Hidden Analytic Structure of Two-Loop Higgs Amplitudes

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Based on: arXiv:1904.07260, arXiv:1910.09384, and in progress;
in collaboration with Dr. Qingjun Jin and Dr. Ke Ren

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Last Year: East Asia Joint Workshop on Fields and Strings
KIAS, Nov 5-9, 2018

A direct relation between N=4 SYM and QCD

for two-loop Higgs-gluons Amplitudes

$$(H \to 3 \text{ gluons})|_{\text{QCD}} \leftrightarrow (\text{3-gluon form factor})|_{\mathcal{N}=4}$$

Q.Jin and GY, PRL 121 101603 (2018) [arXiv:1804.04653],

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New questions we ask:

How about Higgs amplitudes with **fundamental quark** external states?

$$(H \to q\bar{q}g)\,|_{\text{QCD}}$$

What is the **explanation and implication**?

Outline

- Motivations
- Higgs amplitudes
- Hidden analytic structure
- Explanation and implication

Generic strategy of loop computation

Feynman integrals → **Expansion in a basis of integrals** → **Final result in functional form**

Feynman diagrams, On-shell unitarity method, …

$\sum$ (integrand)

Integral reductions: PV reduction, IBP reduction, …

$\sum c_i M_i$

Solving integrals, functional identities to simplify the result, …

$\sum$ functions

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Solving integrals, functional identities to simplify the result, …

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HUGE intermediate expressions

Much simpler analytic form

MHV gluon tree amplitudes

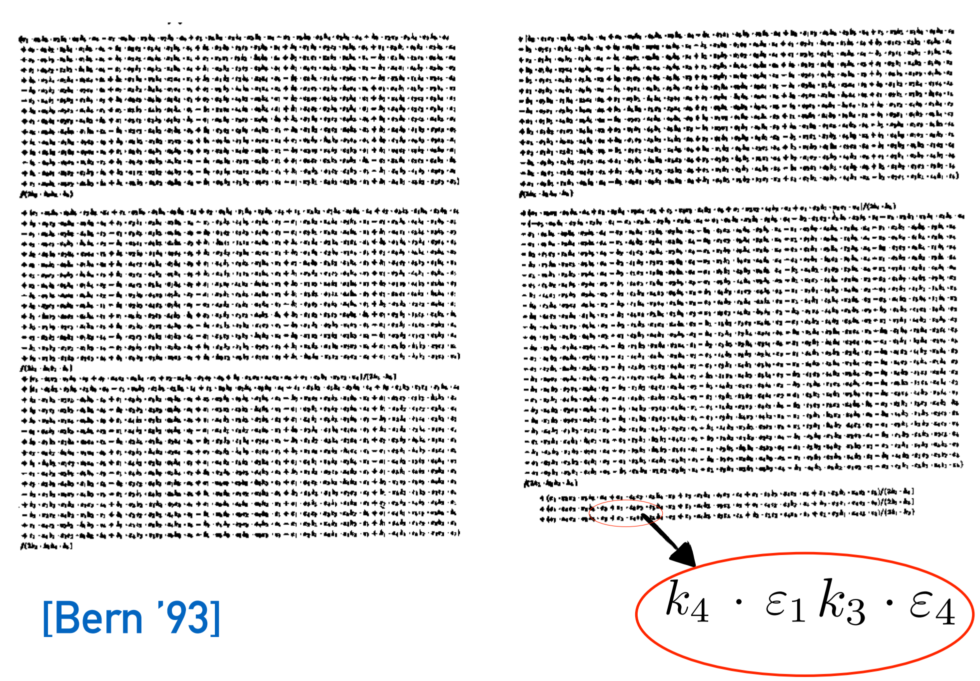
[Parke, Taylor '86]

