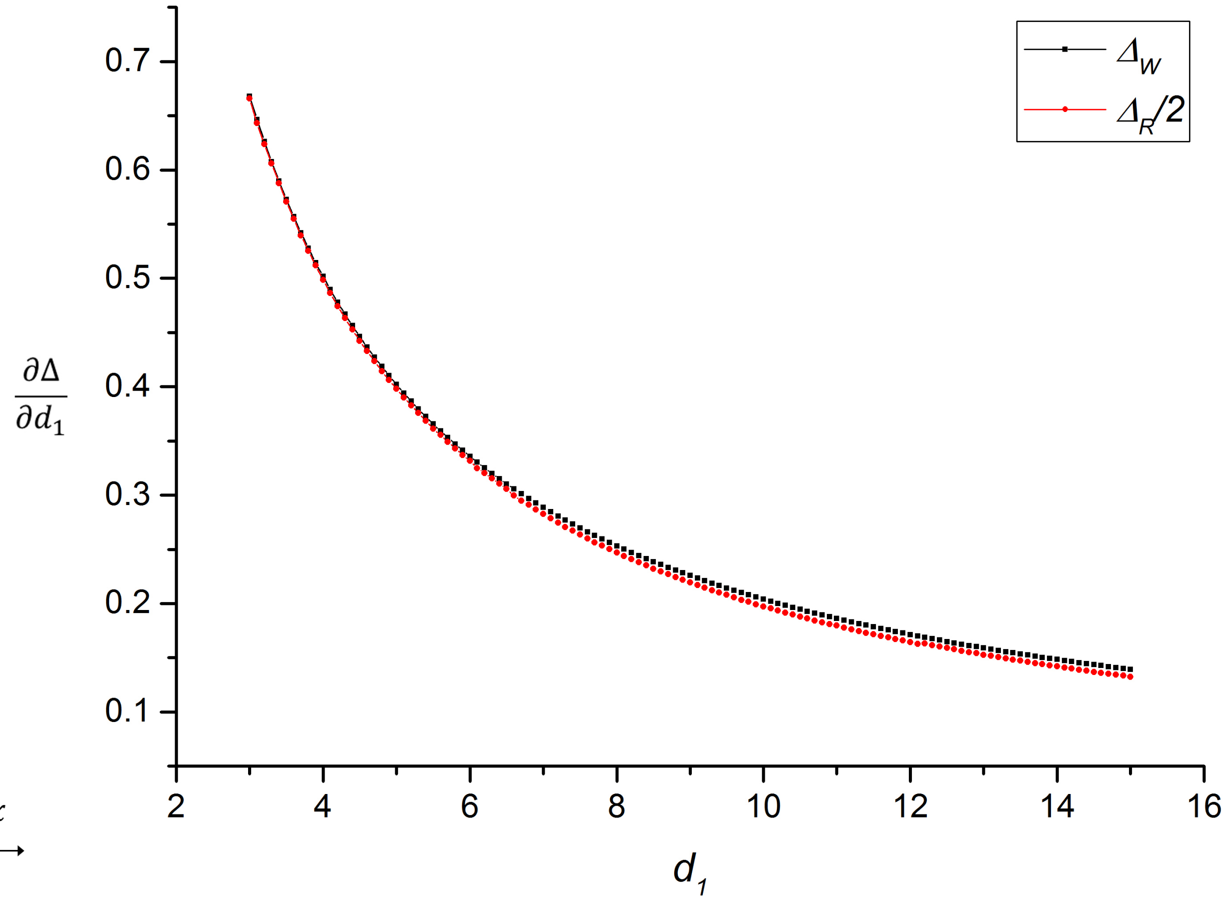




$$\frac{\partial \Delta_R(A:B:C)}{\partial x_i} = \lim_{m,\boldsymbol{n}\to 1} \frac{1}{1-\boldsymbol{n}} \left[-\frac{c}{3} \frac{\partial f\left(\frac{1}{6}(\boldsymbol{n} - \frac{1}{\boldsymbol{n}}), h_i, x_i\right)}{\partial x_i} \right]$$

Comparison (b): to d_1, with $r = 0.5, d_2 = 100$





Conclusion

- Infinite many concrete CFT entropies and its AdS duals have been found
- Some of the dualities has been tested/proved
- Black hole background and time dependent generalization