Complete Prepotential
of
5d Superconformal Field Theories

Kimyeong Lee
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H.Hayashi,S.-S.Kim,KL,F.Yagi to appear soon

5 SCFTs

Complete prepotential

Conclusion

5-dim SCFTs

5-dim SCFTs

Seiberg (96): 5d $\mathcal{N}$=1 Super Yang-Mills theories with SU(2) gauge group and n fundamental hypers have a UV fixed point with enhanced global symmetry $E_{n+1} \supset SO(2n) \times U(1)_I$

Instantons in R^{1+4} are solitons in 4+1 of mass $8\pi^2/g^2 = m_0/2$

There is a conserved current $J^\mu \sim \varepsilon^{\mu\alpha\beta\gamma\delta}\ \mathrm{tr}\,(F_{\alpha\beta}F_{\gamma\delta})$ with a conserved $U(1)_I$ global symmetry

superconformal conformal group $F(4) \supset SO(2,5) \times Sp(1)_R$

5-dim SCFTs

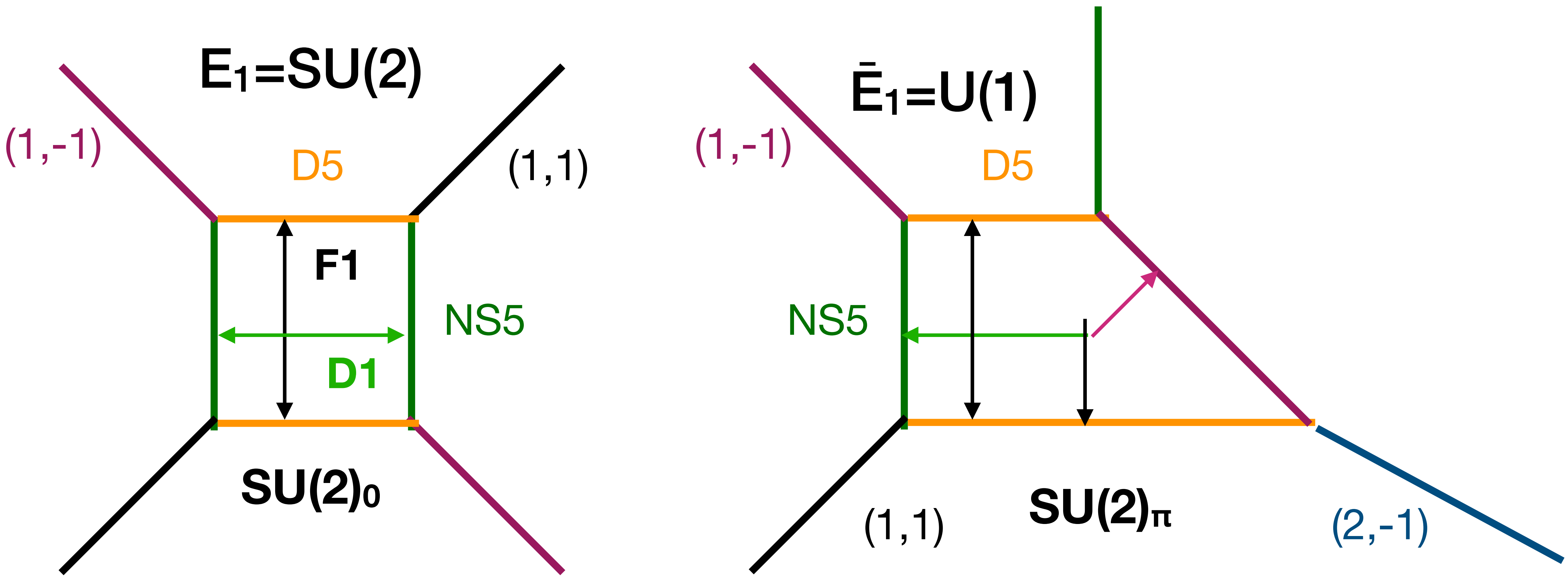
Morrison, Seiberg ('96): For n=0, one can have two options, $\theta=0,\pi$, with E_1, $\bar{E}_1$ enhanced symmetry due to $\pi_4(Sp(N))=Z_2$. There exists a non-Lagrangian theory of rank 1 E_0 without any global symmetry.

E_0, $\bar{E}_1$=U(1), E_1=SU(2), E_2=SU(2)xU(1), E_3=SU(3)xSU(2), E_4=SU(5), E_5=SO(10), E_6, E_7, E_8,

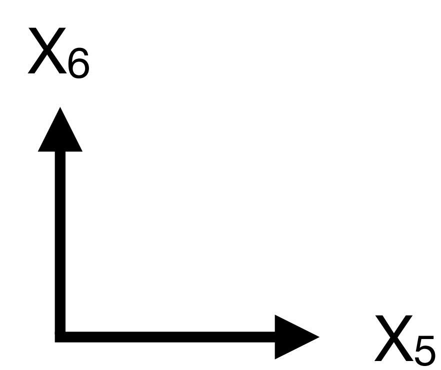
$\tilde{E}_8$: SU(2) + 8 hyper: completed in 6d (1,0) SCFT

- 5d pure SU(2) gauge theory with θ=0,π

Aharony, Hanany 97
Aharony, Hanany, Kol 97



- A (p,q) 5-brane (=p D5 + q NS5 branes) is a line with slop q/p.