$$A_n^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}$$

Feynman diagrams

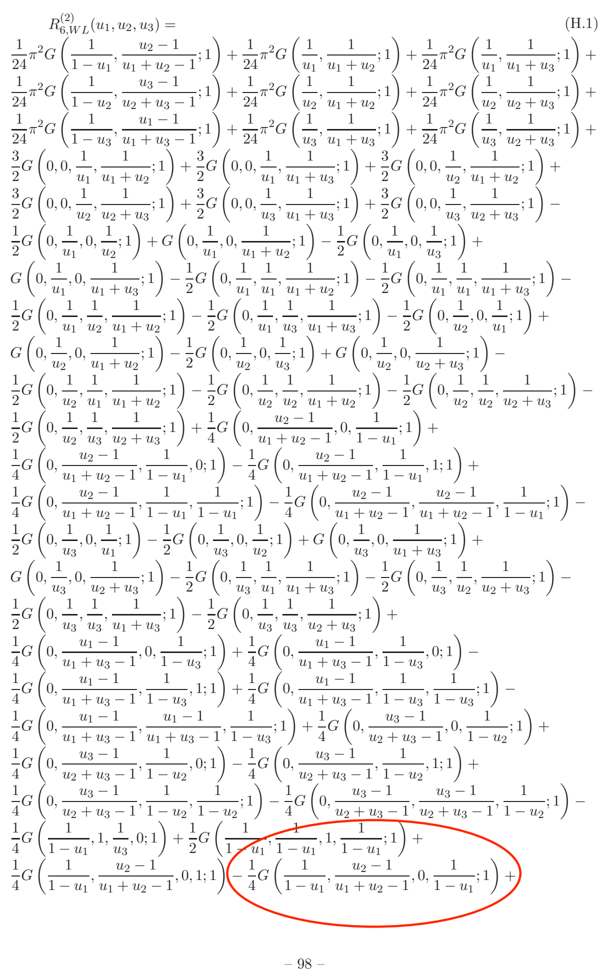
n-gluon	4	5	6	7	8	9	10
# graphs	4	25	220	2485	34300	559405	10525900

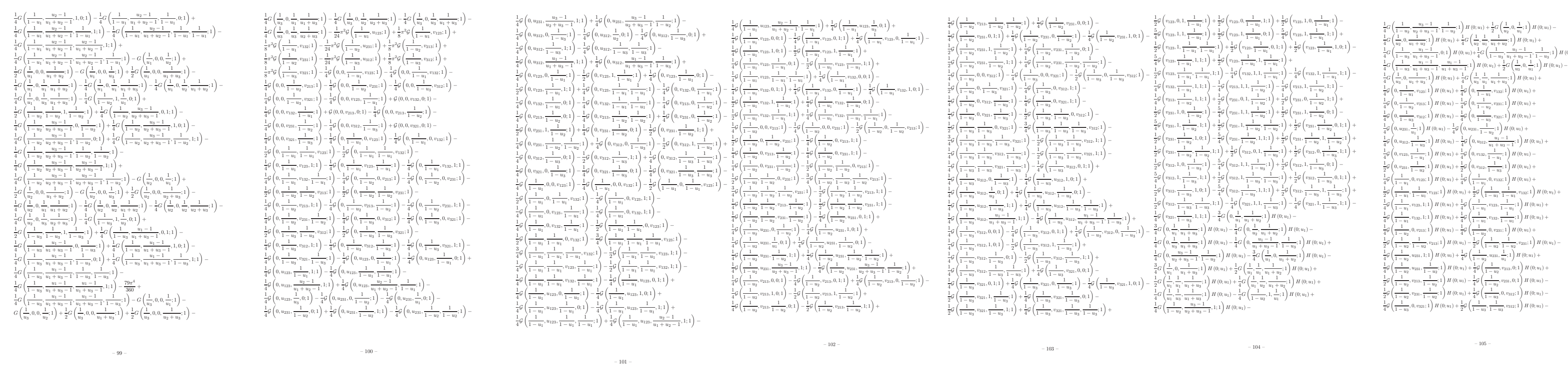
Five-gluon expression:



[Bern '93]

