#### Hidden Analytic Structure of Two-Loop Higgs Amplitudes

#### **Gang Yang**

Institute of Theoretical Physics, Chinese Academy of Sciences



Based on: arXiv:1904.07260, arXiv:1910.09384, and in progress; in collaboration with Dr. Qingjun Jin and Dr. Ke Ren

East Asia Joint Workshop on Fields and Strings 2019 And 12th Taiwan String Theory Workshop Oct 28-Nov 01, 2019 @ NCTS Last Year: East Asia Joint Workshop on Fields and Strings KIAS, Nov 5-9, 2018

#### A direct relation between N=4 SYM and QCD

#### for two-loop Higgs-gluons Amplitudes

 $(H \rightarrow 3 \text{ gluons})|_{\text{QCD}} \leftrightarrow (3\text{-gluon form factor})|_{\mathcal{N}=4}$ 

Q.Jin and GY, PRL 121 101603 (2018) [arXiv:1804.04653],

Last Year: East Asia Joint Workshop on Fields and Strings KIAS, Nov 5-9, 2018

#### A direct relation between N=4 SYM and QCD

for two-loop Higgs-gluons Amplitudes

 $(H \rightarrow 3 \text{ gluons})|_{\text{OCD}} \leftrightarrow (3\text{-gluon form factor})|_{\mathcal{N}=4}$ 

New questions we ask:

How about Higgs amplitudes with **fundamental quark** external states?  $(H \rightarrow q\bar{q}g)|_{OCD}$ 

What is the **explanation and implication**?

# Outline

- Motivations
- Higgs amplitudes
- Hidden analytic structure
- Explanation and implication

## Generic strategy of loop computation



## Generic strategy of loop computation



HUGE intermediate expressions

Much simpler analytic form

#### MHV gluon tree amplitudes

[Parke, Taylor '86]

$$A_n^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}$$

Feynman diagrams								
n-gluon 4 5 6 7 8 9 10								
# graphs	4	25	220	2485	34300	559405	10525900	

Five-gluon expression:



#### Two-loop six-gluon amplitudes in N=4

$$\begin{split} & R_{0,VL}^{(2)}(u_1, u_2, u_3) = (\text{H.1}) \\ & \frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}; 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_1}, \frac{u_1}{u_1+u_2}; 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) + \frac{3}{2}G\left(0, 0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) + \frac{3}{2}G\left(0, 0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) + \frac{3}{2}G\left(0, 0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) + \frac{3}{2}G\left(0, 0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) + \frac{1}{2}G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) + \frac{1}{2}G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) + \frac{1}{2}G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{u_1+u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_2-1}{u_2+u_3}; \frac{1}{u_2+u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{u_1+u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{u_1+u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{u_1+u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_3}; \frac{1}{u_2+u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{u_1+u_2-1}; \frac{1}{u_1+u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_3}; \frac{1}{u_3}; \frac{1}{u_3}; \frac{1}{u_3}; \frac{1}{u_3}; \frac{1}{u_3}; \frac{1}{u_3}; \frac{1}{u_$$

$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1,0;1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right) + $	$\frac{1}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_1}\right)$
$\frac{1}{4}G\left(\frac{1}{1-u}, \frac{u_2-1}{u-1}, \frac{1}{1-u}, 1; 1\right) - \frac{1}{4}G\left(\frac{1}{1-u}, \frac{u_2-1}{u-1}, \frac{1}{1-u}, \frac{1}{1-u}, 1\right) -$	$\frac{1}{2}G\left(\frac{1}{2}, 0, \frac{1}{2}\right)$
$\frac{1}{2}\left(\frac{1}{1-u_1-u_1-u_1-u_1-u_1-u_1-u_1-u_1-u_1-u_$	1 -30 ( -1
$4^{(n)} (1 - u_1 \cdot u_1 + u_2 - 1 \cdot u_1 + u_2 - 1 \cdot 1 \cdot 1)$ $1 \cdot (1 - u_2 - 1 - u_2 - 1 - 1 - 1 - 1 - (1 - 1 - 1))$	8 1 - n
$\overline{4}^{G}(\overline{1-u_{1}}, \overline{u_{1}+u_{2}-1}, \overline{u_{1}+u_{2}-1}, \overline{1-u_{1}}, 1) - G(\overline{u_{1}}, 0, 0, \overline{u_{2}}, 1) +$	$\frac{1}{8}e^{-2}G(1-u_2)$
$\frac{1}{2}G\left(\frac{1}{n_{1}}, 0, 0, \frac{1}{n_{1}+n_{2}}; 1\right) - G\left(\frac{1}{n_{2}}, 0, 0, \frac{1}{n_{2}}; 1\right) + \frac{1}{2}G\left(\frac{1}{n_{1}}, 0, 0, \frac{1}{n_{1}+n_{2}}; 1\right) -$	$\frac{1}{8}e^2\mathcal{G}\left(\frac{1}{1-2}\right)$
$\frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	1 <sub>c</sub> (0.0 1
$4^{-}(u_1 \cdots u_1 \cdot u_1 + u_2) = 4^{-}(u_1 \cdots u_1 \cdot u_1 + u_1) = 4^{-}(u_1 \cdots u_2 \cdot u_1 + u_2) = 1_O(1 - 1 - 1 - 1_O(1 - 1 - 1_O)).$	1. (
$\frac{1}{4} \left( \frac{1}{10}, 0, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10} \right) - \frac{1}{4} \left( \frac{1}{1-10}, \frac{1}{10}, \frac{1}{10}, 0, 1 \right) +$	$\frac{1}{4} = \begin{pmatrix} 0, 0, \frac{1}{1-w} \\ 0, 0, \frac{1}{1-w} \end{pmatrix}$
$\frac{1}{2}G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1, \frac{1}{1-u_2}, 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_2-1}{u_2+u_3-1}, 0, 1; 1\right) -$	$\frac{1}{4}G(0, 0, v_{122}, \frac{1}{3})$
$\frac{1}{4}G\left(\frac{1}{1-u}, \frac{u_2-1}{u-1-v}, 0, \frac{1}{1-u}, 1\right) + \frac{1}{4}G\left(\frac{1}{1-u}, \frac{u_2-1}{u-1-v}, 1, 0, 1\right) -$	$\frac{1}{2}G(0, 0, v_{212}, \frac{1}{2})$
$\frac{1}{2}\left(\frac{1}{2}, \frac{u_2-1}{2}, \frac{1}{2}, \frac{u_1-1}{2}, \frac{1}{2}, \frac{u_2-1}{2}, u_$	1.000-
$4^{\circ}$ $(1 - u2^{\circ}u2 + u3 - 1^{\circ}1 - u2^{\circ}u^{\circ})^{\circ}$ $4^{\circ}$ $(1 - u2^{\circ}u2 + u3 - 1^{\circ}1 - u2^{\circ}u^{\circ})^{\circ}$	1. ( 1
$\frac{1}{4}G\left(\frac{1-u_2}{1-u_2}, \frac{1-u_2}{u_2+u_3-1}, \frac{1-u_2}{1-u_2}, \frac{1}{1-u_2}, 1$	$\frac{1}{2}G\left(0, \frac{1}{1-u_1}, 1$
$\frac{1}{4}G\left(\frac{1}{1-y_{2}}, \frac{y_{2}-1}{y_{2}+y_{3}-1}, \frac{y_{2}-1}{y_{3}+y_{3}-1}, 1; 1\right) +$	$\frac{1}{4}G\left(0, \frac{1}{1-w}\right)$
$\frac{1}{G}\left(\frac{1}{1}, \frac{u_2-1}{1}, \frac{u_2-1}{1}, \frac{1}{1}, 1\right) - G\left(\frac{1}{1}, 0, 0, \frac{1}{1}, 1\right) +$	$\frac{1}{2}g\left(0, \frac{1}{2}\right)$
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	1.00 1
$\frac{1}{2} O \left( \frac{1}{u_2}, 0, 0, \frac{1}{u_1 + u_2}, 1 \right) = O \left( \frac{1}{u_2}, 0, 0, \frac{1}{u_1}, 1 \right) + \frac{1}{2} O \left( \frac{1}{u_2}, 0, 0, \frac{1}{u_2 + u_1}, 1 \right) =$	2 ( 1 - 12
$\frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_1}, \frac{1}{u_1+u_2}, 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2}, \frac{1}{u_1+u_2}, 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2}, \frac{1}{u_2+u_3}, 1\right) -$	$\frac{1}{4}G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}\right)$
$\frac{1}{t}G\left(\frac{1}{t}, 0, \frac{1}{t}, \frac{1}{t-1}, 1\right) - \frac{1}{t}G\left(\frac{1}{t-1}, 1, \frac{1}{t}, 0, 1\right) +$	$\frac{1}{d}G\left(0, \frac{1}{1-N}\right)$
$\frac{1}{2}\left(\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}\right) + \frac{1}{2}\left(\frac{1}{1}, \frac{u_1 - 1}{u_1 - 1}, 0 + 1\right) =$	1 c ( a 1
$2^{(n)} \begin{pmatrix} 1 - u_1 & 1 - u_1 & 1 \\ 1 & u_1 & -1 \\ 1 & 1 & u_1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 &$	2° ( 1 - u1
$\frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{1}{u_1+u_2-1}, 0, \frac{1}{1-u_2}, 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{1}{u_1+u_2-1}, 1, 0, 1\right) -$	$\frac{1}{4} \left( 0, \frac{1}{1-u_1} \right)$
$\frac{1}{4}G\left(\frac{1}{1-y_0}, \frac{y_0-1}{y_0+y_0-1}, \frac{1}{1-y_0}, 0; 1\right) + \frac{1}{4}G\left(\frac{1}{1-y_0}, \frac{y_0-1}{y_0+y_0-1}, \frac{1}{1-y_0}, 1; 1\right) -$	$\frac{1}{4}G\left(0, \frac{1}{1-u_{3}}, 1\right)$
$\frac{1}{2}G\left(\frac{1}{1}, \frac{m-1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}G\left(0, u_{122}, \frac{1}{2}\right)$
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	1.0
$\frac{1}{4}$ $\left(\frac{1-u_1}{u_1+u_2-1}, \frac{u_1+u_2-1}{u_1+u_2-1}, \frac{1}{u_1}\right) = \frac{300}{300}$	4 1 1
$\frac{^{4}G}{4}G\left(\frac{u_{1}}{1-u_{2}}, \frac{u_{1}}{u_{1}+u_{2}-1}, \frac{u_{1}-u}{u_{1}+u_{3}-1}, \frac{u}{1-u_{3}}; 1\right) - G\left(\frac{u}{u_{2}}, 0, 0, \frac{u}{u_{1}}; 1\right) -$	$\frac{1}{4}$ $G \left( 0, u_{123}, \overline{u_2} \right)$
$G\left(\frac{1}{y_{1}}, 0, 0, \frac{1}{y_{1}}; 1\right) + \frac{1}{2}G\left(\frac{1}{y_{1}}, 0, 0, \frac{1}{y_{1}+y_{2}}; 1\right) + \frac{1}{2}G\left(\frac{1}{y_{1}}, 0, 0, \frac{1}{y_{1}+y_{2}}; 1\right) -$	$\frac{1}{4}G(0, u_{212}, \frac{1}{1-}$