Nekrasov Partition Function

- $R^4_{\varepsilon_1 \varepsilon_2} x S^1$ partition function
  - $Z=Z_{pert}\,Z_{inst}$, $Z_{inst}=1+qZ_1+q^2Z_2+q^3Z_3+\ldots$

$$q = e^{-m_0}, \text{ instanton fugacity}$$

  - counting BPS dyonic instanton index with with Ω-deformation
  - topological vertex
- $Z= \exp(F)=PE(F_s)$

Gopakumar-Vafa Invariant for SU(2) gauge theory

small $q=e^{-m0/2}$, e^{-a} expansion

Mitev,Pomoni,Taki,Yagi'14

- Invariant Coulomb parameter $\tilde{a} = a + m_0/(8 - N_f)$
- Enhanced global symmetry $\tilde{A} = e^{-\tilde{a}} = q^{-\frac{2}{8-N_f}} e^{-a}$
- Z= exp(F)=PE(F_s) $\mathscr{F} = \sum_{n=1}^{\infty} \frac{1}{n} \mathscr{F}_s(u^n, v^n, y_l^n, \tilde{A}^n)$

$$\mathscr{F}_s(u, v, \tilde{A}) = \frac{u}{(1-uv)(1-u/v)} \sum_{n,j_+,R}^{\infty} N^{R,n}_{j_-,j_+} \ \chi_R(y_l) \ [j_-, j_+] \tilde{A}^n$$

$$[j_+, j_-] = (-1)^{2j_+ + 2j_- + 1}(u^{2j_-} + \cdots u^{-2j_-})(v^{2j_+} + \cdots v^{-2j_+}) \quad u = e^{-\epsilon_-}, \quad v = e^{-\epsilon_+}$$

$$\text{Spin Structure} = \ [(\frac{1}{2},0) + 2(0,0)] \otimes [(j_-, j_+)]$$

more hypers and vectors

$$[0,0]=-1,\ [0,\frac{1}{2}]=v+v^{-1}$$

$[0,0]\tilde{A}$: perturbative hyper + instanton

$[0,1/2]\tilde{A}^2$: perturbative vector + instantons

$$\mathcal{F}_s^{E_6}=\overline{\mathbf{27}}[0,0]\tilde{A}+\mathbf{27}[0,\frac{1}{2}]\tilde{A}^2+([0,0]+\mathbf{78}[0,1]+[\frac{1}{2},\frac{3}{2}])\tilde{A}^3+\cdots$$

$$\overline{\mathbf{27}}[0,0]\tilde{A}=(10q^{-\frac{2}{3}}+\overline{16}q^{\frac{1}{3}}+q^{\frac{4}{3}})[0,0]\tilde{A}$$

$$\mathbf{27}[0,\frac{1}{2}]\tilde{A}^2=(q^{-\frac{4}{3}}+16q^{-\frac{1}{3}}+10q^{\frac{2}{3}})[0,\frac{1}{2}]\tilde{A}^2$$

Complete Prepotential

IMS prepotential

- Intriligator-Morrison-Seiberg '97
- gauge group G → $U(1)^{r_G}$ in the Coulomb branch
- possible Chern-Simons for SU(N) gauge theory

$$\mathcal{F}(\phi) = \frac{1}{2} m_0 h_{ij} \phi_i \phi_j + \frac{\kappa}{6} d_{ijk} \phi_i \phi_j \phi_k + \frac{1}{12} \left(\sum_{r \in \text{roots}} |r \cdot \phi|^3 - \sum_f \sum_{w \in R_f} |w \cdot \phi - m_f|^3 \right)$$

IMS prepotential

- IMS prepotential is cubic and one-loop exact
  - valid in small gauge coupling regime, perturbative
  - first derivative =monopole tension, second derivative=the Coulomb branch metric

$$T_a = \partial\mathcal{F}/\partial\phi_a, \tau_{ab} = \partial^2\mathcal{F}/\partial\phi_a\partial\phi_b$$

  - enhanced global symmetry may be not manifest.

Complete prepotential

- Improve the IMS prepotential by requiring
  - valid in all flop-transition
  - enhanced global symmetry is manifest
- Complete prepotential = IMS prepotential + additional terms which could be cubic in the Coulomb parameters

$$\mathscr{F}_{total} = \mathscr{F}_{\text{IMS}} + \mathscr{F}_{\text{additional}}$$

IMS prepotential for SU(2)+N_f

$$\mathcal{F}_{SU(2)} = \frac{1}{2}m_0 a^2 + \frac{4}{3}a^3 - \frac{1}{12}\sum_f |a \pm m_f|^2$$
$$= \frac{8-N_f}{6}a^3 + \frac{m_0}{2}a^2 - \frac{1}{2}\sum_f m_f^2 a + \frac{1}{6}\sum_f ||a \pm m_f||^3$$
$$= \frac{8-N_f}{6}\tilde{a}^3 - \frac{1}{2}\left(\frac{m_0^2}{8-N_f} + \sum_f m_f^2\right)\tilde{a} + \frac{1}{6}\sum_f ||\tilde{a} - \frac{m_0}{8-N_f} \pm m_f||^3$$

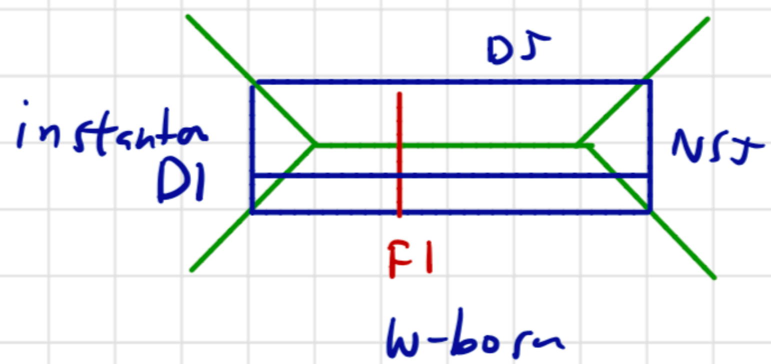
$$\tilde{a} = a + m_0/(8-N_f), \quad \|f\| = \theta(-f)f,\ |f| = f - 2\|f\|$$

Invariant Coulomb

$$y_l = \{e^{\pm m_1}, e^{\pm m_2}, \cdots e^{\pm m_f}, q\}$$ **for SO($2N_f$)xU(1)$_I$**

$E_1 = SU(2)$

$$\mathscr{F}_{E_1} = \mathbf{2}[0, \frac{1}{2}]\tilde{A}^2 + \cdots \qquad \mathbf{2}[0, \frac{1}{2}]\tilde{A}^2 = (q^{-\frac{1}{2}} + q^{\frac{1}{2}})[0, \frac{1}{2}]\tilde{A}^2$$



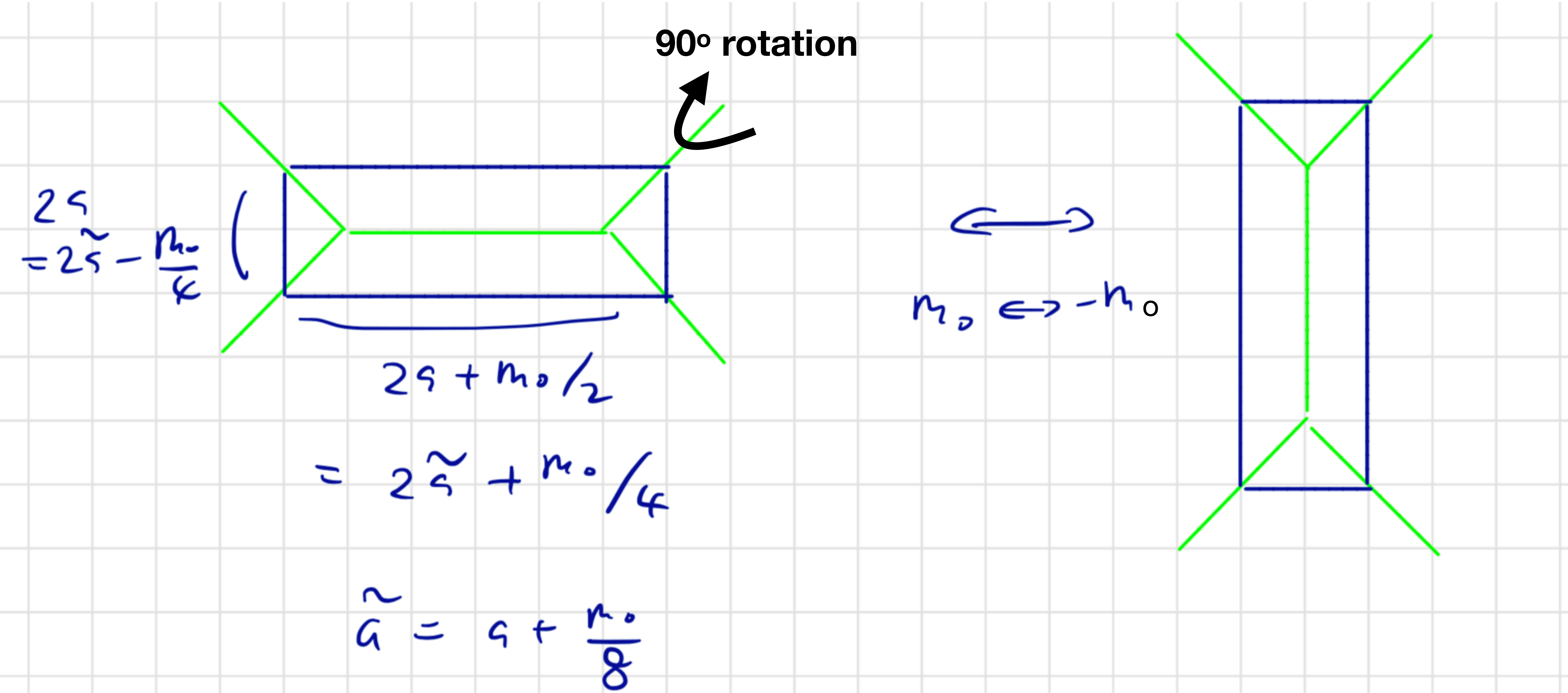
$$\tilde{A}^2 = A^2 q^{\frac{1}{2}} \qquad q = e^{\frac{m_0}{2}}, \quad \tilde{a} = a + \frac{m_0}{8}$$

$$(q^{-\frac{1}{2}} + q^{\frac{1}{2}})\tilde{A}^2 = (1+q)e^{-a}$$

Under the Weyl of $SU(2)$, keep $\tilde{A}$ invariant, and $q \leftrightarrow q^{-1}$

invariant under $m_0 \to -m_0$

E_1=SU(2)



$$\mathcal{F}^{E_1} = \frac{4}{3}a^3 + \frac{1}{2}m_{E_1}a^2 = \frac{4}{3}\tilde{a}^3 - \frac{1}{8}m_{E_1}^2\tilde{a}$$

invariant under m→ -m

role of BPS objects in prepotential

- IMS prepotential for $SU(2)_0$ is cubic and one-loop exact
  - only single W-boson contribution
  - matter contribution
  - enhanced global symmetry

E_2=SU(2)xU(1)

SU(2)+N_f=1

IMS

$$
\begin{aligned}
\mathcal{F}_{E_2} &= \frac{1}{2}m_0 a^2 + \frac{4}{3}a^3 - \frac{1}{12}|a \pm m_1|^2 \\
&= \frac{4}{3}a^3 + \frac{1}{2}m_0 a^2 - \frac{1}{6}a^3 - \frac{1}{2}m_1^2 a + \frac{1}{6}||a \pm m_1||^3
\end{aligned}
$$

E_2=SU(2)xU(1)