$$\begin{split} & \frac{1}{2} ( \left\{ \frac{1}{2}, \frac{$$

$\frac{1-n}{n(1-1)}$ , 1; 1) + $\frac{1}{4}$ $\mathcal{G}\left(0, u_{223}, \frac{n(1-n)}{n(1+n(1-1))}, \frac{1}{(1-n(1-1))}, 1\right)$ -	$\frac{1}{G}\left(\frac{1}{1}, y_{12}, \frac{y_{2}-1}{1}, \frac{1}{1}, 1\right) + \frac{1}{G}\left(\frac{1}{1}, y_{12}, \frac{y_{2}-1}{1}, \frac{1}{1}, 1$
$\frac{1}{2}$ 1) $-\frac{1}{2}$ (0 m $\frac{1}{2}$ 0 1) $-\frac{1}{2}$ (0 m $\frac{1}{2}$ 0 1) +	$4^{-}(1-u_1, \dots, u_1+u_2-1, 1-u_1, f) = 4^{-}(1-u_1, \dots, u_1, f)$
$-u_1(1) = 4^{2} \left( \left( 1 - u_{22} \right) + u_2 \left( 1 - u_1 \right) \right) + 4^{2} \left( \left( 1 - u_{22} \right) + u_1 \left( 1 - u_2 \right) \right) \right)$	$-\frac{1}{4}G\left(\frac{1}{1-u_1}, v_{122}, 0, 0; 1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1}, v_{122}, 0, 1; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2}, v_{122}, 0, 1; 1; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2}, v_{122}, 0, 1; 1; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2}, v_{122}, 0, 1; 1; 1; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2}, v_{122}, 0, 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; $
$\frac{1}{n_1}$ , 1; 1) $-\frac{1}{4}G\left(0, u_{222}, \frac{1}{1-n_2}, \frac{1}{1-n_2}; 1\right) -$	$\frac{1}{2}\left(\frac{1}{1}, \operatorname{cun}, 1, 0; 1\right) - \frac{1}{2}\left(\frac{1}{1}, \operatorname{cun}, 1, \frac{1}{1}; 1\right) +$
$\frac{1}{1}$ $\frac{1}$	$1_{c}(1 - 1_{c}) + 1_{c}(1 - 1_{c})$
$u_1 - 1^{-1} - 1^{-1} - 1^{-1} - u_1 + u_2 - 1^{-1} - u_2^{-1} - u_2^{-1$	$\frac{1}{4}^{p}\left(\frac{1-u_{1}}{1-u_{1}}, \frac{v_{123}}{1-u_{1}}, \frac{1-u_{1}}{1-u_{1}}, \frac{v_{12}}{1-u_{1}}, \frac{v_{123}}{1-u_{1}}, \frac{1-u_{1}}{1-u_{1}}, \frac{v_{123}}{1-u_{1}}, \frac{1-u_{1}}{1-u_{1}}, \frac{v_{123}}{1-u_{1}}, \frac{1-u_{1}}{1-u_{1}}, \frac{v_{123}}{1-u_{1}}, \frac{v_{123}}{1-u_{$
$(-u_1, 1) = \frac{2}{2} \mathcal{G} \left( 0, v_{123}, 1, \frac{1}{1-u_1}, 1 \right) + \frac{2}{4} \mathcal{G} \left( 0, v_{123}, \frac{1}{1-u_1}, 0, 1 \right) = 0$	$-\frac{i}{4}\mathcal{G}\left(\frac{i}{1-u_1}, v_{122}, \frac{i}{1-u_1}, \frac{i}{1-u_1}; 1\right) + \frac{i}{4}\mathcal{G}\left(\frac{i}{1-u_1}, v_{122}, 0, 0; \right)$
$-, 1; 1$ + $\frac{1}{2}$ $\varphi$ $\left(0, v_{120}, \frac{1}{1}, \frac{1}{1}; 1\right) - \frac{1}{2}$ $\varphi$ $\left(0, v_{120}, 0, \frac{1}{1}; 1\right) -$	$\frac{1}{2}G\left(\frac{1}{1-r_1}, r_{122}, 0, 1; 1\right) + \frac{1}{2}G\left(\frac{1}{1-r_1}, r_{122}, 0, \frac{1}{1-r_1}; 1\right) -$
$u_1 \langle 4 \rangle$ $1-u_1 1-u_1 \langle 4 \rangle$ $1-u_1 \langle 1-u_1 \rangle$	$1_{a}(1, 1, 1, 1), 1_{a}(1, 1, 1)$
$\overline{u_1}^{(0)}(1) = \overline{4} \mathcal{G} \left( 0, v_{122}, \frac{1}{1-u_1}, \frac{1}{1-u_1}(1) - \overline{4} \mathcal{G} \left( 0, v_{213}, 0, \frac{1}{1-u_2}(1) - \frac{1}{u_1} \right) \right)$	$\frac{2}{2} \left( \frac{1-u_1}{1-u_1}, v_{122}, 1, \frac{1-u_1}{1-u_1}, 1 \right) + \frac{4}{4} \left( \frac{1-u_1}{1-u_1}, v_{122}, \frac{1-u_1}{1-u_1}, 0 \right)$
$=$ , 0; 1) $-\frac{1}{2}\mathcal{G}\left(0, v_{213}, \frac{1}{1-v}, \frac{1}{1-v}, 1\right) + \frac{1}{2}\mathcal{G}\left(0, v_{223}, 0, \frac{1}{1-v}, 1\right) -$	$-\frac{1}{2}G\left(\frac{1}{1-u_1}, v_{122}, \frac{1}{1-u_1}, 1; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_1}, v_{122}, \frac{1}{1-u_1}, \frac{1}{1-u_2}, \frac{1}{1-$
	$\frac{1}{2}G\left(\frac{1}{1}, 0, 0, v_{213}; 1\right) - \frac{1}{2}G\left(\frac{1}{1}, 0, 0, v_{223}; 1\right) - \frac{1}{2}G\left(\frac{1}{1}, 0, 0, v_{233}; 1\right) - \frac{1}{2}G\left(\frac{1}{1}, 0, 0, v_{23}; 1\right) - \frac{1}{2}G\left(\frac{1}{1}, 0, 0, v_{23}; 1\right) - \frac{1}{2}G\left(\frac{1}{1}, 0, 0, v_{23}; 1\right) - \frac{1}{2}G\left(\frac{1}{1}, 0, v_{23}; 1\right) - \frac{1}{2}G\left(\frac$
$-\frac{1}{12}$ $+ \int -\frac{1}{4} \left( \left( 0, 0.223, \frac{1}{1-12}, 0, 1 \right) - \frac{1}{2} \left( \left( 0, 0.223, \frac{1}{1-12}, 0, 1 \right) \right) \right)$	$4 (1-u_2) / 4 (1-u_2) / 2 (1)$ $1_a (1, 1, 1) / 1_a (1,)$
$\frac{1}{2^{n}}$ , $\frac{1}{1-2^{n}}$ ; 1 + $\frac{1}{4}$ $\mathcal{G}\left(0, v_{212}, 0, \frac{1}{1-2^{n}}; 1\right) - \frac{1}{2}\mathcal{G}\left(0, v_{212}, 1, \frac{1}{1-2^{n}}; 1\right) +$	$=\frac{2}{2} \left[ \frac{1-u_2}{1-u_2}, 0, \frac{1-u_2}{1-u_2}, v_{223}; 1 \right] = \frac{4}{4} \left[ \frac{1-u_2}{1-u_2}, 0, v_{223}; 1; 1 \right] =$
$= 0.1$ $- \frac{1}{6} \left( 0. rm - \frac{1}{1} + 1.1 \right) + \frac{1}{6} \left( 0. rm - \frac{1}{1} - \frac{1}{1} + 1 \right) - \frac{1}{6} \left( 0. rm - \frac{1}{1} - \frac{1}{1} + 1 \right) - \frac{1}{6} \left( 0. rm - \frac{1}{1} - \frac{1}{1} + 1 \right) - \frac{1}{6} \left( 0. rm - \frac{1}{1} + \frac{1}{1} + 1 \right) - \frac{1}{6} \left( 0. rm - \frac{1}{1} + \frac{1}{1} + 1 \right) - \frac{1}{6} \left( 0. rm - \frac{1}{1} + \frac{1}{1} + 1 \right) - \frac{1}{6} \left( 0. rm - \frac{1}{1} + \frac{1}{1} + 1 \right) - \frac{1}{6} \left( 0. rm - \frac{1}{1} + \frac{1}{1} + 1 \right) - \frac{1}{6} \left( 0. rm - \frac{1}{1} + \frac{1}{1} +$	$-\frac{4}{4}G\left(\frac{4}{1-u_1}, 0, v_{213}, \frac{4}{1-u_1}; 1\right) - \frac{4}{4}G\left(\frac{4}{1-u_2}, 0, v_{213}, 1; 1\right) -$
$u_1 \stackrel{(1)}{=} \left\{ \begin{array}{c} 2^n \\ 1 \end{array} \right\} \stackrel{(1)}{=} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} \stackrel{(2)}{=} \left\{ \begin{array}{c} 1 \end{array} \right\} \stackrel{(2)}{=} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} \stackrel{(2)}{=} \left\{ \begin{array}{c} 1 \end{array}$	$\frac{1}{2} \mathcal{G} \left( \frac{1}{1}, 0, v_{223}, \frac{1}{1}; 1 \right) - \frac{1}{2} \mathcal{G} \left( \frac{1}{1}, \frac{1}{1}, 0, v_{223}; \right)$
$\left(\frac{1}{-u_1}, 1\right) = \frac{1}{4}\mathcal{G}\left(0, v_{22}, \frac{1}{1-u_1}, 0, 1\right) = \frac{1}{4}\mathcal{G}\left(0, v_{22}, \frac{1}{1-u_1}, \frac{1}{1-u_2}, 1\right) =$	$\begin{pmatrix} 4 \\ 1 \\ - u_2 \end{pmatrix}$ $\begin{pmatrix} 1 - u_2 \\ 1 \\ - u_1 \end{pmatrix}$ $\begin{pmatrix} 2 \\ - u_2 \\ 1 \\ - u_2 \end{pmatrix}$ $\begin{pmatrix} 1 - u_2 \\ - u_2 \end{pmatrix}$
$p_{mc1} - \frac{1}{6} \left( \frac{1}{1000}, 0.0, p_{mc1} \right) - \frac{1}{6} \left( \frac{1}{1000}, 0, \frac{1}{1000}, p_{mc1} \right) - \frac{1}{6} \left( \frac{1}{10000}, 0, \frac{1}{10000}, 0, \frac{1}{100000} \right)$	$\frac{2}{2}$ $\left(\frac{1-u_2}{1-u_2}, \frac{1-u_2}{1-u_2}, 0, v_{211}; 1\right) = \frac{4}{4}$ $\left(\frac{1-u_2}{1-u_2}, \frac{1-u_2}{1-u_2}, \frac{1-u_2}{1-u_2}; 1-u_2; 1$
$1 \rightarrow 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - $	$-\frac{3}{4}G\left(\frac{1}{1-w_1}, \frac{1}{1-w_1}, \frac{1}{1-w_1}, w_{221}; 1\right) - \frac{1}{2}G\left(\frac{1}{1-w_1}, \frac{1}{1-w_1}, v_{222}; 1\right)$
$-u_1$ , $v_{122}$ , $1 - \frac{1}{4} \left( \frac{1}{1 - u_1}, 0, v_{122}, 1; 1 \right) - \frac{1}{4}$	$\frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, \frac$
$m \cdot \frac{1}{1 - m} \cdot 1 - \frac{1}{2} \mathcal{G} \left( \frac{1}{1 - m} \cdot 0, v_{122}, 1; 1 \right) -$	$\begin{pmatrix} 4^{+} \\ 1 \\ - u_{2}^{-1} \\ 1 \\ - u_{2}^{-1} \\ -$
	$\frac{1}{4} \mathcal{G} \left( \frac{1-u^2}{1-u^2}, \frac{1-u^2}{1-u^2}, \frac{1-u^2}{1-u^2}, \frac{1}{1-u^2}, \frac{1}{u^2} \right) = \frac{1}{4} \mathcal{G} \left( \frac{1-u^2}{1-u^2}, \frac{u^2}{u^2}, \frac{1}{u^2}, $
$\frac{1}{1-u_1} \left( 1 - \frac{1}{2} \right) - \frac{1}{2} \left( \frac{1-u_1}{1-u_1}, \frac{1-u_1}{1-u_1}, 0, 1_{122}, 1 \right) - \frac{1}{2}$	$-\frac{1}{4}G\left(\frac{1}{1-u_1}, u_{211}, 0, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1}, u_{211}, 1, 0; 1\right) +$
$\frac{1}{y_1}$ , 0, $v_{123}$ ; 1) $-\frac{3}{4}$ $\mathcal{G}\left(\frac{1}{1-y_1}, \frac{1}{1-y_2}, \frac{1}{1-y_2}, v_{123}$ ; 1) $-$	$\frac{1}{2}g\left(\frac{1}{1}, y_{m}, \frac{1}{2}, y_{1}\right) + \frac{1}{2}g\left(\frac{1}{1}, y_{m}, \frac{1}{2}, y_{1}\right)$
1	$\frac{1}{1}$ $\begin{pmatrix} 1 - u_2 & \cdots & u_1 & \cdots \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & -u_2 & \cdots & 1 - u_2 & \cdots \\ 1 & 1 & 1 & 1 \\ \end{pmatrix}$
$u_1 (1 - u_1) $	$= \frac{4}{4} \left( \frac{1-u_2}{1-u_2}, \frac{u_{211}}{1-u_2}, \frac{1-u_1}{1-u_2} \right) + \frac{4}{4} \left( \frac{1-u_2}{1-u_2}, \frac{u_{211}}{1-u_2}, \frac{1-u_2}{1-u_2}, \frac{1-u_2}{1-u_2} \right)$
$\frac{1}{y_1}$ , $v_{123}$ , $\frac{1}{1-y_1}$ ; $1 = \frac{1}{2} \mathcal{G} \left( \frac{1}{1-y_1}, \frac{1}{1-y_1}, v_{123}, 1; 1 \right) =$	$-\frac{1}{d}G\left(\frac{1}{1-w_1}, u_{221}, \frac{u_2-1}{w_1+w_2-1}, 2; 1\right) - \frac{1}{d}G\left(\frac{1}{1-w_2}, u_{221}, \frac{u_2}{w_1+w_2}\right)$
$-\frac{1}{1}$ $(1) - \frac{1}{2} \left(\frac{1}{1} - \frac{1}{2} \left(\frac{1}{1} - \frac{1}{2} - \frac{1}{2} \right) + \frac{1}{2} \left(\frac{1}{1} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} \right)$	$\frac{1}{2}\left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}$
$u_1 = 1 - u_1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 +$	$4^{\nu}$ $\begin{pmatrix} 1-u_2 \\ -u_2 \end{pmatrix}$ $\begin{pmatrix} -u_2 \\ -u_2$
$(0, \frac{1}{1-u_1}; 1) = \frac{1}{4} \mathcal{G}\left(\frac{1}{1-u_1}, u_{122}, 1, 0; 1\right) + \dots$	$-\frac{1}{4}\mathcal{G}\left(\frac{1-u_2}{1-u_2}, v_{213}, 1, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1-u_2}{1-u_2}, v_{213}, 1, \frac{1-u_2}{1-u_2}; 1\right) +$
$(\frac{1}{1}, 0; 1) - \frac{1}{2} \mathcal{G} \left( \frac{1}{1}, u_{123}, \frac{1}{1}, 1; 1 \right) +$	$-\frac{1}{d}\mathcal{G}\left(\frac{1}{1-m}, v_{213}, \frac{1}{1-m}, 0, 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-m}, v_{213}, \frac{1}{1-m}, 1\right)$
1-u1 / 4 1-u1 1-u1 /	- ( / - (1

19	$\left(\frac{1}{1-y_1}, v_{213}, \frac{1}{1-y_1}, \frac{1}{1-y_1}, 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-y_1}, v_{223}, 0, 0; 1\right) -$	<u>₹</u> ₽(	$\left(v_{123}, 0, 1, \frac{s}{1-u_1}, 1\right) + \frac{s}{2}\mathcal{G}\left(v_{123}, 0, \frac{s}{1-u_1}, 1-u_1, 1-u$
19	$\left(\frac{1}{1}, v_{211}, 0, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(\frac{1}{1}, v_{211}, 0, \frac{1}{1}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1}, v_{211}, 1, 0; 1\right) - $	- <u>†</u> e(	$v_{123}$ , 1, 1, $\frac{1}{1-u_1}$ ; 1) + $\frac{1}{2}G\left(v_{123}$ , 1, $\frac{1}{1-u_1}$
10	$\begin{pmatrix} 1 \\ -ki \end{pmatrix} = \begin{pmatrix} 1 \\ -ki \end{pmatrix} + $	10 (	$v_{122}, 1, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, 1 + \frac{1}{2}G\left(v_{122}, \frac{1}{1}, \frac{1}{1}, \frac{1}{2}\right)$
1	$\begin{pmatrix} 1 - u_2 & \cdots & 1 - u_2 \end{pmatrix} = \begin{pmatrix} 4^{\circ} & (1 - u_2 & \cdots & 1 - u_2 & \cdots \end{pmatrix}$ $\begin{pmatrix} 1 & \dots & 1 & \dots & 1 & \dots & 1 \end{pmatrix}$	200	$r_{pm} = \frac{1}{1 + 1} + \frac{1}{2} \left( r_{pm} - \frac{1}{1 + 1} \right)$
7	$\left(\frac{1-u_2}{1-u_2}, \frac{v_{222}}{1-u_2}, \frac{1-u_2}{1-u_2}, \frac{v_{222}}{1-u_2}, \frac{v_{222}}{1-u_2}, \frac{1-u_2}{1-u_2}, \frac{1-u_2}{1-u_$	1.1	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$
79	$\left(\frac{1-u_3}{1-u_3}, 0, 0, v_{212}; 1\right) - \frac{2}{4}\mathcal{G}\left(\frac{1-u_3}{1-u_3}, 0, 0, v_{221}; 1\right) - \frac{2}{2}\mathcal{G}\left(\frac{1-u_3}{1-u_3}, 0, \frac{1-u_3}{1-u_3}, v_{222}; 1\right) - $	75	$(120, \frac{1-u_1}{1-u_1}, \frac{1-u_1}{1-u_1}, \frac{1}{u_1}) - \frac{1}{4} ((120, 1, \frac{1}{u_1}))$
29	$\left(\frac{1}{1-u_3}, 0, \frac{1}{1-u_3}, v_{221}; 1\right) - \frac{1}{4}G\left(\frac{1}{1-u_3}, 0, v_{222}, 1; 1\right) -$	79	$v_{133}, \frac{1}{1-u_1}, 1, 1; 1 = \frac{1}{4} \mathcal{G}\left(v_{213}, 1, 1, \frac{1}{1-u_1}\right)$
19	$\left(\frac{1}{1-u_1}, 0, v_{223}, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{221}, 1; 1\right) -$	÷9(	$\left(v_{211}, \frac{1}{1-u_2}, 1, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{211}, 0, 1, \frac{1}{1-u_2}, 1, 1; 1\right)$
10	$\left(\frac{1}{1-s}, 0, v_{223}, \frac{1}{1-s}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-s}, \frac{1}{1-s}, 0, v_{223}; 1\right) -$	1 <u>2</u> 9	$v_{211}, 1, 0, \frac{1}{1-w}, 1 - \frac{5}{4}\mathcal{G}\left(v_{211}, 1, 1, \frac{1}{1-w}\right)$
10	$\begin{pmatrix} 1 & -u_1 \\ -u_2 \\ -u_1 \\ -u_2 \\ -u_1 \\ -u_2 \\ -u_1 \\ -$	÷¢(	$ezn, 1, \frac{1}{1-1}, 1; 1 + \frac{1}{2} \mathcal{G}\left(ezn, 1, \frac{1}{1-1}\right)$
3,	$\begin{pmatrix} 1 - u_1 & 1 - u_2 & \cdots & r \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & r \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & r & r \end{pmatrix}$	100	$(221, \frac{1}{2}, 1, 0; 1) - \frac{1}{2}C(121, \frac{1}{2}, 1)$
Ŧ.	$\begin{pmatrix} 1 - u_1 \\ 1 - u_2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ $	1.0	$1-u_2$ $1-u_2$ $1-u_2$ $1-u_1$ $1-u_2$ $1-u_2$ $1-u_2$
79	$\left(\frac{1-u_3}{1-u_3}, \frac{1-u_3}{1-u_3}, \frac{1}{1-u_3}, 1\right) = \frac{2}{2} \left(\frac{1-u_3}{1-u_3}, \frac{1-u_3}{1-u_3}, \frac{1}{1-u_3}, \frac{1}{1-u$	1.	$\frac{1}{1-u^2}$ $\frac{1-u^2}{1-u^2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{1-u^2}$ $\frac{1}$
79	$\left(\frac{1}{1-u_1}, \frac{1}{1-u_2}, u_{223}, \frac{1}{1-u_2}, 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{223}, 0, 1; 1\right) +$	3	$\left(v_{312}, 1, 0, \frac{1-u_3}{1-u_3}, 1\right) = \frac{4}{4} \left(v_{312}, 1, 1, \frac{1-u_3}{1-u_3}, 1\right)$
2	$\left(\frac{1}{1-u_{1}}, u_{322}, 0, \frac{1}{1-u_{1}}, 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{1}}, u_{332}, 1, 0; 1\right) +$	₹₽(	$v_{312}, 1, \frac{1}{1-u_3}, 2; 1 + \frac{1}{2}G\left(v_{312}, 1, \frac{1}{1-u_3}\right)$
2	$\left(\frac{1}{1-\epsilon}, u_{222}, \frac{1}{\epsilon}, 0; 1\right) + \frac{1}{2} \mathcal{G}\left(\frac{1}{1-\epsilon}, u_{222}, \frac{1}{1-\epsilon}, 0; 1\right) -$	- <u>+</u> 20(	$v_{312}, \frac{1}{1-u_3}, 1, 0; 1 - \frac{3}{4}\mathcal{G}\left(v_{312}, \frac{1}{1-u_3}, 1\right)$
10	$\begin{pmatrix} 1 & u_1 \\ 1 & u_{12} \end{pmatrix}$ , $\begin{pmatrix} 1 & u_1 \\ 1 & u_{12} \end{pmatrix}$ , $\begin{pmatrix} 1 & u_1 \\ 1 & u_{12} \end{pmatrix}$ , $\begin{pmatrix} 1 & u_{12} \\ 1 & u_{12} \end{pmatrix}$ , $\begin{pmatrix} 1 & u$	±¢(	$v_{312}, \frac{1}{1-w_1}, \frac{1}{1-w_2}, 1; 1 - \frac{1}{4}\mathcal{G}\left(v_{321}, 1, 1\right)$
1.	$\begin{pmatrix} 1 - u_1 & 1 - u_1 \end{pmatrix} = \begin{pmatrix} 4 & (1 - u_1 & 1 - u_1 & 1 - u_1 \end{pmatrix}$ $\begin{pmatrix} 1 & u_1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 - 1 & 1 \end{pmatrix}$	100	$r_{121}$ , $\frac{1}{1}$ , 1, 1; 1) $-\frac{3}{3}G\left(0, \frac{1}{1}, \frac{1}{1}\right)$
Ŧ.	$\begin{pmatrix} 1 - u 1 & u 1 + u - 1 & 1 \\ 1 & u - u 1 & 1 & 1 \\ \end{pmatrix} \begin{pmatrix} 1 - u 1 & u - 1 & 1 & 1 \\ 1 & u - 1 & 1 & 1 \\ 1 & u - 1 & 1 & 1 \\ \end{pmatrix}$	3al	$\begin{pmatrix} 1 - u_1 \\ 0 \\ - \\ - \\ - \\ - \\ - \\ - \\ - \\ - \\ -$
7	$\left(\frac{1-u_2}{1-u_2}, v_{212}, 0, 0; 1\right) = \frac{4}{4} \left(\frac{1-u_2}{1-u_2}, v_{212}, 0, 1; 1\right) + \frac{4}{4} \left(\frac{1-u_2}{1-u_2}, v_{212}, 0, \frac{1-u_2}{1-u_2}; 1\right) =$	1.	$\begin{pmatrix} -u_1 & u_1 + u_2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -u_1 & -u_2 \\ -u_1 & -u_2 \end{pmatrix}$
79	$\left(\frac{1-u_3}{1-u_3}, v_{312}, 1, 0; 1\right) - \frac{2}{2}G\left(\frac{1-u_3}{1-u_3}, v_{312}, 1, \frac{1-u_3}{1-u_3}; 1\right) +$	-75	$(0, \frac{u_1}{u_2}, \frac{u_1 + u_2}{u_1 + u_2}; 1) H(0; u_1) - \frac{1}{4} U(0, \frac{u_1 + u_2}{u_1 + u_2}; 1)$
÷9	$\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_2}, 0; 1\right) - \frac{1}{2}G\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}, 1; 1\right) +$	70	$(0, \frac{1}{u_2 + u_1 - 1}, \frac{1}{1 - u_2}, 1) H(0; u_1) - \frac{1}{4}G$
19	$\left(\frac{1}{1-y_1}, v_{212}, \frac{1}{1-y_1}, \frac{1}{1-y_1}, 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-y_1}, v_{221}, 0, 0; 1\right) -$	$\frac{2}{4}G($	$\left(\frac{1}{u_1}, 0, \frac{1}{u_1 + u_2}; 1\right) H(0; u_1) + \frac{1}{2}G\left(\frac{1}{u_1}, \frac{1}{u_1}; u_2\right)$
10	$\left(\frac{1}{1-c}, v_{221}, 0, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-c}, v_{221}, 0, \frac{1}{1-c}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-c}, v_{221}, 1, 0; 1\right) -$	$\frac{1}{2}a($	$\left(\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_2}, \frac{1}{u_2}; 1\right) H(0; u_1) + \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_2}; 1\right)$
10	$\left(\frac{1}{1-v_{11}}, v_{221}, 1, \frac{1}{1-v_{11}}, 1\right) + \frac{1}{2}\rho\left(\frac{1}{1-v_{22}}, v_{222}, \frac{1}{1-v_{11}}, 0, 1\right) -$	$\frac{1}{2}G($	$\left(\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_2 + u_3}; 1\right) H(0; u_1) - \frac{1}{4}G\left(\frac{1}{1-1}\right)$
1.0	$\begin{pmatrix} 1 - u_1 & \cdots & 1 - u_1 \\ 1 & \cdots & 1 & 1 + 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & \cdots & 1 & - u_1 \\ 1 & \cdots & 1 & 1 \end{pmatrix} + \frac{1}{2}$	10	$\left(\frac{1}{1}, \frac{u_2 - 1}{1}, 1; 1\right) H(0; u_1) -$