Complete

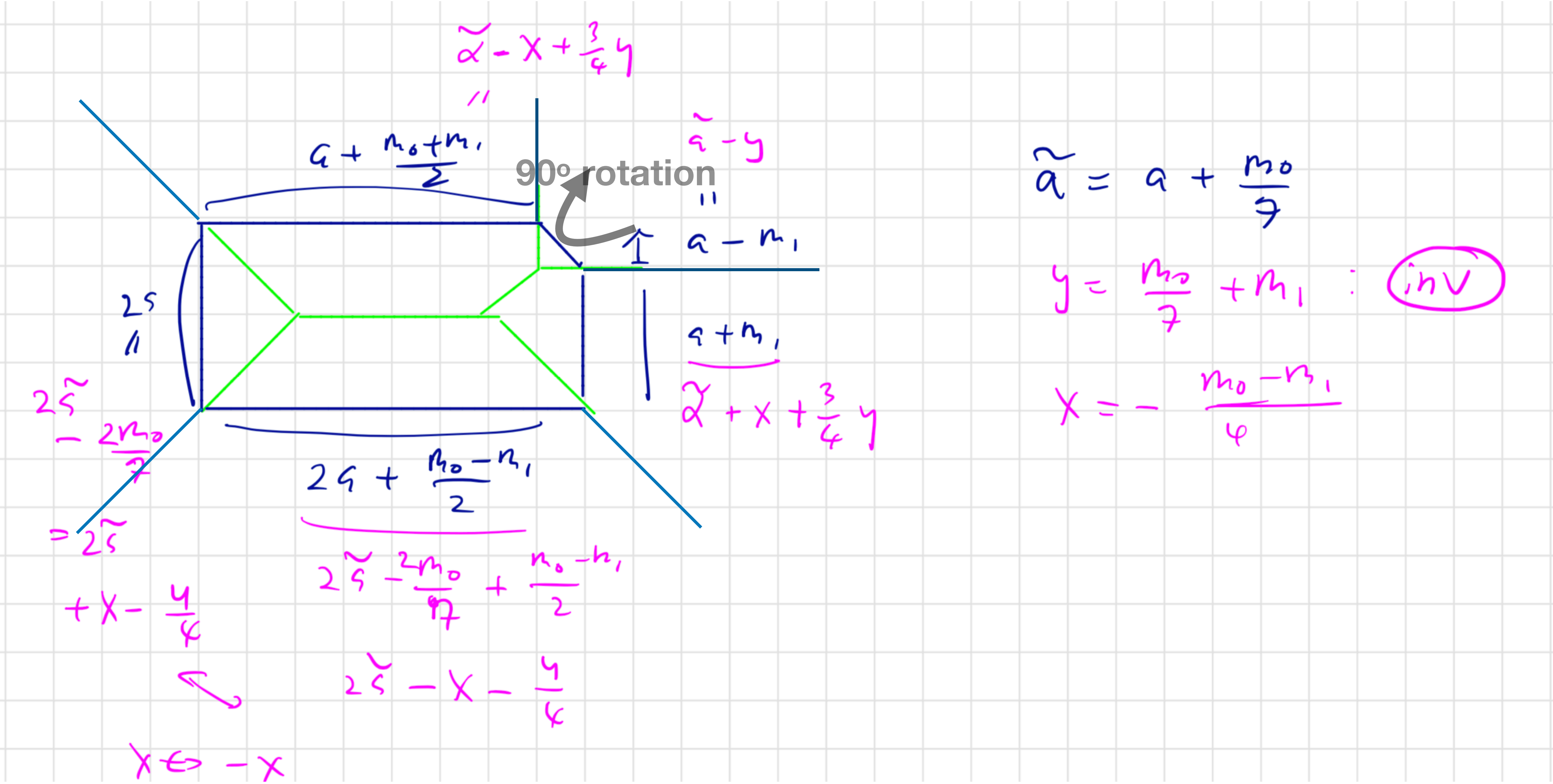
SU(2) doublet

$$\begin{aligned}
\mathcal{F}_{E_2} &= \frac{1}{2}m_0a^2+\frac{4}{3}a^3-\frac{1}{12}|a\pm m_1|^2 \\
&= \frac{4}{3}a^3+\frac{1}{2}m_0a^2-\frac{1}{6}a^3-\frac{1}{2}m_1^2a+\frac{1}{6}||a\pm m_1||^3 \\
&= \frac{7}{6}a^3+\frac{1}{2}m_0a^2-\frac{1}{2}m_1^2a+\frac{1}{6}||a+m_1||^3+\frac{1}{6}||a-m_1||^3+\frac{1}{6}||a+\frac{1}{2}(m_0+m_1)||^3 \\
&= \frac{7}{6}\tilde{a}^3-(4x^2+\frac{y^2}{16})\tilde{a}+\frac{1}{6}||\tilde{a}-\frac{4}{7}y||^3+\frac{1}{6}||\tilde{a}\pm x+\frac{3}{4}y||^3
\end{aligned}$$

$$||f||=\theta(-f)f,\quad \tilde{a}=a+\frac{1}{7}m_0,\quad x=-\frac{1}{4}(m_0-m_1),\ y=\frac{m_0}{7}+m_1$$

E_2 = SU(2)xU(1) **invariant under x→-x,y→y**

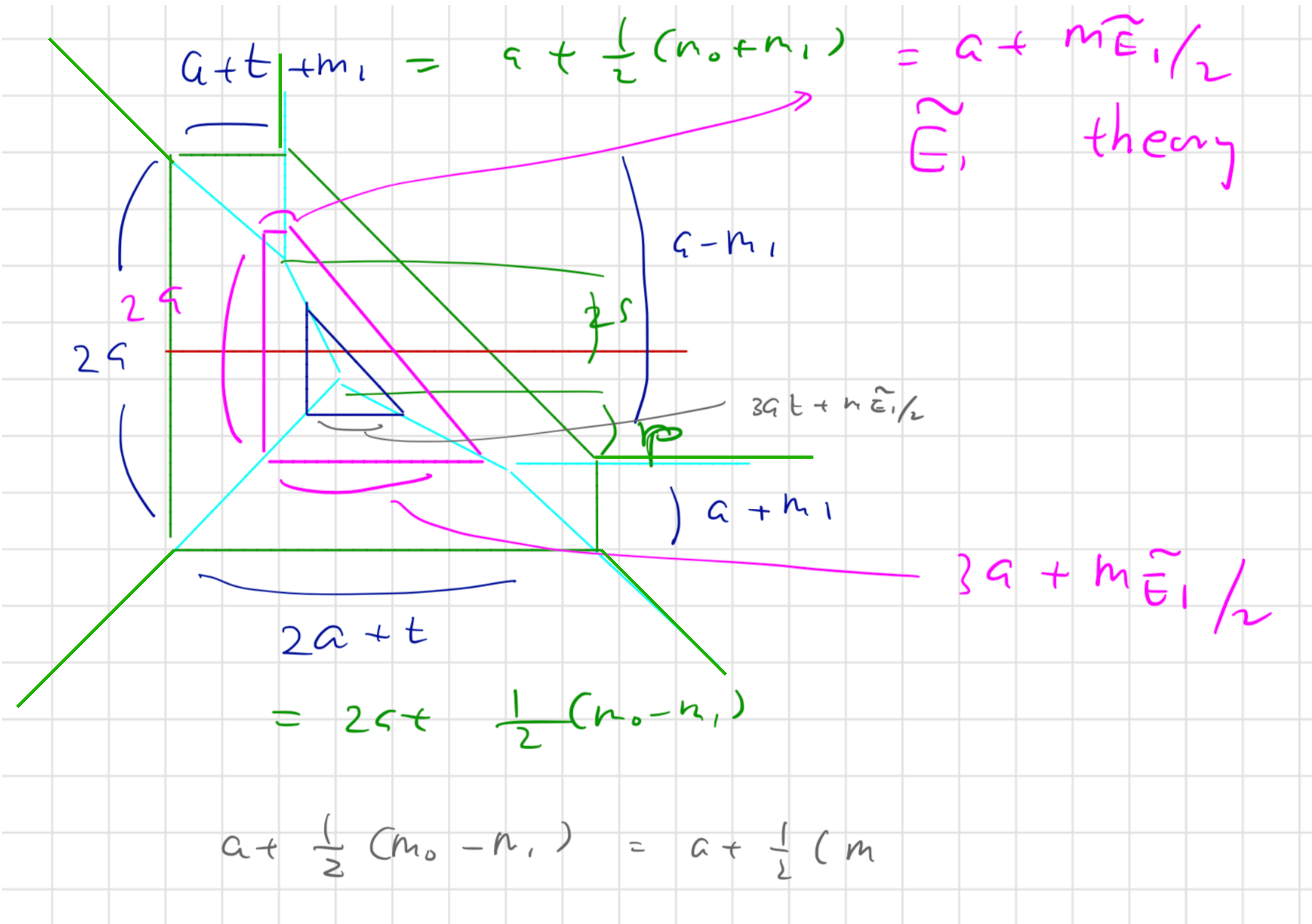
E_2 S-duality



E_2 = SU(2)xU(1): x→-x, y→y,

$E_2 \to \tilde{E}_1 \to E_0$

SU(2)xU(1) → U(1) → 0



$E_2 \rightarrow \tilde{E}_1 \rightarrow E_0$

$$\mathcal{F}_{E_2} = \frac{7}{6}a^3 + \frac{1}{2}m_0 a^2 - \frac{1}{2}m_1^2 a + \frac{1}{6}||a - m_1||^3 + \frac{1}{6}||a + m_1||^3 + \frac{1}{6}||a + \frac{1}{2}(m_0 + m_1)||^3$$

$a>0, m_1<0$

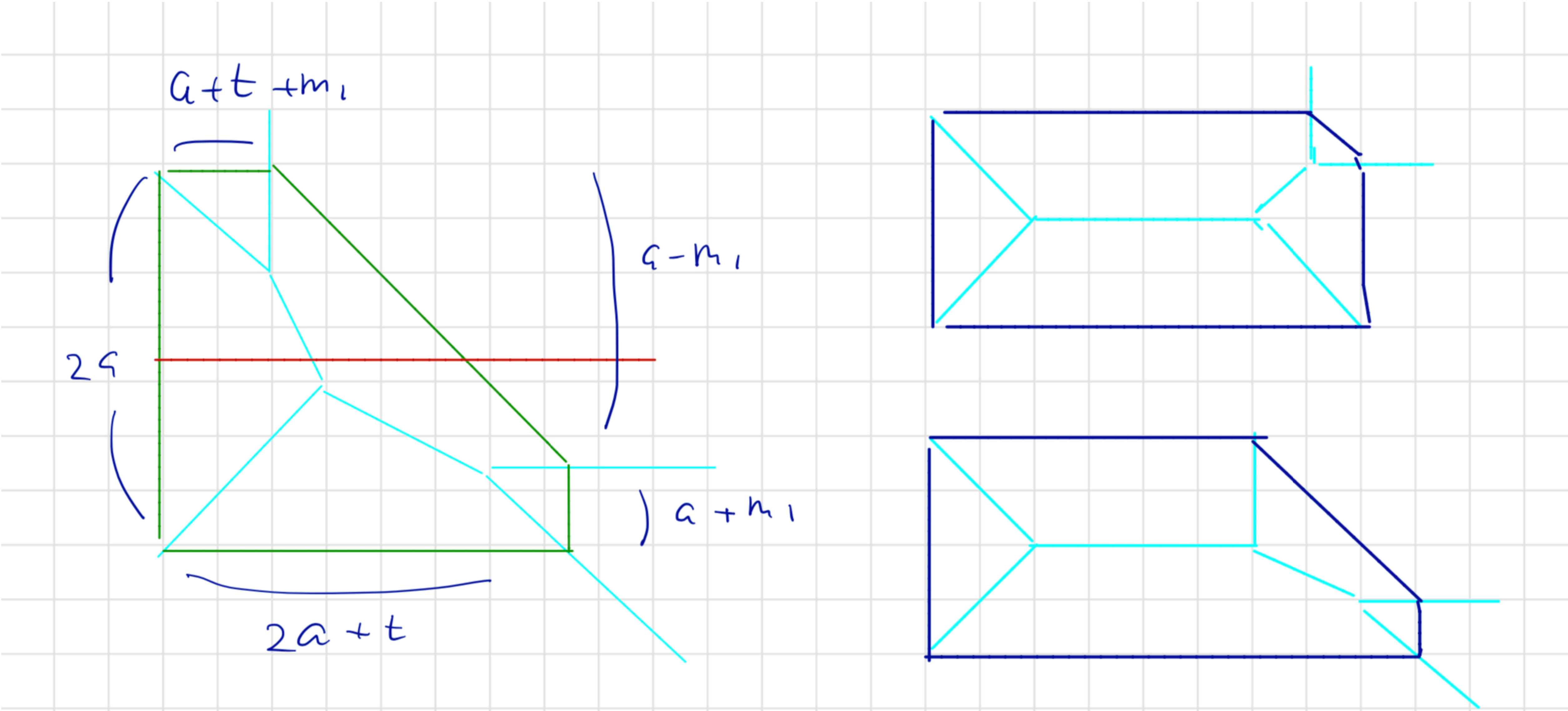
As $a + m_1 < 0$ and $m_0 + m_1$ fixed

$$\begin{aligned}\mathcal{F}_{\tilde{E}_1} &= \frac{7}{6}a^3 + \frac{1}{2}m_0 a^2 - \frac{1}{2}m_1^2 a + \frac{1}{6}(a + m_1)^3 + \frac{1}{6}||a + \frac{1}{2}(m_0 + m_1)||^3 \\ &= \frac{4}{3}a^3 + \frac{1}{2}m_{\tilde{E}_1} a^2 + \frac{1}{6}||a + \frac{1}{2}m_{\tilde{E}_1}||^3, \quad m_{\tilde{E}_1} = m_0 + m_1\end{aligned}$$