 $\begin{array}{l} & = & \left( - \left( \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} \right) B(\mathbf{n}_{1}, \mathbf{n}_{1} + \left( \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} \right) B(\mathbf{n}_{1}, \mathbf{n}_{1} + \frac{1}{\sqrt{1-1}} \right) B(\mathbf{n}_{1}, \mathbf{n}_{1} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} \right) B(\mathbf{n}_{1}, \mathbf{n}_{1} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} \right) B(\mathbf{n}_{1}, \mathbf{n}_{1} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} \right) B(\mathbf{n}_{1}, \mathbf{n}_{1} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} \right) B(\mathbf{n}_{1}, \mathbf{n}_{1} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} \left( \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1$ 

#### [Del Duca, Duhr, Smirnov 2010]

#### "multiple(Goncharov)-polylogrithm function"



$$\begin{split} & \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \left( \frac{1}{1-\alpha} \right) B(0,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \left( \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \left( \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \left( \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \left( \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \left( \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \frac{1}{1-\alpha} \right) B(1,\alpha) - \frac{1}{2} \left( \frac{1}{$$

 $\begin{array}{l} \left\{ \left( \sum_{i=1}^{n} (\sum_{j=1}^{n} (\sum_{i=1}^{n} (\sum_{j=1}^{n} (\sum_{j$ 

$$\begin{split} & \frac{1}{4} \widetilde{G} \left( z_{223}, 1, \frac{1}{1-u_0}, 1 \right) H(0, u_1) + \frac{1}{4} \widetilde{G} \left( z_{223}, \frac{1}{1-u_0}, 1, 1 \right) \\ & \frac{1}{4} \widetilde{G} \left( u_{23}, 1, \frac{1}{1-u_0}, 2 \right) H(0, u_2) + \frac{1}{4} \widetilde{G} \left( u_{23}, \frac{1}{1-u_0}, 2, 1 \right) \\ & \frac{1}{4} \widetilde{G} \left( \frac{1}{u_1}, \frac{1}{u_1+u_1} \right) H(0, u_2) H(0, u_2) + \\ & \frac{1}{4} \widetilde{G} \left( \frac{1}{u_1}, \frac{1}{u_1+u_1} \right) \\ & \frac{1}{4} \widetilde{G} \left( \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1+u_1} \right) \\ & \frac{1}{4} \widetilde{G} \left( \frac{1}{u_1$$

$$\begin{split} & \frac{1}{2} \left( \left( \frac{1}{|||_{q=1}^{-1} - q_{1}||} \right) \left( \frac{1}{||_{q=1}^{-1} - q_{1}||_{q=1}^{-1} + q_{1}||_{q=1}^{$$

 $\begin{array}{c} \frac{1}{1-(m+1)} \\ \frac{1}{1$ 

 $\begin{array}{l} \frac{1}{2} \exp (\log \left( \frac{1}{2} - \frac{1}{2} + \exp \left( \frac{1}{2} + \frac{1}{2} + - \frac{1}{2} + \exp \left( \frac{1}{2} + + \frac{1}{2} + \exp \left( \frac{1}{2} + + \frac{1}{2} + \exp \left( \frac{1}{2} + + \frac{1}{2} + + \exp \left( \frac{1$ 

 $\frac{1}{4}H(0; u_2) \mathcal{H}\left(0, 1, 1; \frac{1}{u_{122}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(0, 1, 1; \frac{1}{u_{122}}\right) + \frac{1}{4}H(0; u_1) \mathcal{H}\left(0, 1, 1; \frac{1}{u_{122}}\right)$  $\frac{1}{4}H(0; u_3) \mathcal{H}\left(0, 1, 1; \frac{1}{u_{012}}\right) - \frac{1}{4}H(0; u_1) \mathcal{H}\left(0, 1, 1; \frac{1}{u_{021}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(0, 1, 1; \frac{1}{u_{022}}\right)$  $H(0; u_1) \mathcal{H}(0, 1, 1; \frac{1}{u_1}) - \frac{1}{4} H(0; u_2) \mathcal{H}(0, 1, 1; \frac{1}{u_1}) - \frac{1}{4} H(0; u_1) \mathcal{H}(0, 1, 1; \frac{1}{u_1})$  $\frac{1}{4}H(0; u_2) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{321}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{u_{123}}\right)$  $\frac{1}{4}H(0; u_1)\mathcal{H}(1, 0, 1;$  $\frac{1}{4}H(0; u_2) \mathcal{H}\left(1, 0, 1; \frac{1}{u_{312}}\right) + \frac{1}{4}H(0; u_2) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{123}}\right) - \frac{1}{4}H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{123}$  $\frac{1}{4}H(0; u_2) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{122}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{122}}\right)$  $\frac{1}{4}H(0; u_1) \mathcal{H}(1, 0, 1;$  $\frac{1}{4}H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{213}}\right) - \frac{1}{4}H(0; u_1) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{223}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{223}$  $\frac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\frac{1}{v_{viv}}\right) - \frac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\frac{1}{v_{3i2}}\right) - \frac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\frac{1}{v_{3i2}}\right) - \frac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\frac{1}{v_{3i2}}\right) - \frac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\frac{1}{v_{3i2}}\right) - \frac{1}{4}H(0;u_1)\mathcal{H}$  $\frac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right) + H(0;u_2)\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right) - H(0;u_3)\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)$  $H(0; u_1) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{231}}\right) + H(0; u_3) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{231}}\right) + H(0; u_1) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{312}}\right)$  $H(0; u_2) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{312}}\right) - \frac{3}{2} \mathcal{H}\left(0, 0, 0, 1; \frac{1}{u_{123}}\right) - \frac{3}{2} \mathcal{H}\left(0, 0, 0, 1; \frac{1}{u_{233}}\right)$  $\begin{array}{c} \frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{312}}\right) - 3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{132}}\right) - 3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right) - 3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right) - 3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{321}}\right) - \frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{231}}\right) - \frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{231}}\right) - \frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right) - \frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right) - \frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right) + \frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right) + \frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right) - \frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right) + \frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}$  $\frac{1}{4}\mathcal{H}\left(0, 1, 1, 1; \frac{1}{u_{122}}\right) + \frac{1}{4}\mathcal{H}\left(0, 1, 1, 1; \frac{1}{u_{122}}\right) + \zeta_3 H\left(0; u_1\right) + \zeta_3 H\left(0; u_2\right) + \zeta_3 H\left(0; u_3\right) + \zeta_3 H\left(0; u_3\right$  $\frac{5}{2}\zeta_{3}H(1;u_{1}) + \frac{5}{2}\zeta_{3}H(1;u_{2}) + \frac{5}{2}\zeta_{3}H(1;u_{3}) + \frac{1}{2}\zeta_{3}\mathcal{H}\left(1;\frac{1}{u_{123}}\right) + \frac{1}{2}\zeta_{3}\mathcal{H}\left(1;\frac{1}{u_{231}}\right)$  $\frac{1}{2}\zeta_{3}\mathcal{H}\left(1;\frac{1}{u_{312}}\right) - \frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{123}}\right) - \frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{231}}\right) - \frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{312}}\right)$  $\frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{123}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{132}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{213}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{213}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{231}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{312}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{31$  $\frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{\upsilon_{321}}\right) + \frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{\upsilon_{213}}\right) + \frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{\upsilon_{213}}\right) + \frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{\upsilon_{231}}\right) + \frac{1}{4}\mathcal{H}\left(0,1,$  $\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{221}}\right) + \frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{123}}\right) + \frac{1}{4}\mathcal{H}\left(1,0$  $\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{231}}\right) + \frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{312}}\right) + \frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{312}}\right) + \frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{321}}\right) + \frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{132}}\right) + \frac{1}{4}\mathcal{H}\left(1,1$  $\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{132}}\right) + \frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{213}}\right) + \frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{213}}\right) + \frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{231}}\right) + \frac{1}{4}\mathcal{H}\left(1,1$  $\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{221}}\right) + \frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{123}}\right) + \frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{123}}\right) + \frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{231}}\right) + \frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{131}}\right) + \frac{3}{2}\mathcal{H}\left(1,1$ 

#### - 114 -

#### Result can be remarkably simple

#### 17 pages =

[Goncharov, Spradlin, Vergu, Volovich 2010]

$$\begin{split} &\sum_{i=1}^{3} \left( L_4 \left( x_i^+, x_i^- \right) - \frac{1}{2} \operatorname{Li}_4 \left( 1 - 1/u_i \right) \right) - \frac{1}{8} \left( \sum_{i=1}^{3} \operatorname{Li}_2 (1 - 1/u_i) \right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72} J^4 + \frac{\pi^4}{12} J$$

#### a line result in terms of classical polylogarithms!

Such simplicity is totally unexpected using traditional Feynman diagrams!

### Lessons from modern amplitudes

Conceptually: New structures and new formulations in QFT

Witten's twistor theory, Double-copy, Grassmannian and Amplituhedron, CHY formalism, new geometric structure...

#### Methodologically: New powerful computational techniques

Spinor helicity formalism

BCFW recursion relation

(Generalized) unitarity cuts

Developed based on studying N=4 SYM, now applicable to general theories

New integral reduction and integration methods

#### N=4 super Yang-Mills theory

<u>N=4 SYM theory</u> : -> <u>QCD's maximally supersymmetric cousin</u>

$$\mathcal{L} = -\frac{1}{g_{\rm YM}^2} \operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) + \text{fermions} + \text{scalars}$$

where all fields are the in the adjoint representation of the gauge group SU(Nc).

- conformal invariant, UV finite
- prime model in AdS/CFT correspondence
- solvable in the planar limit due to Integrability

What (more) can we learn from N=4 SYM for realistic QCD?



Maximally transcendental parts are equal between the two theories

(conjecture)

### Transcendental number / function

A root of a nonzero polynomial equation with integer coefficients?



Similarly, a transcendental function is an analytic function that does not satisfy a polynomial equation, in contrast to an algebraic function.