when $a + \frac{m_{\tilde{E}_1}}{2} < 0$

$$\begin{aligned}\mathcal{F}_{E_0} &= \frac{4}{3}a^3 + \frac{1}{2}m_{\tilde{E}_1} a^2 + \frac{1}{6}(a + \frac{1}{2}m_{\tilde{E}_1})^3 \\ &= \frac{3}{2}a_{E_0}^3, \quad a_{E_0} = a + \frac{1}{6}m_{\tilde{E}_1}\end{aligned}$$

$E_2, E_1, \tilde{E}_1, E_0$



$$T = (2a+t)(a+m_1) + \frac{1}{2}(a-m_1)(a+t+m_1+2a+t) = \frac{7}{2}a^2 + (2t+m_1)a - \frac{1}{2}m_1^2$$

$$T = \frac{d\mathcal{F}}{da} = \frac{7}{2}a^2 + m_0 a - \frac{1}{2}m_1^2 \qquad t = \frac{1}{2}(m_0 - m_1)$$

SU(2)+N_f=7: E_8 global

$$\tilde{a} = a + m_0,\ \tilde{A} = q^{-2}A, \qquad e^{-\tilde{a}} = e^{2\cdot\frac{m_0}{2}} e^{-a} \qquad y_i = e^{-m_i}, i = 0,1\cdots 7$$

$$6\mathcal{F}_{IMS} = a^3 + 3m_0 a^2 - 3\sum_{k=1}^{7} m_i^2 a + \sum_{k=1}^{7} \| a \pm m_a \|^3$$

$$6\mathcal{F} = \tilde{a}^3 - 3\sum_{k=0}^{7} m_i^2 \tilde{a} + \sum_{k=1}^{7} \| \tilde{a} - m_0 \pm m_a \|^3 + \text{a lot more terms expected}$$

14 terms

$$248 \to 14_{-2} + 64_{-1} + 91_0 + 1_0 + 64_1 + 14_2 \text{ of } SO(14)$$

GV insvariants

$$\mathcal{F}_s = \mathbf{248}[0,0]\tilde{A} + \cdots$$

$$\mathcal{F}_s = (\mathbf{14}q^{-2} + \mathbf{64}q^{-1} + \mathbf{91} + \mathbf{1} + \overline{\mathbf{64}}q + \mathbf{14}q^2)[0,0]\tilde{A} + \cdots$$

SU(2)+N_f=7: E_8 global

Add Weyl Reflections of SO(14) $\pm m_0 \pm m_i,\ m_i \pm m_j\ (a \neq b)$

$$\mathcal{F}_1 = \frac{1}{6}\tilde{a}^3 - \frac{1}{2}\sum_{k=0}^{7} m_i^2 \tilde{a} + \sum_{0\le i<j\le 7} \frac{1}{6} \| \tilde{a} \pm m_i \pm m_j \|^3$$

$SO(16)$ $120 \rightarrow 14_{-2} + +91_0 + 1_0 + 14_2$ of $SO(14$

$$248 \rightarrow 14_{-2} + 64_{-1} + 91_0 + 1_0 + 64_1 + 14_2 \text{ of } SO(14)$$

Add E_8 Weyl Reflections $m_i \rightarrow m_i - \frac{1}{4}\sum_{j=0}^{7} m_j$ $\quad m_0 + m_1 \rightarrow \frac{m_0 + m_1 - m_2 \cdots m_7}{2}$

$$\mathcal{F}_2 = \frac{1}{6}\tilde{a}^3 - \frac{1}{2}\sum_{k=0}^{7} m_i^2 \tilde{a} + \frac{1}{6}\sum_{0\le i<j\le 7} \| \tilde{a} \pm m_i \pm m_j \|^3 + \frac{1}{6}\sum_{\text{even}+} \| \tilde{a} \pm \frac{\pm m_0 \cdots m_7}{2} \|^3$$

$$\mathcal{F}_2 = \frac{1}{6}\tilde{a}^3 - \frac{1}{2}\sum_{k=0}^{7} m_i^2 \tilde{a} + \frac{1}{6}\sum_{\mathbf{w}\in\mathbf{248}} \| \tilde{a} + \mathbf{w}\cdot\mathbf{m}_j \|^3$$

Complete Prepotential

SU(2)+N_f=7: E_8 global

Additional Checks:
pq 5 brane webs,
geometry, GV invariant
and prepotential in small
ϵ_1,ϵ_2 limit

$Sp(2)+N_f=9$: $SO(18)\to SO(20)$

$$\tilde{a}_1 = a_1 + m_0,\ a_2,\ M_i = \{m_0, m_1 \cdots m_9, -m_0, \cdots - m_9\}$$

$$\mathcal{F}_1 = \frac{1}{6}\tilde{a}_1^3 - \frac{1}{3}\tilde{a}_2^3 + \tilde{a}_1\tilde{a}_2 - \frac{1}{2}\sum_{k=0}^{9} m_i^2(\tilde{a}_1 + \tilde{a}_2)$$

$$\mathcal{F}_2 = \frac{1}{6}\sum_{0\le i<j\le 9} \| \tilde{a}_1 \pm m_i \pm m_j \|^3$$

SO(20) adjoint

$$\mathcal{F}_3 = \frac{1}{6}\sum_{i=0}^{9} \| \tilde{a}_2 \pm m_i \|^3$$

SO(20) vector

$$\mathcal{F}_4 = \frac{1}{6}\sum_{s_i=\pm 1/2} \| \tilde{a}_1 + \tilde{a}_2 + \sum_{i=0}^{9} s_i m_i \|^3$$

SO(20) spinors

$$\mathcal{F}_5 = \frac{1}{6}\sum_{i_1<i_2\cdots i_5} \| 2\tilde{a}_1 + \tilde{a}_2 + \sum_{k=1}^{5} \pm m_{i_k} \|^3$$

rank 5 anti-symmetric

Sp(2)+N_f=9: [3]-SU(2)-SU(2)-[4]

$$\tilde{a}_1 = m_0^{(1)} + m_0^{(2)} + a^{(2)},\ \tilde{a}_2 = \frac{1}{2} m_0^{(1)} + a^{(1)} - a^{(2)}$$

$$\mathcal{F}_1 = \frac{1}{6}\tilde{a}_1^3 - \frac{1}{3}\tilde{a}_2^3 + \tilde{a}_1\tilde{a}_2 - \frac{1}{2}\sum_{k=0}^{9} m_i^2(\tilde{a}_1 + \tilde{a}_2)$$

$$\mathcal{F}_2 = \frac{1}{6}\sum_{0\le i<j\le 9} \| \tilde{a}_1 \pm m_i \pm m_j \|^3$$

IMS for $a^{(2)}$

$$\mathcal{F}_3 = \frac{1}{6}\sum_{i=0}^{9} \| \tilde{a}_2 \pm m_i \|^3$$

IMS for bifundamental

$$\mathcal{F}_4 = \frac{1}{6}\sum_{s_i=\pm 1/2} \| \tilde{a}_1 + \tilde{a}_2 + \sum_{i=0}^{9} s_i m_i \|^3$$

IMS for $a^{(1)}$

$$\mathcal{F}_5 = \frac{1}{6}\sum_{i_1<i_2\cdots i_5} \| 2\tilde{a}_1 + \tilde{a}_2 + \sum_{k=1}^{5} \pm m_{i_k} \|^3$$

IMS for bi-fundamental

[5]-SU(2)-SU(2)$_0$: E$_8$ global symmetry

Use the duality map: Sp(2)+AS+7F = [1]-SU(2)-SU(2)-[5]

Apruzzi,Lawri,Lin,Schafer-Nameki,Wang'19

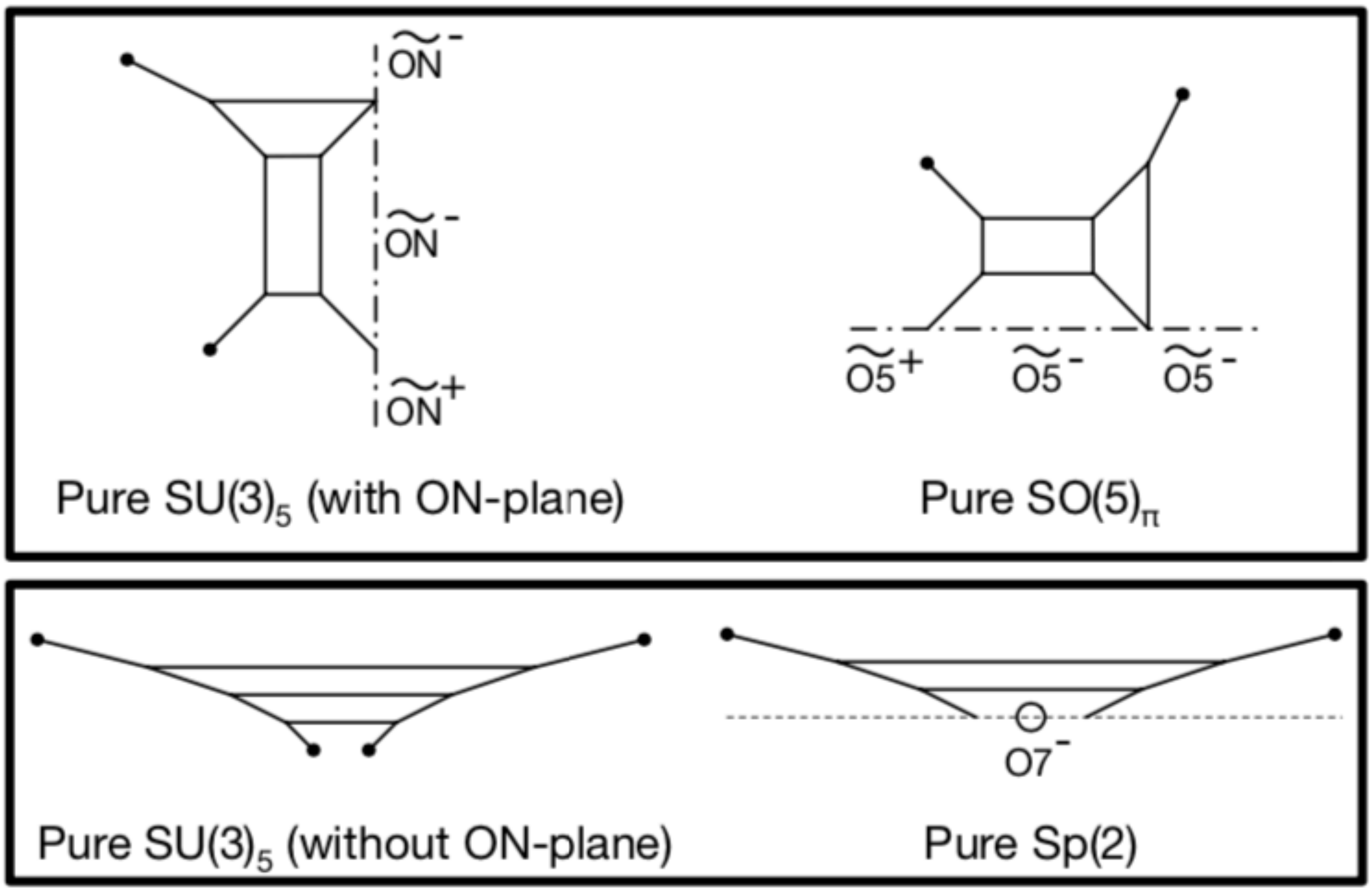
Brane picture

Prepotential $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$

$$\mathcal{F}_1 = \frac{1}{6}(\tilde{a}_1^3 + \tilde{a}_2^3) - \frac{1}{2}\sum_{i=0}^{7} m_i^2(\tilde{a}_1 + \tilde{a}_2) + \| \tilde{a}_1 + \tilde{a}_2 + \sum_{i=0}^{9} s_i m_i \|^3 + \frac{1}{6} \| \tilde{a}_1 - \tilde{a}_2 \|^3$$

$$\mathcal{F}_2 = \frac{1}{6}\sum_{\mathbf{w}\in\mathbf{248}} \| \tilde{a}_2 + \mathbf{w}\cdot\mathbf{m} \|^3 + \frac{1}{6}\sum_{\mathbf{w}\in\mathbf{3875}} \| \tilde{a}_2 + \mathbf{w}\cdot\mathbf{m} \|^3 + \frac{1}{6}\sum_{\mathbf{w}\in\mathbf{30380}} \| \tilde{a}_2 + \mathbf{w}\cdot\mathbf{m} \|^3$$