### Transcendentality degree

Number	Function	Transcendental degree
$2/3, \sqrt{2}$	rational function	0
π	Log(x)	1
ζĸ	Lik(x)	k
$\epsilon$ (D = 4	$-2\epsilon$ )	-1

Riemann zeta value:

$$\zeta_k = \sum_{n=1}^{\infty} \frac{1}{n^k}, \qquad k \ge 2$$
$$S_k(j) = \sum_{n=1}^j \frac{1}{n^k} \xrightarrow{j \to \infty} \zeta_k$$

Polylogarithms:

$$\operatorname{Li}_{k}(z) = \sum_{n=1}^{\infty} \frac{z^{n}}{n^{k}} = \int_{0}^{z} \frac{\operatorname{Li}_{k-1}(t)}{t} dt$$
$$\operatorname{Li}_{1}(z) = -\log(1-z) \qquad \operatorname{Li}_{k}(1) = \zeta_{k}$$



• Anomalous dimension of twist-2 operators:

$$\gamma^{\mathcal{N}=4}(j) = \gamma^{\text{QCD}}(j)|_{\text{max. trans}}$$

[Kotikov, Lipatov, Onishchenko, Velizhanin]



• Anomalous dimension of twist-2 operators:

 $\gamma^{\mathcal{N}=4}(j) = \gamma^{\text{QCD}}(j)|_{\text{max. trans}}$ 

[Kotikov, Lipatov, Onishchenko, Velizhanin]

• Higgs-3gluon two-loop amplitudes: [Brandhuber, Travaglini, GY 2012]

N=4 SYM  
form factorHiggs plus 3-gluon  
amplitudes 
$$m_t \to \infty$$
 $\mathscr{O}_0 = \operatorname{tr}(F_{\mu\nu}F^{\mu\nu})$ 

## Maximal transcendental part of Higgs amplitudes: $m_t \rightarrow \infty$

-G

+2

 $+\epsilon$ 

[Gehrmann, Jaquier, Glover, Koukoutsakis 2011]



Multiple polyLogarithm

 $+2G(1 - v, 0, u)H(1, 0, v) - 2G(1 - v, 1 - v, u)H(1, 0, v) + G(1 - v, -v, u)H(1, 0, v) - G(-v, 0, u)H(1, 0, v) + 2G(-v, 1 - v, u)H(1, 0, v) + G(0, 0, u)H(1, 1, v) \\ -2G(0, -v, u)H(1, 1, v) - 2G(-v, 0, u)H(1, 1, v) + 4G(-v, -v, u)H(1, 1, v) + G(0, u)H(0, 0, 1, v) - 3G(1 - v, u)H(0, 0, 1, v) + 4G(-v, u)H(0, 0, 1, v) \\ +G(0, u)H(0, 1, 0, v) + G(1 - v, u)H(0, 1, 0, v) - G(0, u)H(0, 1, 1, v) + 2G(-v, u)H(0, 1, 1, v) + G(0, u)H(1, 0, 0, v) + G(1 - v, u)H(1, 0, 0, v) + H(1, 1, 0, 0, v) \\ -G(0, u)H(1, 0, 1, v) + 2G(-v, u)H(1, 0, 1, v) - G(0, u)H(1, 1, 0, v) + 4G(1 - v, u)H(1, 1, 0, v) - 2G(-v, u)H(1, 1, 0, v) + H(0, 0, 1, 1, v) + H(0, 1, 0, 1, v) \\ +G(1 - v, 1 - v, u)H(0, 0, v) + 2G(1 - v, 1 - v, -v, u)H(1, v) - G(1 - v, -v, 0, 1 - v, u) + H(0, 1, 1, 0, v) + G(1 - v, 0, 1 - v, 0, u) - G(0, 1 - v, 1, 0, u) \\ +4G(-v, 1 - v, -v, 1 - v, u)$ 

-G(-v,0,u)H(0,1,v) - 2G(-v,1-v,u)H(0,1,v) + 4G(-v,-v,u)H(0,1,v) - G(0,0,u)H(1,0,v) + G(0,-v,u)H(1,0,v) - G(1,0,u)H(1,0,v) - G

-2G(0,0,1,0,u) + G(0,0,1-v,1-v,u) + 2G(0,0,-v,1-v,u) - G(0,1,0,1-v,u) + 4G(0,1,1,0,u) - G(0,1,1-v,0,u) + G(0,1-v,0,1-v,u) + G(0,1-v,0,1-v,u) + G(0,1-v,0,1-v,u) + 2G(0,-v,1-v,0,u) - 2G(0,-v,1-v,1-v,u) - 2G(1,0,0,1-v,u) + 2G(1,0

-2G(1,0,1-v,0,u) + 4G(1,1,0,0,u) - 4G(1,1,1,0,u) - 2G(1,1-v,0,0,u) + G(1-v,0,0,1-v,u) - G(1-v,0,1,0,u) - 2G(-v,1-v,1-v,u)H(0,v) - 2G(1-v,1,0,0,u) + 2G(1-v,1,0,0,u) + 2G(1-v,1-v,0,0,u) + 2G(1-v,1-v,0,0,u) + 2G(1-v,1-v,1,0,u) - 2G(1-v,1-v,0,0,u) + 2G(1-v,0,0,u) + 2G

-G(1-v,1,0,u)H(0,v) - G(1-v,1-v,0,u)H(0,v) - G(1-v,-v,1-v,u)H(0,v) + G(-v,0,1-v,u)H(0,v) + G(-v,1-v,0,u)H(0,v) + H(1,0,0,1,v) + G(-v,1-v,0,u)H(0,v) + H(1,0,0,1,v) + G(-v,1-v,0,u)H(0,v) + H(1,0,0,1,v) + G(-v,1-v,0,u)H(0,v) + G(-v,1-v,0,u)H(0

-G(0,0,u)H(0,1,v) + G(0,-v,u)H(0,1,v) - G(1,0,u)H(0,1,v) + 2G(1-v,0,u)H(0,1,v) + 2G(1-v,1-v,u)H(0,1,v) - 3G(1-v,-v,u)H(0,1,v) - 3G(1-v,

-G(0,0,-v,u)H(1,v) + G(0,1,0,u)H(1,v) - G(0,1-v,0,u)H(1,v) + G(0,1-v,-v,u)H(1,v) - 2G(0,-v,0,u)H(1,v) - 2G(0,-v,

-4G(1-v, -v, -v, u)H(1, v) + 2G(-v, 0, 1-v, u)H(1, v) + 2G(-v, 1-v, 0, u)H(1, v) - 4G(-v, 1-v, -v, u)H(1, v) + 4G(-v, -v, -v, u)H(1, v) + G(0, 0, u)H(0, 0, v) + G(0, 1-v, u)H(0, 0, v) + G(1-v, 0, u)H(0, 0, v) + H(1, 0, 1, 0, v)

y) + 2G(1, 0, 0, u)H(1, v) - G(1 - v, 0, 0, u)H(1, v) + G(1 - v, 0, -v, u)H(1, v) - 2G(1 - v, 1, 0, u)H(1, v) - G(1 - v, 0, -v, 1 - v, u)H(1, v) + G(1 - v, 0, -v, 1 - v, u)H(1, v) + G(1 - v, 0, -v, 1 - v, u)H(1, v) + G(1 - v

[Brandhuber, Travaglini, GY 2012]

$$N=4 -2\left[J_4\left(-\frac{uv}{w}\right) + J_4\left(-\frac{vw}{u}\right) + J_4\left(-\frac{wu}{v}\right)\right] - 8\sum_{i=1}^3 \left[\operatorname{Li}_4\left(1-\frac{1}{u_i}\right) + \frac{\log^4 u_i}{4!}\right] - 2\left[\sum_{i=1}^3 \operatorname{Li}_2\left(1-\frac{1}{u_i}\right)\right]^2 + \frac{1}{2}\left[\sum_{i=1}^3 \log^2 u_i\right]^2 + 2(J_2^2 - \zeta_2 J_2) - \frac{\log^4(uvw)}{4!} - \zeta_3 \log(uvw) - \frac{123}{8}\zeta_4$$

$$J_4(x) = \text{Li}_4(x) - \log(-x)\text{Li}_3(x) + \frac{\log^2(-x)}{2!}\text{Li}_2(x) - \frac{\log^3(-x)}{3!}\text{Li}_1(x) - \frac{\log^4(-x)}{48}, \quad J_2 = \sum_{i=1}^3 \left(\text{Li}_2(1-u_i) + \frac{1}{2}\log(u_i)\log(u_{i+1})\right), \quad u = \frac{s_{12}}{q^2}, \quad v = \frac{s_{23}}{q^2}, \quad w = \frac{s_{13}}{q^2}, \quad w = \frac{s_{13}}{q^2}, \quad w = \frac{s_{13}}{q^2}$$



Questions that we will address:

- (1) To which extend is the correspondence correct?
- (2) What is the origin of the correspondence?
- (3) Can we go beyond N=4? How about lower trans. parts?

# Outline

- Motivations
- Higgs amplitudes
- Hidden analytic structure
- Explanation and implication

### Higgs Effective Field Theory



There have been computations for inclusive Higgs production to NNNLO orders in the heavy quark limit.

[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger 2016]

### Higgs Effective Field Theory



Effective Higgs-gluon vertices:

 $\mathscr{L}_{\text{eff}} = C_0 H \text{tr}(F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}(\frac{1}{m_t^2})$ 

Higgs plus jet production is sensitive to new physics.

EFT description is not good when  $p_T \sim 2m_t$ 



High dimension operators contributions become important.

### Higgs Effective Field Theory

$$O_{1} = H \operatorname{tr}(F_{\mu}^{\nu}F_{\nu}^{\rho}F_{\rho}^{\mu})$$

$$O_{2} = H \operatorname{tr}(D_{\rho}F_{\mu\nu}D^{\rho}F^{\mu\nu})$$

$$O_{3} = H \operatorname{tr}(D^{\rho}F_{\rho\mu}D_{\sigma}F^{\sigma\mu})$$

$$O_{4} = H \operatorname{tr}(F_{\mu\rho}D^{\rho}D_{\sigma}F^{\sigma\mu})$$

We consider two-loop S-matrix of Higgs plus 3 partons with high dimensional operators.



#### Higgs amplitudes as form factors



Higgs amplitudes are equivalent to form factors:

$$\mathcal{F}_{\mathcal{O}_{i},n} = \int d^{4}x \, e^{-iq \cdot x} \langle p_{1}, p_{2}, p_{3} | \mathcal{O}_{i}(x) | 0 \rangle$$

$$\begin{split} \mathcal{O}_{1} &= \operatorname{tr}(F_{\mu}^{\nu}F_{\nu}^{\rho}F_{\rho}^{\mu}) \\ \mathcal{O}_{2} &= \operatorname{tr}(D_{\rho}F_{\mu\nu}D^{\rho}F^{\mu\nu}) \\ \mathcal{O}_{3} &= \operatorname{tr}(D^{\rho}F_{\rho\mu}D_{\sigma}F^{\sigma\mu}) \\ \mathcal{O}_{4} &= \operatorname{tr}(F_{\mu\rho}D^{\rho}D_{\sigma}F^{\sigma\mu}) \end{split}$$

 $\mathcal{O}_0 = \operatorname{tr}(F_{\mu\nu}F^{\mu\nu})$ 

Operator relation:

$$\mathcal{O}_{2} = \frac{1}{2} \partial^{2} \mathcal{O}_{0} - 4 g_{\mathrm{YM}} \mathcal{O}_{1} + 2 \mathcal{O}_{4} \longrightarrow \mathcal{F}_{\mathcal{O}_{2}} = \frac{1}{2} q^{2} \mathcal{F}_{\mathcal{O}_{0}} - 4 g_{\mathrm{YM}} \mathcal{F}_{\mathcal{O}_{1}} + 2 \mathcal{F}_{\mathcal{O}_{4}}$$
$$D_{\rho} F^{\rho \mu} = -g \sum_{i=1}^{n_{f}} \bar{\psi}_{i} \gamma^{\mu} \psi_{i} \longrightarrow \mathcal{O}_{3} \rightarrow \mathcal{O}_{3}' = g^{2} \sum_{i,j=1}^{n_{f}} (\bar{\psi}_{i} \gamma^{\mu} \psi_{i}) (\bar{\psi}_{j} \gamma_{\mu} \psi_{j})$$
$$\mathcal{O}_{4} \rightarrow \mathcal{O}_{4}' = g F_{\mu \nu} D^{\mu} \sum_{i,j=1}^{n_{f}} (\bar{\psi}_{i} \gamma^{\nu} T^{A} \psi_{i})$$

## Two-loop QCD computation

Our strategy:



### On-shell unitarity method

Unitarity requires that loop amplitudes/form factors have consistent discontinuities by cutting propagators. [Bern, Dixon, Dunbar, Kosower 1994]

[Britto, Cachazo, Feng 2004]

On the cut, the loop quantity factorizes into a product of tree-level or lower-loop results.



The form factors are guarantee to be correct once they satisfy all cut constraints.

#### Integration by part reduction

 $p_2$ 

[Chetyrkin, Tkachov 1981]

$$\int d^D l_1 \dots d^D l_L \ \frac{\partial}{\partial l_i^{\mu}} (\text{integrand}) = 0$$

A set of linear relations between different integrals.

Any integrand = (coefficients) x (Master integrals)

Public packages: Reduze 2, FIRE, LiteRed, etc

### Unitarity + IBP

• Usual strategy (blue color):



### Unitarity + IBP

• Improved strategy (red color):



#### Improved strategy



Advantages:

- no need to reconstruct full integrand
- IBP is simplified
- Different cuts provide self-consistency checks

#### Higgs plus three gluons

All cuts that are needed:





Master integrals are known in terms of 2d Harmonic polylogarithms.



[Gehrmann, Remiddi 2001]

#### Loop structure of form factors

Form factors have divergences:

IR divergences

soft and collinear divergences

UV divergences

renormalization of coupling g and operators O

The IR and UV are mixed in general in a non-trivial way.

General structure of (bare) amplitudes/form factors:

#### full result = IR + UV + finite remainder

#### Loop structure of form factors

In the case of N=4 SYM:



Similar (but a little more complicated) structure is also known for QCD.

## Operator mixing

$$\hat{\mathcal{O}}_I^{\mathrm{R}} = Z_I^{\ J} \hat{\mathcal{O}}_J^{\mathrm{B}}$$

No mixing at one-loop:

$$(Z_{\hat{\mathcal{O}}}^{(1)}) = \frac{1}{\epsilon} \begin{pmatrix} \frac{N_c}{2} + n_f & 0 & 0 & 0\\ 0 & -\beta_0 & 0 & 0\\ 0 & 0 & (Z^{(1)})_3^{-3} & 0\\ 0 & 0 & 0 & \frac{8C_F}{3} + \frac{2n_f}{3} \end{pmatrix}$$

Starting to mix at two-loop:

[Jin, GY 1910.09384]

$$\begin{split} (Z^{(2)})\big|_{\frac{1}{\epsilon^2}\text{-part.}} &= \frac{1}{2} \Big( Z^{(1)} \Big)^2 - \frac{1}{2\epsilon} Z^{(1)} \beta_0 \\ (Z^{(2)}_{\hat{\mathcal{O}}})\big|_{\frac{1}{\epsilon}\text{-part.}} &= \frac{1}{\epsilon} \begin{pmatrix} \frac{25N_c^2}{12} + \frac{5N_c n_f}{12} - \frac{3n_f}{4N_c} & -N_c^2 & (Z^{(2)})_1^3 & \frac{5}{9} + \frac{5N_c^2}{12} \\ 0 & -\beta_1 & 0 & 0 \\ 0 & 0 & (Z^{(2)})_3^3 & n_f \Big( \frac{5N_c}{72} + \frac{1}{18N_c} \Big) + \frac{1}{36} + \frac{7}{72N_c^2} \\ 0 & \left( -\frac{5N_c}{6} + \frac{2}{9N_c} \right) n_f (Z^{(2)})_4^3 & \frac{80N_c^2}{27} - \frac{20}{9} + \frac{7}{27N_c^2} + \left( \frac{25N_c}{27} + \frac{13}{18N_c} \right) n_f \Big) \end{split}$$

We also study one- and two-loop mixing up to operators of dim-14. [Jin, Ren, GY in prep.]

Dim-6 operator basis:  $\hat{\mathcal{O}}_I = \{\mathcal{O}_1, \hat{\mathcal{O}}_2, \mathcal{O}_3, \mathcal{O}_4\}$   $\mathcal{O}_1 = \operatorname{tr}(F_{\mu}^{\ \nu}F_{\nu}^{\ \rho}F_{\rho}^{\ \mu})$   $\hat{\mathcal{O}}_2 \equiv \partial^2 \mathcal{O}_0$   $\mathcal{O}_3 = \operatorname{tr}(D^{\rho}F_{\rho\mu}D_{\sigma}F^{\sigma\mu})$  $\mathcal{O}_4 = \operatorname{tr}(F_{\mu\rho}D^{\rho}D_{\sigma}F^{\sigma\mu})$ 

$$\mathcal{O}_{2} = \frac{1}{2} \partial^{2} \mathcal{O}_{0} - 4 g_{\text{YM}} \mathcal{O}_{1} + 2 \mathcal{O}_{4}$$
$$\mathcal{O}_{2} = \text{tr}(D_{\rho} F_{\mu\nu} D^{\rho} F^{\mu\nu})$$
$$\mathcal{O}_{0} = \text{tr}(F_{\mu\nu} F^{\mu\nu})$$

#### Loop structure of form factors

General structure of (bare) amplitudes/form factors:

full result = universal IR + UV + finite remainder

The non-trivial information is contained in the **finite remainder**:

$$F^{(1)} = I^{(1)}(\epsilon)F^{(0)} + F^{(1),\text{fin}} + \mathcal{O}(\epsilon),$$
  

$$F^{(2)} = I^{(2)}(\epsilon)F^{(0)} + I^{(1)}(\epsilon)F^{(1)} + F^{(2),\text{fin}} + \mathcal{O}(\epsilon)$$

# Outline

- Motivations
- Higgs amplitudes
- Hidden analytic structure
- Explanation and implication

### Higgs and 3-parton results

external particles	$(1^-, 2^-, 3^-)$	$(1^-, 2^-, 3^+)$	$(1^q, 2^{\overline{q}}, 3^-)$
form factors	$\mathcal{F}^{(l)}_{\mathcal{O}_i,lpha}$	$\mathcal{F}_{\mathcal{O}_i,eta}^{(l)}$	$\mathcal{F}^{(l)}_{\mathcal{O}_i,\gamma}$

There are six different color factors:  $\mathscr{R}^{(l)}_{\mathscr{O}} = \mathscr{F}^{(l), \text{fin}}_{\mathscr{O}} / \mathscr{F}^{(0)}_{\mathscr{O}}$ 

$$\mathscr{R}_{\mathscr{O}}^{(2)} = N_{c}^{2} \mathscr{R}_{\mathscr{O}}^{(2),N_{c}^{2}} + N_{c}^{0} \mathscr{R}_{\mathscr{O}}^{(2),N_{c}^{0}} + \frac{1}{N_{c}^{2}} \mathscr{R}_{\mathscr{O}}^{(2),N_{c}^{-2}} + n_{f} N_{c} \mathscr{R}_{\mathscr{O}}^{(2),n_{f}N_{c}} + \frac{n_{f}}{N_{c}} \mathscr{R}_{\mathscr{O}}^{(2),n_{f}/N_{c}} + n_{f}^{2} \mathscr{R}_{\mathscr{O}}^{(2),n_{f}^{2}}$$

$$3\text{-gluon case}$$

#### Finite remainder for $\mathcal{O}_1 \rightarrow ggg$ $\mathcal{O}_1 = tr(F^3)$

#### Organized according to **transcendentality degree**:



#### Degree-4 part:

 $\Omega_{O_2;4}^{(2)} = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{$ 

#### Finite remainder

Simplified results: (using e.g. "symbol")

$$\Omega_{\mathcal{O}_{1};4}^{(2)} = -\frac{3}{2}\mathrm{Li}_{4}(u) + \frac{3}{4}\mathrm{Li}_{4}\left(-\frac{uv}{w}\right) - \frac{3}{2}\log(w)\mathrm{Li}_{3}\left(-\frac{u}{v}\right) + \frac{\zeta_{2}}{8}\left[5\log^{2}(u) - 2\log(v)\log(w)\right] \\ + \frac{\log^{2}(u)}{32}\left[\log^{2}(u) + 2\log^{2}(v) - 4\log(v)\log(w)\right] - \frac{1}{4}\zeta_{4} - \frac{1}{2}\zeta_{3}\log(-q^{2}) + \mathrm{perms}(u, v, w)\right]$$

$$\Omega_{\mathcal{O}_{1};4}^{(2)} = \Omega_{\mathcal{O}_{1};4}^{(2),\mathcal{N}=4}$$

[Q.Jin and GY, PRL 121 101603 (2018)]

[Brandhuber, Kostacinska, Penante, Travaglini 2017]

### Higgs plus 3-parton results

external particles	$(1^-, 2^-, 3^-)$	$(1^-, 2^-, 3^+)$	$(1^q, 2^{\overline{q}}, 3^-)$
form factors	$\mathcal{F}^{(l)}_{\mathcal{O}_i,lpha}$	$\mathcal{F}^{(l)}_{\mathcal{O}_i,eta}$	$\mathcal{F}^{(l)}_{\mathcal{O}_i,\gamma}$

There are six different color factors:  $\mathscr{R}^{(l)}_{\mathscr{O}} = \mathscr{F}^{(l), \text{fin}}_{\mathscr{O}} / \mathscr{F}^{(0)}_{\mathscr{O}}$ 

$$\mathscr{R}_{\mathscr{O}}^{(2)} = N_{c}^{2} \mathscr{R}_{\mathscr{O}}^{(2),N_{c}^{2}} + N_{c}^{0} \mathscr{R}_{\mathscr{O}}^{(2),N_{c}^{0}} + \frac{1}{N_{c}^{2}} \mathscr{R}_{\mathscr{O}}^{(2),N_{c}^{-2}} + n_{f} N_{c} \mathscr{R}_{\mathscr{O}}^{(2),n_{f}N_{c}} + \frac{n_{f}}{N_{c}} \mathscr{R}_{\mathscr{O}}^{(2),n_{f}/N_{c}} + n_{f}^{2} \mathscr{R}_{\mathscr{O}}^{(2),n_{f}^{2}}$$

A different expansion:

 $\mathcal{R}_{\mathcal{O}}^{(2)} = C_{A}^{2} \mathcal{R}_{\mathcal{O}}^{(2), C_{A}^{2}} + C_{A} C_{F} \mathcal{R}_{\mathcal{O}}^{(2), C_{A} C_{F}} + C_{F}^{2} \mathcal{R}_{\mathcal{O}}^{(2), C_{F}^{2}} + n_{f} C_{A} \mathcal{R}_{\mathcal{O}}^{(2), n_{f} C_{A}} + n_{f} C_{F} \mathcal{R}_{\mathcal{O}}^{(2), n_{f} C_{F}} + n_{f}^{2} \mathcal{R}_{\mathcal{O}}^{(2), n_{f}^{2}}$ 

$$C_A = N_c, \qquad C_F = \frac{N_c^2 - 1}{2N_c}$$

 $\mathscr{R}^{(2)}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) = C_A^2 \mathscr{R}^{(2), C_A^2}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_A C_F \mathscr{R}^{(2), C_A C_F}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_F^2 \mathscr{R}^{(2), C_F^2}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_A C_F \mathscr{R}^{(2), C_A C_F}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_F^2 \mathscr{R}^{(2), C_F^2}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_A C_F \mathscr{R}^{(2), C_A C_F}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_A C_F \mathscr{R}^{(2), C_A C_F}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_F \mathscr{R}^{(2), C_A C_F}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_F \mathscr{R}^{(2), C_A C_F}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_A C_F \mathscr{R}^{(2), C_A C_F}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_F \mathscr{R}^{(2), C_A C_$ 

 $\mathcal{R}^{(2),C_{F}^{2}}_{\mathcal{O}_{0};4} = \left(-G(0,z)^{2} + 2G(1,z)G(0,z) + 2G(1-z,y)G(0,z) + 2G(1,y)^{2} + G(1,y)(-4G(1,z) - 4G(1-z,y))\right)G(0,y)^{2}$ 

 $+\left(-\frac{4}{3}G(1,y)^3 - 2G(0,z)G(1,z)^2 + 2G(0,z)^2G(1,z) + \left(2G(0,z)^2 - 4G(0,z)G(1,z)\right)G(1-z,y)\right)G(0,y)$  $-\,12G(1-z,1,1,y)G(0,y)-16G(1-z,1-z,1,y)G(0,y)-G(0,z)^2G(1,z)^2$ 

 $+2G(0,1,y)^{2}-8G(0,1,z)^{2}+(2G(0,z)G(1,z)^{2}-2G(0,z)^{2}G(1,z))G(1-z,y)$  $+G(1,y)\left(G(1-z,y)(8G(1,z)G(-z,y)-4G(0,z)G(1,z))-2G(0,z)G(1,z)^2\right)$ 

 $+ \left(2G(0,y)^2 + (-4G(0,z) + 4G(1,y) - 8G(1,z) - 4G(1-z,y))G(0,y) + 8G(1,z)^2 + 8G(0,z)G(1,z)\right)$ + (4G(0,z) + 4G(1,z))G(1-z,y) + G(1,y)(4G(0,z) + 12G(1,z) + 12G(1-z,y)))G(0,1,z)

 $+ \left(2G(0,z)^2 - 4G(0,y)G(0,z) - 4G(1,y)G(0,z) - 8G(1-z,y)G(0,z) - 8G(1,z)G(-z,y)\right)G(0,1-z,y)$ + 8G(1, y)G(1, z)G(0, -z, y)

 $+G(1-z,1,y)\left(2G(0,y)^{2}+(-4G(0,z)+4G(1,y)+8G(1,z)+8G(1-z,y))G(0,y)+4G(0,z)G(1,z)\right)$ -8G(1,z)G(-z,y) - 12G(0,1,z) - 8G(-z,1-z,y))

 $+ G(0,1,y) \left(4G(1,y)^2 + 4G(0,z)G(1-z,y) + G(0,y)(4G(1,y) + 4G(1,z) + 4G(1-z,y)) - 8G(1,z)G(-z,y) + G(1,z)G(-z,y) + G(1,z)G($ -4G(0,1,z) + 8G(1-z,1,y) - 8G(-z,1-z,y) + G(1,y)(8G(1-z,y) - 8G(1,z))G(-z,1-z,y) + G(1,z)G(1-z,y) + G(1-z,y) + G(1-z,y)G(1-z,y) + G(1-z,y)G(1-z,y)G(1-z,y) + G(1-z,y)G(1-z,y)G(1-z,y) + G(1-z,y)+(-8G(0, y)-16G(1, y))G(0, 0, 1, y)+(4G(0, y)-8G(0, z)-16G(1, y)-12G(1, z)-4G(1-z, y))G(0, 0, 1, z)

+ 4G(0,z)G(0,0,1-z,y) + 8G(1,z)G(0,0,-z,y) + (-8G(0,y) - 8G(1,y))G(0,1,1,y)

+(20G(0, y) - 4G(0, z) - 20G(1, y) - 32G(1, z) - 12G(1 - z, y))G(0, 1, 1, z)

+ (4G(0, y) + 4G(0, z) - 4G(1, y) - 8G(1, z) - 8G(1 - z, y))G(0, 1 - z, 1, y) + 16G(0, z)G(0, 1 - z, 1 - z, y)+(8G(1, y) + 8G(1, z))G(0, -z, 1 - z, y) + (-4G(1, y) - 8G(1, z) - 8G(1 - z, y))G(1 - z, 0, 1, y)

 $+\left(8G(1,z)+8G(1-z,y)\right)G(-z,0,1,y)+8G(1,z)G(-z,0,1-z,y)$ 

+(8G(1,z)+8G(1-z,y))G(-z,1-z,1,y)-16G(1,y)G(-z,1-z,1-z,y)+24G(0,0,0,1,y)

+32G(0,0,0,1,z) + 16G(0,0,1,1,y) + 36G(0,0,1,1,z) - 12G(0,0,1-z,1,y) + 8G(0,0,-z,1-z,y)

 $+ \, 8G(0,1,1,1,y) + 48G(0,1,1,1,z) - \, 8G(0,1,1-z,1,y) - \, 8G(0,1-z,0,1,y) - \, 4G(0,1-z,1,1,y) \\$ 

+ 16G(0, 1 - z, 1 - z, 1, y) - 8G(0, 1 - z, -z, 1 - z, y) - 4G(1 - z, 0, 0, 1, y) - 4G(1 - z, 0, 1, y) - 4G(1 - z, 0, 1, y) - 4G(1 - z, 0, 1, y) - 4+16G(1-z, 0, 1-z, 1, y) + 16G(1-z, 1-z, 0, 1, y) - 8G(1-z, -z, 0, 1, y) - 8G(1-z, -z, 1-z, 1, y) - 8G(1-z, -z, 1-z, 1, y) - 8G(1-z, -z, 1, y) -

 $+ (2G(0, y)^{2} + (12G(0, z) - 8G(1, y) - 8G(1, z) - 4G(1 - z, y))G(0, y) + 2G(0, z)^{2} + 4G(1, y)^{2} + 8G(1, z)^{2} + 8G($ 

 $-12G(0,z)G(1,z) + (4G(1,z) - 4G(0,z))G(1-z,y) + G(1,y)(-4G(0,z) + 4G(1,z) + 4G(1,z,y)))\zeta(2)$  $+ 69\zeta(4) + (-16G(0, y) - 16G(0, z) + 12G(1, y) + 44G(1, z) + 32G(1 - z, y))\zeta(3).$ 

 $\mathcal{R}^{(2),C_AC_F}_{\mathcal{O}_{0;4}} = -\frac{4}{3}G(0,z)G(1,z)^3 + 3G(0,z)^2G(1,z)^2 - 8G(0,0,-z,y)G(1,z) - 10G(-z,0,1-z,y)G(1,z) - 1$ 

 $-2G(0, 1, y)^{2} + 11G(0, 1, z)^{2} + (2G(0, z)^{2}G(1, z) - 2G(0, z)G(1, z)^{2})G(1 - z, y)$ 

 $+ G(0,y)^2 \left( 3G(0,z)^2 - 2G(1,z)G(0,z) - 2G(1-z,y)G(0,z) + G(1,y) (-4G(0,z) + 4G(1,z) + 4G(1-z,y)) \right)$ +  $G(0, y) \left(-6G(1, z)G(0, z)^{2} + 3G(1, z)^{2}G(0, z) + (4G(0, z)G(1, z) - 2G(0, z)^{2})G(1 - z, y)\right)$ 

+ G(1, y)(2G(0, z)G(1, z) + 2G(0, z)G(1 - z, y)))+ G(1, y) (3 $G(0, z)G(1, z)^{2}$  + G(1 - z, y)(6G(0, z)G(1, z) - 10G(1, z)G(-z, y)))

+  $(-2G(0, y)^{2} + (8G(0, z) - 4G(1, y) + 10G(1, z) + 4G(1 - z, y))G(0, y) - 5G(1, z)^{2} - 8G(0, z)G(1, z)$ +G(1, y)(-6G(0, z) - 16G(1, z) - 16G(1 - z, y)) + (-4G(0, z) - 4G(1, z))G(1 - z, y))G(0, 1, z)

 $+(-2G(0,z)^{2}+2G(0,y)G(0,z)+6G(1,y)G(0,z)+8G(1-z,y)G(0,z)+10G(1,z)G(-z,y)$ 

 $+ 2G(0,1,z) \Big) G(0,1-z,y) + (2G(0,y)G(1,z) - 2G(0,z)G(1,z) - 10G(1,y)G(1,z))G(0,-z,y) \\$ +G(1,y)(10G(1,z)-10G(1-z,y))G(-z,1-z,y)

 $+G(1-z,1,y)(-2G(0,y)^2+(2G(0,z)-4G(1,y)-8G(1,z)-8G(1-z,y))G(0,y)-6G(0,z)G(1,z))$ + 10G(1,z)G(-z,y) + 16G(0,1,z) + 10G(-z,1-z,y))

+G(0,1,y)(-2G(0,z)G(1,z)+10G(-z,y)G(1,z)+G(0,y)(6G(0,z)-4G(1,y)-6G(1,z)-6G(1-z,y))-8G(0,z)G(1-z,y) + 4G(0,1,z) - 8G(1-z,1,y) + 10G(-z,1-z,y))

+ (4G(0, y) - 4G(0, z) + 8G(1, y) + 4G(1, z) + 4G(1 - z, y))G(0, 0, 1, y)

+(-10G(0, y) + 6G(0, z) + 22G(1, y) + 2G(1, z) + 4G(1 - z, y))G(0, 0, 1, z) - 4G(0, z)G(0, 0, 1 - z, y) $+ \, 8G(0,y)G(0,1,1,y) + (-26G(0,y) + 26G(1,y) + 28G(1,z) + 12G(1-z,y))G(0,1,1,z) \\$ 

+(-4G(0, y) - 2G(0, z) + 4G(1, y) + 8G(1, z) + 8G(1 - z, y))G(0, 1 - z, 1, y)

 $-\ 16G(0,z)G(0,1-z,1-z,y) + (2G(0,y)-2G(0,z)-10G(1,y)-10G(1,z))G(0,-z,1-z,y) \\ - (2G(0,z)-2G(0,z)-10G(1,y)-10G(1,z))G(0,-z,1-z,y) \\ - (2G(0,z)-2G(0,z)-10G(1,z))G(0,-z,1-z,y) \\ - (2G(0,z)-2G(1,z))G(0,-z,1-z,y) \\ - (2G(0,z)-2G(1,z))G(0,-z,y) \\ - (2G(0,z)-2G(1,z))G(0,-z,y) \\ - (2G(0,z)-2G(1,z))G(0,-z,y) \\ - (2G(0,z)-2G(1,z))G(0,z) \\ - (2G(0,z)-2G(1,z))G(1,z)G(1,z))G(1,z)G$ +(2G(0, y) + 4G(0, z) + 4G(1, y) + 8G(1, z) + 8G(1 - z, y))G(1 - z, 0, 1, y) + 12G(0, y)G(1 - z, 1, 1, y)