$SU(3)_5 \longleftrightarrow Sp(2)_\pi$



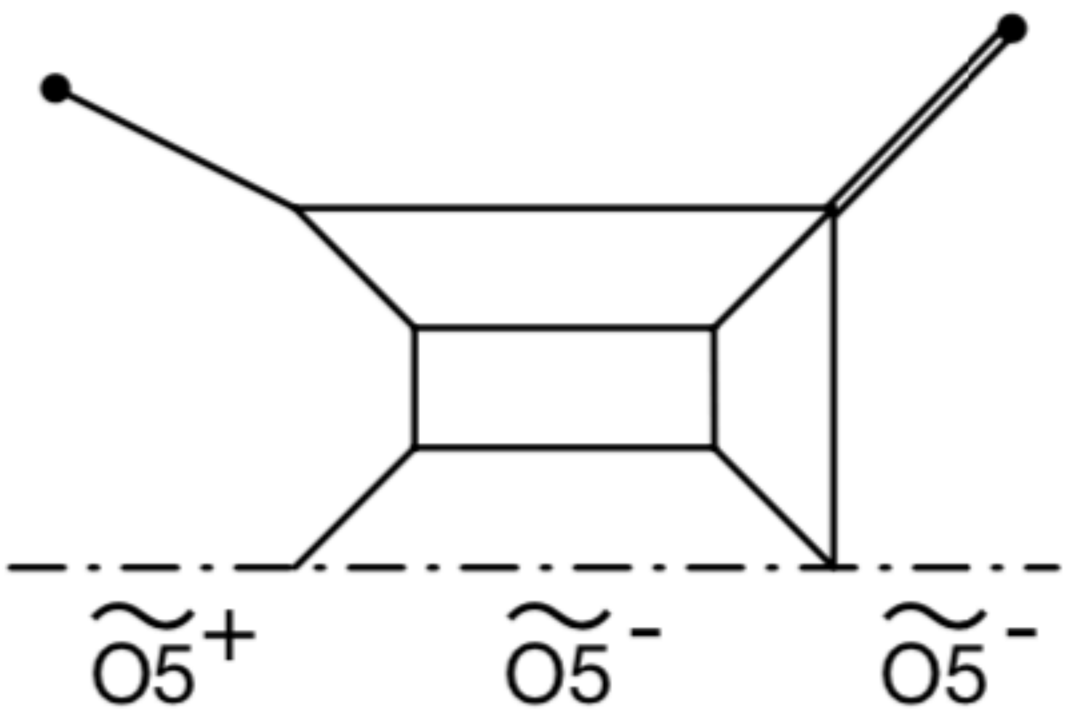
Pure $SU(3)_5$ (with ON-plane)

Pure $SO(5)_\pi$

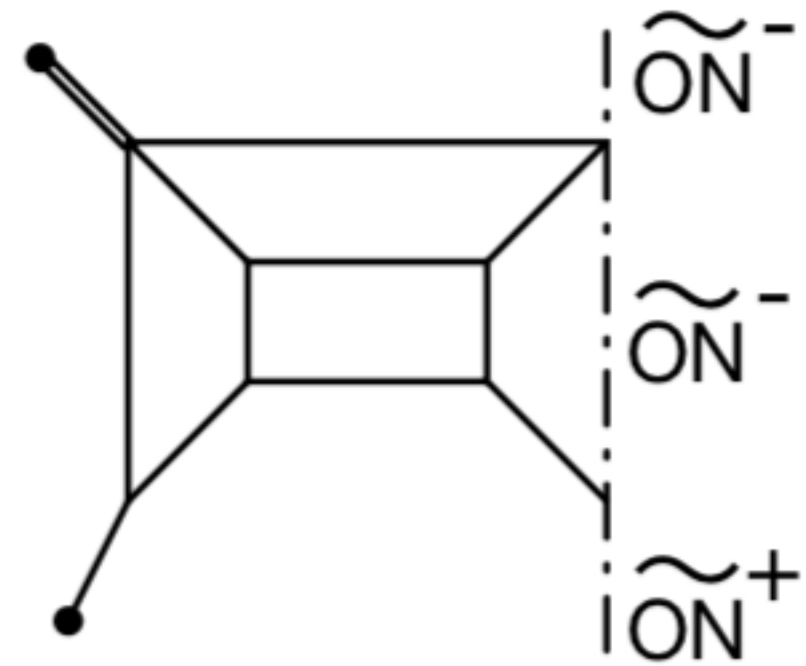
Pure $SU(3)_5$ (without ON-plane)

Pure $Sp(2)$

$SU(3)_7 \longleftrightarrow G_2$



Pure G_2



Pure $SU(3)_7$

Conclusion

Complete prepotential

- All the regions of the flop transition is included.
- the detail relation between complete prepotential and GV invariant is found.
  - the vector contribution is S-invariant
  - only BPS hypermultiplets which flop contribute to the additional expression
- The complete prepotential is S-dual invariant and has manifest global symmetry.

$$\mathcal{F}_{complete} = \mathcal{F}_{\mathrm{IMS}} + \mathcal{F}_{\mathrm{add}}$$

Complete prepotential

- It would provide an additional tool for analysis of 5d, 6d SCFTs and LSTs.
- A great tool to test dualities between the various IR descriptions of a given UV theory.