+ 16G(0, y)G(1 - z, 1 - z, 1, y) + (-10G(1, z) - 10G(1 - z, y))G(-z, 0, 1, y)

 $+ \ (-10G(1,z) - 10G(1-z,y))G(-z,1-z,1,y) + 20G(1,y)G(-z,1-z,1-z,y)$ 

-12G(0, 0, 0, 1, y) - 20G(0, 0, 0, 1, z) - 8G(0, 0, 1, 1, y) - 28G(0, 0, 1, 1, z) + 12G(0, 0, 1 - z, 1, y)

-8G(0,0,-z,1-z,y) - 46G(0,1,1,1,z) + 8G(0,1,1-z,1,y) + 6G(0,1-z,0,1,y)

+4G(0, 1-z, 1, 1, y) - 16G(0, 1-z, 1-z, 1, y) + 10G(0, 1-z, -z, 1-z, y) + 4G(1-z, 0, 1, 1, y)-16G(1-z, 0, 1-z, 1, y) - 16G(1-z, 1-z, 0, 1, y) + 10G(1-z, -z, 0, 1, y) + 10G(1-z, -z, 1-z, 1, y)

 $+(-2G(0, y)^{2}+(-8G(0, z)+2G(1, y)+10G(1, z)+4G(1-z, y))G(0, y)-2G(0, z)^{2}-5G(1, z)^{2}$ 

+ 6G(0,z)G(1,z) + G(1,y)(6G(0,z) - 6G(1,z) - 6G(1-z,y)) + (4G(0,z) - 4G(1,z))G(1-z,y) $-2G(0, 1, y) + 2G(0, 1 - z, y) + 2G(1 - z, 1, y)) \zeta(2)$ 

 $-\frac{311\zeta(4)}{4} + (22G(0, y) + 22G(0, z) - 18G(1, y) - 52G(1, z) - 34G(1 - z, y))\zeta(3) + (22G(0, z) - 18G(1, z) - 52G(1, z) - 34G(1 - z, y))\zeta(3) + (22G(0, z) - 18G(1, z) - 52G(1, z) - 34G(1 - z, y))\zeta(3) + (22G(0, z) - 18G(1, z) - 52G(1, z) - 34G(1 - z, y))\zeta(3) + (22G(0, z) - 18G(1, z) - 52G(1, z) - 34G(1 - z, y))\zeta(3) + (22G(0, z) - 18G(1, z) - 52G(1, z) - 34G(1 - z, y))\zeta(3) + (22G(0, z) - 18G(1, z) - 52G(1, z) - 34G(1 - z, y))\zeta(3) + (22G(0, z) - 18G(1, z) - 52G(1, z) - 34G(1 - z, y))\zeta(3) + (22G(0, z) - 52G(1, z) - 52$ 

 $\mathcal{R}^{(2),C_{A}}_{\mathcal{O}_{0};4} = \frac{4}{3}G(0,z)G(1,z)^{3} - \frac{3}{2}G(0,z)^{2}G(1,z)^{2} + 2G(-z,y)^{2}G(1,z)^{2} - 2G(0,z)G(-z,y)G(1,z)^{2} - \frac{4}{3}\sum_{z,y}y^{3}G(1,z) + 2G(-z,y)^{2}G(1,z)^{2} - 2G(0,z)G(-z,y)G(1,z)^{2} - 2G(0,z)G(-z,y)G(1,z)^{2} - 2G(-z,y)^{2}G(1,z)^{2} - 2G(-z,y)G(-z,y)G(1,z)^{2} - 2G(-z,y)G($  $-4G(0, 0, -z, y)G(1, z) + 2G(-z, 0, 1 - z, y)G(1, z) - 3G(0, 1, z)^{2} + G(0, y)^{2} \left(-\frac{3}{2}G(0, z)^{2} + G(1, z)G(0, z)\right) + G(1, z)G(0, z) - G(1, z)G(0, z)$  $+\frac{1}{2}G(1,z)^{2}+\frac{1}{2}G(1-z,y)^{2}+G(1,y)(2G(0,z)-2G(1,z)-2G(1-z,y))+(G(0,z)+G(1,z))G(1-z,y))$ 

+  $G(1 - z, y)^2 \left( \frac{1}{2} G(0, z)^2 + 2G(1, z)G(0, z) - 2G(1, z)G(-z, y) \right)$ 

 $+G(1-z,y)\left(-G(1,z)G(0,z)^{2}+4G(1,z)^{2}G(0,z)-2G(1,z)G(-z,y)G(0,z)+4G(1,z)G(-z,y)^{2}\right)$  $+G(0,y)\left(3G(1,z)G(0,z)^2-2G(1,z)^2G(0,z)-G(1-z,y)^2G(0,z)+\left(2G(0,z)G(1,z)-2G(1,z)^2\right)G(-z,y)\right)$ +  $G(1 - z, y) \left(G(0, z)^2 - 4G(1, z)G(0, z) - 2G(1, z)G(-z, y)\right)$ 

 $+ G(1, y) \left( G(1 - z, y)(2G(1, z)G(-z, y) - 2G(0, z)G(1, z)) - G(0, z)G(1, z)^2 \right)$  $+(-3G(1,z)^2+8G(-z,y)G(1,z)-4G(1-z,y)^2-4G(-z,y)^2+G(0,y)(-4G(0,z)-2G(1,z))$ 

+G(1,y)(2G(0,z)+4G(1,z)+4G(1-z,y))+G(1-z,y)(8G(-z,y)-8G(1,z)))G(0,1,z) $+ \left(-2G(0,y)G(0,z) - 2G(1,y)G(0,z) + 4G(1,z)G(0,z) - 2G(1,z)G(-z,y) - 2G(0,1,z)\right)G(0,1-z,y) \\$ +(2G(0, y)G(1, z) - 2G(0, z)G(1, z) + 2G(1, y)G(1, z))G(0, -z, y) + G(0, 1, y)(2G(0, z)G(1 - z, y))

+G(0,y)(-4G(0,z)+4G(1,z)+4G(1-z,y))-2G(1,z)G(-z,y)-2G(-z,1-z,y))+G(1-z,1,y)(2G(0,z)G(1,z)-2G(-z,y)G(1,z)+G(0,y)(-2G(0,z)+4G(1,z)+4G(1-z,y))

 $-4G(0,1,z) - 2G(-z,1-z,y)) + (-2G(1-z,y)^2 + (4G(1,z) - 2G(0,z))G(1-z,y) - 2G(0,z)G(1,z))$ +G(0,y)(2G(0,z)-2G(1,z)-2G(1-z,y))+G(1,y)(2G(1-z,y)-2G(1,z)))G(-z,1-z,y)

+ (4G(0,z) - 4G(1,z) - 4G(1-z,y))G(0,0,1,y)+(6G(0, y) + 2G(0, z) - 6G(1, y) + 10G(1, z) + 8G(1 - z, y) - 8G(-z, y))G(0, 0, 1, z) $+ \, 4G(0,z)G(0,0,1-z,y) + (6G(0,y) + 4G(0,z) - 6G(1,y) + 4G(1,z) + 8G(1-z,y) - 8G(-z,y))G(0,1,1,z) \\ - 6G(1,y) + 6G(1,z) + 6$ +(2G(0,z)-4G(1,z)-4G(1-z,y))G(0,1-z,1,y)+4G(0,z)G(0,1-z,1-z,y)+(2G(0, y) - 2G(0, z) + 2G(1, y) + 2G(1, z))G(0, -z, 1 - z, y)

+ (-2G(0,y) - 4G(1,z) - 4G(1-z,y))G(1-z,0,1,y) - 4G(0,y)G(1-z,1-z,1,y)+(2G(1,z)+2G(1-z,y))G(-z,0,1,y)+(2G(1,z)+2G(1-z,y))G(-z,1-z,1,y) $+ \left(-4G(1, y) - 4G(1, z) + 8G(1 - z, y)\right)G(-z, 1 - z, 1 - z, y) + 8G(1 - z, y)G(-z, -z, 1 - z, y)$ -12G(0, 0, 0, 1, z) - 8G(0, 0, 1, 1, z) - 4G(0, 0, -z, 1-z, y) - 2G(0, 1, 1, 1, z) + 2G(0, 1-z, 0, 1, y)+4G(0, 1-z, 1-z, 1, y) - 2G(0, 1-z, -z, 1-z, y) + 4G(1-z, 0, 0, 1, y) + 4G(1-z, 0, 1-z, 1, y) $+ \ 4G(1-z,1-z,0,1,y) - 2G(1-z,-z,0,1,y) - 2G(1-z,-z,1-z,1,y) - 12G(-z,1-z,1-z,1-z,y) - 2G(1-z,-z,1-z,1-z,y) - 2G(1-z,-z,1-z,y) - 2G(1-z,-z,1-z,1-z,y) - 2G(1-z,-z,1-z,y) - 2G(1-z,-z,y) - 2G(1-z, -8G(-z,-z,1-z,1-z,y) - 8G(-z,-z,-z,1-z,y) + \left(-3G(1,z)^2 + 3G(0,z)G(1,z) + 2G(-z,y)G(1,z) + 2G(-z,y)G(1,z)$ 

 $-4G(1-z,y)^{2} + (3G(0,z) - 8G(1,z))G(1-z,y) + G(1,y)(-2G(0,z) + 2G(1,z) + 2G(1-z,y))$  $+ \, G(0,y) (-G(0,z) + G(1,z) + 3G(1-z,y)) - 2G(0,1-z,y) - 2G(1-z,1,y) + 2G(-z,1-z,y) \big) \, \zeta(2)$  $+ \frac{119\zeta(4)}{2} + \left(-7G(0,y) - 7G(0,z) + 6G(1,y) + 7G(1,z) + G(1-z,y)\right)\zeta(3),$ 

#### Multiple polyLogarithm

G(1-z, -z, 1-z, 1, y)

 $\mathscr{R}^{(2)}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) = C_A^2 \mathscr{R}^{(2), C_A^2}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_A C_F \mathscr{R}^{(2), C_A C_F}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_F^2 \mathscr{R}^{(2), C_F^2}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_A C_F \mathscr{R}^{(2), C_A C_F}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_F^2 \mathscr{R}^{(2), C_F^2}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}) + C_A C_F \mathscr{R}^{(2), C_A C_F}_{\mathrm{tr}(F^2);4}(1^q, 2^{\bar{q}}, 3^{\pm}$ 

+G(1, y)(2G(0, z)G(1, z) + 2G(0, z)G(1 - z, y)))

 $\mathcal{R}^{(2),C_F^2}_{\mathcal{O}_0,4} = \left(-G(0,z)^2 + 2G(1,z)G(0,z) + 2G(1-z,y)G(0,z) + 2G(1,y)^2 + G(1,y)(-4G(1,z) - 4G(1-z,y))\right)G(0,y)^2$ 

 $+ \left(2G(0,y)^2 + (-4G(0,z) + 4G(1,y) - 8G(1,z) - 4G(1-z,y))G(0,y) + 8G(1,z)^2 + 8G(0,z)G(1,z)\right)$ 

 $+ \left(2G(0,z)^2 - 4G(0,y)G(0,z) - 4G(1,y)G(0,z) - 8G(1-z,y)G(0,z) - 8G(1,z)G(-z,y)\right)G(0,1-z,y)$ 

+ (4G(0,z) + 4G(1,z))G(1-z,y) + G(1,y)(4G(0,z) + 12G(1,z) + 12G(1-z,y)))G(0,1,z)

-4G(0,1,z) + 8G(1-z,1,y) - 8G(-z,1-z,y) + G(1,y)(8G(1-z,y) - 8G(1,z))G(-z,1-z,y) + G(1,z)G(1-z,y) + G(1-z,y) + G(1-z,y)G(1-z,y) + G(1-z,y)G(1-z,y)G(1-z,y) + G(1-z,y)G(1-z,y)G(1-z,y) + G(1-z,y)

+(-8G(0, y)-16G(1, y))G(0, 0, 1, y)+(4G(0, y)-8G(0, z)-16G(1, y)-12G(1, z)-4G(1-z, y))G(0, 0, 1, z)

+ (4G(0, y) + 4G(0, z) - 4G(1, y) - 8G(1, z) - 8G(1 - z, y))G(0, 1 - z, 1, y) + 16G(0, z)G(0, 1 - z, 1 - z, y)

+(8G(1, y) + 8G(1, z))G(0, -z, 1 - z, y) + (-4G(1, y) - 8G(1, z) - 8G(1 - z, y))G(1 - z, 0, 1, y)

+(8G(1,z)+8G(1-z,y))G(-z,1-z,1,y)-16G(1,y)G(-z,1-z,1-z,y)+24G(0,0,0,1,y)

+32G(0,0,0,1,z)+16G(0,0,1,1,y)+36G(0,0,1,1,z)-12G(0,0,1-z,1,y)+8G(0,0,-z,1-z,y)

 $+ \, 8G(0,1,1,1,y) + 48G(0,1,1,1,z) - \, 8G(0,1,1-z,1,y) - \, 8G(0,1-z,0,1,y) - \, 4G(0,1-z,1,1,y) \\$ 

+16G(1-z, 0, 1-z, 1, y) + 16G(1-z, 1-z, 0, 1, y) - 8G(1-z, -z, 0, 1, y) - 8G(1-z, -z, 1-z, 1, y)

+  $(2G(0, y)^2 + (12G(0, z) - 8G(1, y) - 8G(1, z) - 4G(1 - z, y))G(0, y) + 2G(0, z)^2 + 4G(1, y)^2 + 8G(1, z)^2 + 8G(1, z$ 

 $-12G(0,z)G(1,z) + (4G(1,z) - 4G(0,z))G(1-z,y) + G(1,y)(-4G(0,z) + 4G(1,z) + 4G(1,z,y)))\zeta(2)$ 

+16G(0, 1-z, 1-z, 1, y) - 8G(0, 1-z, -z, 1-z, y) - 4G(1-z, 0, 0, 1, y) - 4G(1-z, 0, 1, 1, y)

 $+ 69\zeta(4) + (-16G(0, y) - 16G(0, z) + 12G(1, y) + 44G(1, z) + 32G(1 - z, y))\zeta(3).$ 

+ 4G(0,z)G(0,0,1-z,y) + 8G(1,z)G(0,0,-z,y) + (-8G(0,y) - 8G(1,y))G(0,1,1,y)

+(20G(0, y) - 4G(0, z) - 20G(1, y) - 32G(1, z) - 12G(1 - z, y))G(0, 1, 1, z)

+(8G(1,z)+8G(1-z,y))G(-z,0,1,y)+8G(1,z)G(-z,0,1-z,y)

-8G(1,z)G(-z,y) - 12G(0,1,z) - 8G(-z,1-z,y))

 $+\left(-\frac{4}{3}G(1,y)^3 - 2G(0,z)G(1,z)^2 + 2G(0,z)^2G(1,z) + \left(2G(0,z)^2 - 4G(0,z)G(1,z)\right)G(1-z,y)\right)G(0,y)$ 

 $-\,12G(1-z,1,1,y)G(0,y)-16G(1-z,1-z,1,y)G(0,y)-G(0,z)^2G(1,z)^2$ 

 $+2G(0,1,y)^{2}-8G(0,1,z)^{2}+(2G(0,z)G(1,z)^{2}-2G(0,z)^{2}G(1,z))G(1-z,y)$ 

 $+G(1,y)\left(G(1-z,y)(8G(1,z)G(-z,y)-4G(0,z)G(1,z))-2G(0,z)G(1,z)^2\right)$ 

+ 8G(1, y)G(1, z)G(0, -z, y)

 $-2G(0, 1, y)^{2} + 11G(0, 1, z)^{2} + (2G(0, z)^{2}G(1, z) - 2G(0, z)G(1, z)^{2})G(1 - z, y)$ 

 $\mathcal{R}^{(2),C_AC_F}_{\mathcal{O}_{0;4}} = -\frac{4}{2}G(0,z)G(1,z)^3 + 3G(0,z)^2G(1,z)^2 - 8G(0,0,-z,y)G(1,z) - 10G(-z,0,1-z,y)G(1,z) - 1$ 

+ G(1, y) (3 $G(0, z)G(1, z)^{2}$  + G(1 - z, y)(6G(0, z)G(1, z) - 10G(1, z)G(-z, y)))

+G(1,y)(10G(1,z)-10G(1-z,y))G(-z,1-z,y)

 $\mathcal{R}^{(2),C_A^2}_{\mathcal{O}_0;4} = \frac{4}{2}G(0,z)G(1,z)^3 - \frac{3}{2}G(0,z)^2G(1,z)^2 + 2G(-z,y)^2G(1,z)^2 - 2G(0,z)G(-z,y)G(1,z)^2 - \frac{4}{2}G(1,z)^2 - \frac{4}{2}G(1$  $-4G(0, 0, -z, y)G(1, z) + 2G(-z, 0, 1 - z, y)G(1, z) - 3G(0, 1, z)^{2} + G(0, y)^{2} \left(-\frac{3}{2}G(0, z)^{2} + G(1, z)G(0, z)\right) + G(1, z)G(0, z) - G(1, z)G(0, z)$  $+\frac{1}{2}G(1,z)^{2}+\frac{1}{2}G(1-z,y)^{2}+G(1,y)(2G(0,z)-2G(1,z)-2G(1-z,y))+(G(0,z)+G(1,z))G(1-z,y))$ 

+  $G(1 - z, y)^2 \left(\frac{1}{2}G(0, z)^2 + 2G(1, z)G(0, z) - 2G(1, z)G(-z, y)\right)$ 

 $+ G(1-z,y) \left(-G(1,z)G(0,z)^2 + 4G(1,z)^2 G(0,z) - 2G(1,z)G(-z,y)G(0,z) + 4G(1,z)G(-z,y)^2\right)$  $+G(0,y)\left(3G(1,z)G(0,z)^{2}-2G(1,z)^{2}G(0,z)-G(1-z,y)^{2}G(0,z)+\left(2G(0,z)G(1,z)-2G(1,z)^{2}\right)G(-z,y)\right)$ +  $G(1 - z, y) \left(G(0, z)^2 - 4G(1, z)G(0, z) - 2G(1, z)G(-z, y)\right)$ 

 $+G(1,y)(G(1-z,y)(2G(1,z)G(-z,y)-2G(0,z)G(1,z))-G(0,z)G(1,z)^{2})$  $+(-3G(1,z)^2+8G(-z,y)G(1,z)-4G(1-z,y)^2-4G(-z,y)^2+G(0,y)(-4G(0,z)-2G(1,z))$ +G(1,y)(2G(0,z)+4G(1,z)+4G(1-z,y))+G(1-z,y)(8G(-z,y)-8G(1,z)))G(0,1,z)

 $+ \left(-2G(0,y)G(0,z) - 2G(1,y)G(0,z) + 4G(1,z)G(0,z) - 2G(1,z)G(-z,y) - 2G(0,1,z)\right)G(0,1-z,y) \\$ +(2G(0, y)G(1, z) - 2G(0, z)G(1, z) + 2G(1, y)G(1, z))G(0, -z, y) + G(0, 1, y)(2G(0, z)G(1 - z, y))+G(0,y)(-4G(0,z)+4G(1,z)+4G(1-z,y))-2G(1,z)G(-z,y)-2G(-z,1-z,y))

+G(1-z,1,y)(2G(0,z)G(1,z)-2G(-z,y)G(1,z)+G(0,y)(-2G(0,z)+4G(1,z)+4G(1-z,y)) $-4G(0,1,z) - 2G(-z,1-z,y)) + (-2G(1-z,y)^2 + (4G(1,z) - 2G(0,z))G(1-z,y) - 2G(0,z)G(1,z))$ +G(0,y)(2G(0,z)-2G(1,z)-2G(1-z,y))+G(1,y)(2G(1-z,y)-2G(1,z)))G(-z,1-z,y)

+(4G(0,z)-4G(1,z)-4G(1-z,y))G(0,0,1,y)+(6G(0, y) + 2G(0, z) - 6G(1, y) + 10G(1, z) + 8G(1 - z, y) - 8G(-z, y))G(0, 0, 1, z) $+ \, 4G(0,z)G(0,0,1-z,y) + (6G(0,y) + 4G(0,z) - 6G(1,y) + 4G(1,z) + 8G(1-z,y) - 8G(-z,y))G(0,1,1,z) \\ - 6G(1,y) + 6G(1,z) + 6$ 

+(2G(0,z)-4G(1,z)-4G(1-z,y))G(0,1-z,1,y)+4G(0,z)G(0,1-z,1-z,y)+(2G(0, y) - 2G(0, z) + 2G(1, y) + 2G(1, z))G(0, -z, 1 - z, y)+(-2G(0,y)-4G(1,z)-4G(1-z,y))G(1-z,0,1,y)-4G(0,y)G(1-z,1-z,1,y)

+(2G(1,z)+2G(1-z,y))G(-z,0,1,y)+(2G(1,z)+2G(1-z,y))G(-z,1-z,1,y)+(-4G(1, y) - 4G(1, z) + 8G(1 - z, y))G(-z, 1 - z, 1 - z, y) + 8G(1 - z, y)G(-z, -z, 1 - z, y)-12G(0, 0, 0, 1, z) - 8G(0, 0, 1, 1, z) - 4G(0, 0, -z, 1 - z, y) - 2G(0, 1, 1, 1, z) + 2G(0, 1 - z, 0, 1, y)+4G(0, 1-z, 1-z, 1, y) - 2G(0, 1-z, -z, 1-z, y) + 4G(1-z, 0, 0, 1, y) + 4G(1-z, 0, 1-z, 1, y) $+ \ 4G(1-z,1-z,0,1,y) - 2G(1-z,-z,0,1,y) - 2G(1-z,-z,1-z,1,y) - 12G(-z,1-z,1-z,1-z,y) - 2G(1-z,-z,1-z,1-z,y) - 2G(1-z,-z,1-z,y) - 2G(1-z,-z,1-z,1-z,y) - 2G(1-z,-z,1-z,y) - 2G(1-z,-z,y) - 2G(1-z, -8G(-z,-z,1-z,1-z,y)-8G(-z,-z,-z,1-z,y)+\left(-3G(1,z)^2+3G(0,z)G(1,z)+2G(-z,y)G(1,$  $-4G(1-z,y)^{2} + (3G(0,z) - 8G(1,z))G(1-z,y) + G(1,y)(-2G(0,z) + 2G(1,z) + 2G(1-z,y))$ 

 $+ \, G(0,y) (-G(0,z) + G(1,z) + 3G(1-z,y)) - 2G(0,1-z,y) - 2G(1-z,1,y) + 2G(-z,1-z,y) \big) \, \zeta(2)$  $+ \frac{119\zeta(4)}{2} + \left(-7G(0,y) - 7G(0,z) + 6G(1,y) + 7G(1,z) + G(1-z,y)\right)\zeta(3),$ 

(not obvious in Nc expansion!)

 $+G(1-z,1,y)\left(2G(0,y)^{2}+(-4G(0,z)+4G(1,y)+8G(1,z)+8G(1-z,y))G(0,y)+4G(0,z)G(1,z)\right)$  $+G(1-z,1,y)(-2G(0,y)^2+(2G(0,z)-4G(1,y)-8G(1,z)-8G(1-z,y))G(0,y)-6G(0,z)G(1,z))$ + 10G(1,z)G(-z,y) + 16G(0,1,z) + 10G(-z,1-z,y)) $+G(0,1,y)\left(4G(1,y)^{2}+4G(0,z)G(1-z,y)+G(0,y)(4G(1,y)+4G(1,z)+4G(1-z,y))-8G(1,z)G(-z,y)+G(1,z)G(-z$ 

+G(0,1,y)(-2G(0,z)G(1,z)+10G(-z,y)G(1,z)+G(0,y)(6G(0,z)-4G(1,y)-6G(1,z)-6G(1-z,y))-8G(0,z)G(1-z,y) + 4G(0,1,z) - 8G(1-z,1,y) + 10G(-z,1-z,y))

 $+ G(0,y)^2 \left( 3G(0,z)^2 - 2G(1,z)G(0,z) - 2G(1-z,y)G(0,z) + G(1,y) (-4G(0,z) + 4G(1,z) + 4G(1-z,y)) \right)$ 

+  $(-2G(0, y)^{2} + (8G(0, z) - 4G(1, y) + 10G(1, z) + 4G(1 - z, y))G(0, y) - 5G(1, z)^{2} - 8G(0, z)G(1, z)$ 

+G(1,y)(-6G(0,z)-16G(1,z)-16G(1-z,y))+(-4G(0,z)-4G(1,z))G(1-z,y))G(0,1,z)

 $+ \, 2G(0,1,z) \big) \, G(0,1-z,y) + (2G(0,y)G(1,z) - 2G(0,z)G(1,z) - 10G(1,y)G(1,z))G(0,-z,y) \\$ 

+  $G(0, y) \left(-6G(1, z)G(0, z)^{2} + 3G(1, z)^{2}G(0, z) + (4G(0, z)G(1, z) - 2G(0, z)^{2})G(1 - z, y)\right)$ 

 $+(-2G(0,z)^{2}+2G(0,y)G(0,z)+6G(1,y)G(0,z)+8G(1-z,y)G(0,z)+10G(1,z)G(-z,y)$ 

+ (4G(0, y) - 4G(0, z) + 8G(1, y) + 4G(1, z) + 4G(1 - z, y))G(0, 0, 1, y)

+(-10G(0, y) + 6G(0, z) + 22G(1, y) + 2G(1, z) + 4G(1 - z, y))G(0, 0, 1, z) - 4G(0, z)G(0, 0, 1 - z, y)

 $+ \, 8G(0,y)G(0,1,1,y) + (-26G(0,y) + 26G(1,y) + 28G(1,z) + 12G(1-z,y))G(0,1,1,z) \\$ 

+(-4G(0, y) - 2G(0, z) + 4G(1, y) + 8G(1, z) + 8G(1 - z, y))G(0, 1 - z, 1, y)

-16G(0,z)G(0,1-z,1-z,y) + (2G(0,y) - 2G(0,z) - 10G(1,y) - 10G(1,z))G(0,-z,1-z,y)+(2G(0, y) + 4G(0, z) + 4G(1, y) + 8G(1, z) + 8G(1 - z, y))G(1 - z, 0, 1, y) + 12G(0, y)G(1 - z, 1, 1, y)

+ 16G(0, y)G(1 - z, 1 - z, 1, y) + (-10G(1, z) - 10G(1 - z, y))G(-z, 0, 1, y)

+(-10G(1,z)-10G(1-z,y))G(-z,1-z,1,y)+20G(1,y)G(-z,1-z,1-z,y)

-12G(0, 0, 0, 1, y) - 20G(0, 0, 0, 1, z) - 8G(0, 0, 1, 1, y) - 28G(0, 0, 1, 1, z) + 12G(0, 0, 1 - z, 1, y)

-8G(0,0,-z,1-z,y) - 46G(0,1,1,1,z) + 8G(0,1,1-z,1,y) + 6G(0,1-z,0,1,y)

+4G(0, 1-z, 1, 1, y) - 16G(0, 1-z, 1-z, 1, y) + 10G(0, 1-z, -z, 1-z, y) + 4G(1-z, 0, 1, 1, y)-16G(1-z, 0, 1-z, 1, y) - 16G(1-z, 1-z, 0, 1, y) + 10G(1-z, -z, 0, 1, y) + 10G(1-z, -z, 1-z, 1, y)

 $+(-2G(0, y)^{2}+(-8G(0, z)+2G(1, y)+10G(1, z)+4G(1-z, y))G(0, y)-2G(0, z)^{2}-5G(1, z)^{2}$ + 6G(0, z)G(1, z) + G(1, y)(6G(0, z) - 6G(1, z) - 6G(1 - z, y)) + (4G(0, z) - 4G(1, z))G(1 - z, y) $-2G(0, 1, y) + 2G(0, 1 - z, y) + 2G(1 - z, 1, y)) \zeta(2)$ 

 $-\frac{311\zeta(4)}{4} + (22G(0, y) + 22G(0, z) - 18G(1, y) - 52G(1, z) - 34G(1 - z, y))\zeta(3)$ 

Sum three terms together we get the 3-gluon result !

#### Multiple polyLogarithm

G(1-z, -z, 1-z, 1, y)

#### $\mathscr{R}^{(2),C_A^2}_{\operatorname{tr}(F^2):4}(1^q,2^{\bar{q}},3^{\pm}) + \mathscr{R}^{(2),C_AC_F}_{\operatorname{tr}(F^2):4}(1^q,2^{\bar{q}},3^{\pm}) + \mathscr{R}^{(2),C_F^2}_{\operatorname{tr}(F^2):4}(1^q,2^{\bar{q}},3^{\pm}) = \mathscr{R}^{(2)}_{\operatorname{tr}(F^2):4}(1^-,2^-,3^{\pm})$

$$\mathcal{R}_{tr(F^{2});4}^{(2)}(1^{-},2^{-},3^{\pm}) = -2\left[J_{4}\left(-\frac{uv}{w}\right) + J_{4}\left(-\frac{vw}{u}\right) + J_{4}\left(-\frac{wu}{v}\right)\right] - 8\sum_{i=1}^{3}\left[\operatorname{Li}_{4}\left(1-\frac{1}{u_{i}}\right) + \frac{\log^{4}u_{i}}{4!}\right] - 2\left[\sum_{i=1}^{3}\operatorname{Li}_{2}\left(1-\frac{1}{u_{i}}\right)\right]^{2} + \frac{1}{2}\left[\sum_{i=1}^{3}\log^{2}u_{i}\right]^{2} + 2(J_{2}^{2} - \zeta_{2}J_{2}) - \frac{\log^{4}(uvw)}{4!} - \zeta_{3}\log(uvw) - \frac{123}{8}\zeta_{4}$$

For length-3 operator, the correspondence is the same  $! \begin{array}{l} \mathcal{O}_4 = \operatorname{tr}(F_{\mu\rho}D^{\rho}D_{\sigma}F^{\sigma\mu}) \\ \rightarrow \mathcal{O}'_4 = gF_{\mu\nu}D^{\mu}\sum_{i=1}^{n_f}(\bar{\psi}_i\gamma^{\nu}T^A\psi_i) \end{array}$ 

$$\mathcal{R}^{(2)}_{\mathcal{O}_{4};4}(1^{q}, 2^{\bar{q}}, 3^{\pm}) = C^{2}_{A}\mathcal{R}^{(2), C^{2}_{A}}_{\mathcal{O}_{4};4}(1^{q}, 2^{\bar{q}}, 3^{\pm}) + C_{A}C_{F}\mathcal{R}^{(2), C_{A}C_{F}}_{\mathcal{O}_{4};4}(1^{q}, 2^{\bar{q}}, 3^{\pm}) + C^{2}_{F}\mathcal{R}^{(2), C^{2}_{F}}_{\mathcal{O}_{4};4}(1^{q}, 2^{\bar{q}}, 3^{\pm})$$

- $+\zeta(3)(-38G(1-z,y)+6G(0,y)+6G(0,z)-38G(1,z))$
- $+G(0,y)^{2}\left(-G(1,z)G(1-z,y)-\frac{1}{2}G(1-z,y)^{2}-\frac{1}{2}G(1,z)^{2}\right)-2G(0,y)G(0,1,1,z)$

 $-2G(0,y)G(1-z,1,1,y) + 2G(0,y)G(1-z,1-z,1,y) + \left(G(0,z)G(1,z) - \frac{1}{2}G(0,z)^2\right)G(1-z,y)^2$ 

 $\mathcal{R}_{\mathcal{O}_4:4}^{(2),C_F^2} = \zeta(3)(24G(1-z,y)+24G(1,z))-22\zeta(4).$ 

- $+ G(0,1,z) \left( G(0,y) (2G(1-z,y) + 2G(1,z)) G(1-z,y)^2 + G(0,y)^2 \right)$
- $+ \left(2G(0,z)G(1-z,y) + G(0,z)^2 2G(1,z)G(0,z) + 2G(0,1,z)\right)G(0,1-z,y)$
- $+ \left( G(0,y)^2 + 2G(0,1,y) \right) G(1-z,1,y) + G(0,0,1,z) (-2G(1-z,y) 4G(0,y) 2G(1,z))$
- -4G(0,z)G(0,0,1-z,y)+6G(1,z)G(0,0,-z,y)-2G(0,z)G(0,1-z,1-z,y)-2G(0,0,1-z,1,y)+6G(0,0,-z,1-z,y)-2G(0,1,1-z,1,y)-2G(0,1-z,0,1,y)
- -2G(0, 0, 1-z, 1, y) + 0G(0, 0, -z, 1-z, y) 2G(0, 1, 1-z, 1, y) 2G(0, 1-z, 0, 1, y) 2G(0, 1-z, 0, 1, y) 2G(0, 1-z, 0, 1, 1, y) 2G(0, 1-z, 0, 1, 1, y) 2G(1-z, 0, 0, 1, y) 2G(1-z, 0, 1, 1, y)
- $-2G(1-z,0,1-z,1,y)-2G(1-z,1-z,0,1,y)+6G(0,0,0,1,z)+2G(0,0,1,1,z)+\frac{73\zeta(4)}{4}$

$$\begin{split} \mathcal{R}^{(2),C_A^2}_{\mathcal{O}_4;4} &= \zeta(2) \left( G(0,y)(3G(1-z,y)-2G(0,z)+3G(1,z)) - G(1-z,y)^2 + (3G(0,z)-4G(1,z))G(1-z,y) \right. \\ & \left. - 2G(0,1-z,y) - 2G(1-z,1,y) - \frac{1}{2}G(0,y)^2 - \frac{1}{2}G(0,z)^2 - 2G(1,z)^2 + 3G(0,z)G(1,z) - 2G(0,1,z) \right) \right. \\ & \left. + \zeta(3)(11G(1-z,y)-6G(0,y)-6G(0,z)+11G(1,z)) - 3G(1,z)G(0,0,-z,y) \right. \\ & \left. + \left( \frac{1}{4}G(0,z)^2 + \frac{1}{2}G(1,z)G(0,z) \right) G(1-z,y)^2 + \left( \frac{3}{2}G(0,z)G(1,z)^2 - \frac{1}{2}G(0,z)^2G(1,z) \right) G(1-z,y) \right. \\ & \left. + G(0,y)^2 \left( \frac{1}{4}G(1-z,y)^2 + \left( \frac{1}{2}G(0,z) + \frac{1}{2}G(1,z) \right) G(1-z,y) - \frac{1}{4}G(0,z)^2 + \frac{1}{2}G(1,z)G(0,z) + \frac{1}{4}G(1,z)^2 \right) \right] \end{split}$$

 $+G(0,y)\left(-G(0,z)G(1-z,y)^{2}+\left(\frac{1}{2}G(0,z)^{2}-2G(0,z)G(1,z)\right)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1,z)G(0,z)^{2}-2G(0,z)G(1-z,y)+\frac{1}{2}G(1-z,y)+\frac{1$ 

 $-G(1,z)^2G(0,z)$ 

 $+G(0,1,z)\left(G(0,y)(G(1-z,y)+G(1,z))-\frac{1}{2}G(1-z,y)^2-3G(1,z)G(1-z,y)-G(0,y)^2-\frac{3}{2}G(1,z)^2\right)$ 

$$\begin{split} &+ \left(G(0,z)G(1-z,y)-G(0,z)^2+2G(1,z)G(0,z)-2G(0,1,z)\right)G(0,1-z,y) \\ &+ \left(-G(0,y)^2-2G(0,1,y)\right)G(1-z,1,y)+G(0,0,1,z)(2G(1-z,y)+G(0,y)+2G(1,z)) \\ &+ G(0,z)G(0,0,1-z,y)+G(0,1,1,z)(3G(1-z,y)-G(0,y)+3G(1,z))-G(0,z)G(0,1-z,1-z,y) \\ &+ 2G(0,y)G(1-z,1,1,y)+G(0,y)G(1-z,1-z,1,y)+2G(0,0,1-z,1,y)-3G(0,0,-z,1-z,y) \\ &+ 2G(0,1,1-z,1,y)+2G(0,1-z,0,1,y)+2G(0,1-z,1,1,y)-G(0,1-z,1-z,1,y) \\ &+ 2G(1-z,0,0,1,y)+2G(1-z,0,1,1,y)-G(1-z,0,1-z,1,y)-G(1-z,1-z,0,1,y) \\ &+ \frac{1}{2}G(0,z)G(1,z)^3-\frac{1}{4}G(0,z)^2G(1,z)^2-3G(0,0,0,1,z)-2G(0,0,1,1,z)-3G(0,1,1,1,z)-\frac{27\zeta(4)}{2}\,, \end{split}$$

#### Sum of the three terms is same as the 3-gluon case:

$$\mathscr{R}^{(2),C^2_A}_{\mathcal{O}_4;4}(1^q,\!2^{\bar{q}},\!3^{\pm}) + \mathscr{R}^{(2),C_AC_F}_{\mathcal{O}_4;4}(1^q,\!2^{\bar{q}},\!3^{\pm}) + \mathscr{R}^{(2),C^2_F}_{\mathcal{O}_4;4}(1^q,\!2^{\bar{q}},\!3^{\pm}) = \mathscr{R}^{(2)}_{\mathrm{tr}(F^3);4}(1^-,\!2^-,\!3^{\pm})$$

$$\begin{aligned} \mathscr{R}_{\mathrm{tr}(F^{3});4}^{(2)} &= -\frac{3}{2}\mathrm{Li}_{4}(u) + \frac{3}{4}\mathrm{Li}_{4}\left(-\frac{uv}{w}\right) - \frac{3}{2}\log(w)\mathrm{Li}_{3}\left(-\frac{u}{v}\right) + \frac{\log^{2}(u)}{32}\left[\log^{2}(u) + 2\log^{2}(v) - 4\log(v)\log(w)\right] \\ &+ \frac{\zeta_{2}}{8}\left[5\log^{2}(u) - 2\log(v)\log(w)\right] - \frac{1}{4}\zeta_{4} + \mathrm{perms}(u, v, w), \end{aligned}$$

#### General correspondence

[Jin and GY, 1904.07260]

Max. Tran. of  $(H \to q\bar{q}g)|_{C_F \to C_A}$   $c_A = N_c, \quad c_F = \frac{N_c^2 - 1}{2N_c}$ = Max. Tran. of  $(H \to 3g)$ = Max. Tran. of  $\mathcal{N} = 4$  form factors

It applies to form factors with more general operators:

	Leng	$\operatorname{gth}$ -2		Length-3		Higher length	
Operators		F-F		A A	F-A-F		FAAF
Examples	$\operatorname{tr}(F^2)$	$ar{\psi}\psi$	$ar{\phi}\phi$	$     tr(F^3),      tr(F^{\nu}_{\mu}D_{\sigma}F^{\rho}_{\nu}D^{\sigma}F^{\mu}_{\rho}) $	$F_{\mu\nu}D^{\mu}(\bar{\psi}\gamma^{\nu}\psi),$ $F_{\mu\nu}(\bar{\psi}\gamma^{\mu\nu}\psi)$	$\operatorname{tr}(F^L), L \ge 4$	$\bar{\psi}(F^L)\psi, L \ge 2$
External Partons	$(g,g,g),(ar{\psi},\psi,g)$	$(ar{\psi},\psi,g)$	$(ar{\phi},\phi,g)$	(g,g,g)	$(ar{\psi},g,\psi)$	$(g_1,\ldots,g_L)$	$(ar{\psi},g_1,\ldots,g_L,\psi)$
Max. Trans.							
Remainder $R_{L2;4}(u,v,w)$		$R_{\mathrm{L3;4}}(u,v,w)$		$\sum_i \mathcal{R}^{(2)}_{ ext{density};4}(u_i,v_i,w_i)$			
(with $C_F \to C_A$ )							

#### Simplicity at lower transcendental parts

$$\mathcal{O}_1 \to ggg \qquad \mathcal{O}_1 = \operatorname{tr}(F^3)$$

#### Degree-3 part:

$$\mathcal{R}_{\mathcal{O}_{1},\alpha;3}^{(2),N_{c}^{2}} = \left(1 + \frac{u}{w}\right) T_{3}(u,v,w) + \frac{143}{72}\zeta_{3} - \frac{11}{24}\zeta_{2}\log(u) + \operatorname{perms}(u,v,w)$$
$$\mathcal{R}_{\mathcal{N}=4;3}^{(2),N_{c}^{2}} = \left(1 + \frac{u}{w}\right) T_{3}(u,v,w) + \operatorname{perms}(u,v,w)$$

$$T_{3} := \left[ -\operatorname{Li}_{3}\left(-\frac{u}{w}\right) + \log(u)\operatorname{Li}_{2}\left(\frac{v}{1-u}\right) - \frac{1}{2}\log(1-u)\log(u)\log\left(\frac{w^{2}}{1-u}\right) + \frac{1}{2}\operatorname{Li}_{3}\left(-\frac{uv}{w}\right) + \frac{1}{12}\log^{3}(w) + \frac{1}{2}\log(u)\log(v)\log(w) + (u\leftrightarrow v)\right] + \operatorname{Li}_{3}(1-v) - \operatorname{Li}_{3}(u) + \frac{1}{2}\log^{2}(v)\log\left(\frac{1-v}{u}\right) - \zeta_{2}\log\left(\frac{uv}{w}\right).$$

#### Simplicity at lower transcendental parts

$$\mathcal{O}_1 \to ggg \qquad \mathcal{O}_1 = \operatorname{tr}(F^3)$$

#### Degree-2 to 0 parts:

$$\mathcal{R}_{\hat{\mathcal{O}}_{1},\alpha;2}^{(2),N_{c}^{2}} = \left(\frac{u^{2}}{w^{2}} - \frac{1}{2}\right)T_{2}(u,v) - \frac{55}{48}\log^{2}(u) - \frac{73}{72}\log(u)\log(v) + \frac{23}{6}\zeta_{2} + \operatorname{perms}(u,v,w)$$
$$\boxed{T_{2}(u,v) := \operatorname{Li}_{2}(1-u) + \operatorname{Li}_{2}(1-v) + \log(u)\log(v) - \zeta_{2}}$$

$$\mathcal{R}_{\mathcal{O}_1,\alpha;1}^{(2),N_c^2} = \left(\frac{119}{18} + \frac{v}{w} + \frac{u^2}{2vw}\right)\log(u) + \operatorname{perms}(u,v,w)$$

$$\mathcal{R}_{\mathcal{O}_1,\alpha;0}^{(2),N_c^2} = \frac{487}{72} \frac{1}{uvw} - \frac{14075}{216}$$

#### Simplicity at lower transcendental parts

[Jin, GY 1910.09384]

$$\mathcal{O}_4 \to q\bar{q}g \qquad \qquad \mathcal{O}_4 \to \mathcal{O}'_4 = gF_{\mu\nu}D^{\mu}\sum_{i,j=1}^{n_f} (\bar{\psi}_i \gamma^{\nu} T^A \psi_i)$$

Degree-3 part:

$$\begin{aligned} \mathcal{R}_{\mathcal{O}_{4};\gamma;3}^{(2),N_{c}^{2}} &= -\left(1+\frac{w}{u}\right)T_{3}(v,w,u) - \frac{1}{3}\left(1+\frac{v}{u}\right)T_{3}(w,v,u) - \pi^{2}\left[\frac{7\log(v)}{24} + \frac{\log(w)}{72}\right] - \frac{257\zeta_{3}}{18} \\ \mathcal{R}_{\mathcal{O}_{4};\gamma;3}^{(2),N_{c}^{0}} &= \left[\frac{1}{2}\left(\frac{w}{v}+1\right)^{2} + \frac{w}{v}+1\right]T_{3}(u,w,v) - \left[\frac{1}{6}\left(\frac{u}{v}+1\right)^{2} - \frac{4}{3}\frac{u}{v} - 1\right]T_{3}(w,u,v) \\ &+ T_{3}(v,w,u) + \frac{1}{2}T_{3}(w,v,u) - T_{3}(v,u,w) \\ &+ \pi^{2}\left[\frac{31\log(u)}{72} - \frac{5\log(v)}{18} - \frac{7\log(w)}{36}\right] + \frac{137\zeta_{3}}{36}. \\ \mathcal{R}_{\mathcal{O}_{4};\gamma;3}^{(2),N_{c}^{-2}} &= -\frac{w}{u}T_{3}(v,w,u) - \frac{1}{3}\frac{v}{u}T_{3}(w,v,u) - T_{3}(v,u,w) - \frac{1}{3}T_{3}(w,u,v) \\ &+ \frac{1}{18}\pi^{2}(5\log(u) + 3\log(v) + \log(w)) + \frac{15\zeta_{3}}{2}. \\ \mathcal{R}_{\mathcal{O}_{4};\gamma;3}^{(2),N_{c}n_{f}} &= -\frac{2}{3}T_{3}(u,v,w) - \frac{2}{3}T_{3}(u,w,v) + \frac{1}{36}\pi^{2}(8\log(u) - 3\log(v) - 3\log(w)) + \frac{43\zeta(3)}{9}. \end{aligned}$$

$$T_{3} := \left[ -\operatorname{Li}_{3}\left(-\frac{u}{w}\right) + \log(u)\operatorname{Li}_{2}\left(\frac{v}{1-u}\right) - \frac{1}{2}\log(1-u)\log(u)\log\left(\frac{w^{2}}{1-u}\right) + \frac{1}{2}\operatorname{Li}_{3}\left(-\frac{uv}{w}\right) + \frac{1}{12}\log^{3}(w) + \frac{1}{2}\log(u)\log(v)\log(w) + (u\leftrightarrow v) \right] + \operatorname{Li}_{3}(1-v) - \operatorname{Li}_{3}(u) + \frac{1}{2}\log^{2}(v)\log\left(\frac{1-v}{u}\right) - \zeta_{2}\log\left(\frac{uv}{w}\right).$$

# Outline

- Motivations
- Higgs amplitudes
- Hidden analytic structure
- Explanation and implication

#### What is the origin of the correspondence?



The idea is to apply "physical constraints": For example, the universal IR divergences can be used to fix the finite part.

#### What is the origin of the correspondence?

Use a set of "Uniform transcendental integrals"  $I_{i,\text{UT}}$  as integral basis, e.g.:  $I_{\text{tri}}^{(1)} = -\frac{1}{\epsilon^2} + \frac{\pi^2}{12} + \frac{7}{3}\zeta_3\epsilon + \frac{47}{1440}\pi^2\epsilon^2 + \mathcal{O}(\epsilon^3)$ 



The maximal transcendentality part for minimal two-loop form factors can be uniquely fixed by the infrared divergences:

$$\mathscr{F}_{deg-4}^{(2)} = \sum_{i} c_{i,(0)} I_{i,UT}^{(2)} = (\text{IR div})|_{deg-4} + \mathcal{O}(\epsilon^{0})$$

(For N=4 BPS case, this fixes the full result !)

#### What is the origin of the correspondence?

Use a set of "Uniform transcendental integrals"  $I_{i,\text{UT}}$  as integral basis, e.g.:  $I_{\text{tri}}^{(1)} = -\frac{1}{\epsilon^2} + \frac{\pi^2}{12} + \frac{7}{3}\zeta_3\epsilon + \frac{47}{1440}\pi^2\epsilon^2 + \mathcal{O}(\epsilon^3)$ 



Lower transcendentality parts can be partially fixed by infrared divergences, plus additional physical constraints (e.g. unitarity cuts, collinear limit etc).

$$\mathscr{F}_{deg^{-}(4-a)}^{(2)} = \epsilon^{a} \sum_{i} c_{i,(a)} I_{i,\text{UT}}^{(2)} = (\text{IR}/\text{UV div}) |_{deg^{-}(4-a)} + \mathcal{O}(\epsilon^{0})$$

The higher the degree, the simpler the constraints.

### Hierarchy of "complexity"



Similarity at one-loop: the most challenging part is the rational part.



- (1) To which extend is the correspondence correct? Higgs amplitudes of high dim operators, and external quarks
- (2) What is the origin of the correspondence?
  - It can be proved using physical constraints.
- (3) Does it help in practice? How about lower trans. parts?
  - Simple structure also exists and may be fixed in similar ways.
  - The most challenging part is the rational parts, which hopefully may be computed using alternative techniques.

### Summary and outlook

The study of analytic structures and the maximal transcendentality correspondence can be very useful.

#### A new strategy:

Divide and conquer (according transcendentality degree) and bootstrap results (using physical constraints) without intermediate complicated steps.

Towards a better way of computation (in realistic QCD)?!

### Summary and outlook

The study of analytic structures and the maximal transcendentality correspondence can be very useful.

#### A new strategy:

Divide and conquer (according transcendentality degree) and bootstrap results (using physical constraints) without intermediate complicated steps.

Towards a better way of computation (in realistic QCD)?!

## Thank you for your attention