Two-loop six-gluon amplitudes in N=4

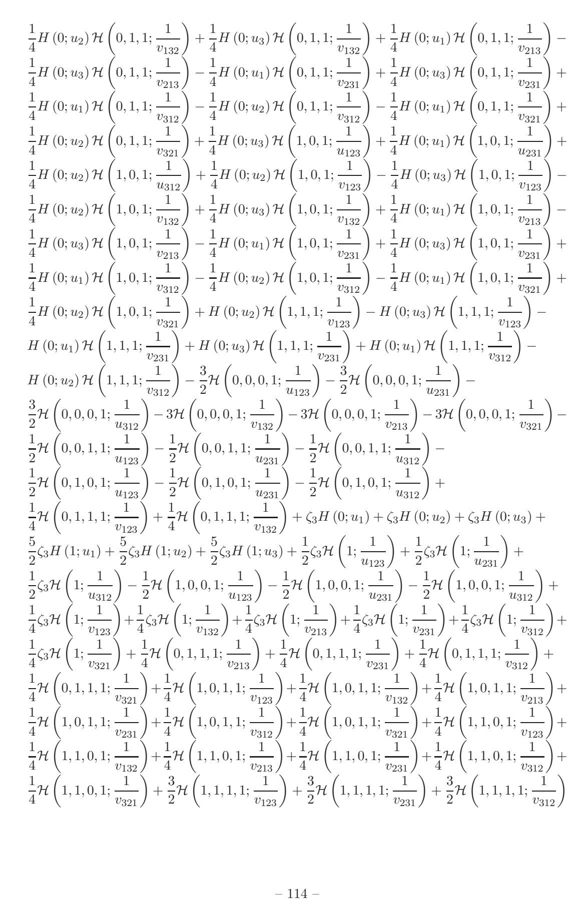


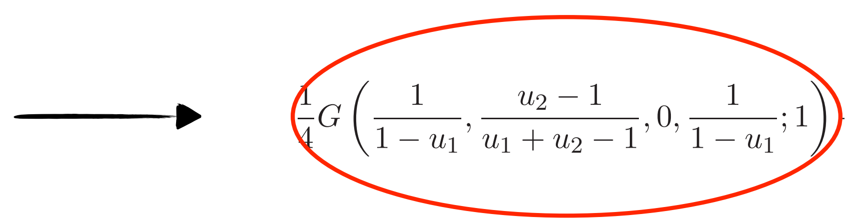


[Del Duca, Duhr, Smirnov 2010]

“multiple(Goncharov)-polylogrithm function”

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)$$





Result can be remarkably simple

17 pages =

[Goncharov, Spradlin, Vergu, Volovich 2010]

$$\sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

$$L_4(x^+, x^-) = \frac{1}{8!!}\log(x^+x^-)^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+) + \ell_{4-m}(x^-)) \qquad \ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n(1/x)\right) \qquad J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-)).$$

a line result in terms of classical polylogarithms!

Such simplicity is totally unexpected using traditional Feynman diagrams!

Lessons from modern amplitudes

Conceptually: New structures and new formulations in QFT

Witten's twistor theory, Double-copy, Grassmannian and Amplituhedron, CHY formalism, new geometric structure…

Methodologically: New powerful computational techniques

Spinor helicity formalism

BCFW recursion relation

(Generalized) unitarity cuts

New integral reduction and integration methods

Developed based on studying N=4 SYM, now applicable to general theories

N=4 super Yang-Mills theory

N=4 SYM theory : -> QCD's maximally supersymmetric cousin

$$\mathcal{L} = -\frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \mathrm{fermions} + \mathrm{scalars}$$

where all fields are the in the adjoint representation of the gauge group SU(N_c).

- conformal invariant, UV finite
- prime model in AdS/CFT correspondence
- solvable in the planar limit due to Integrability

What (more) can we learn from N=4 SYM for realistic QCD?

Maximal Transcendentality Principle

N=4 SYM ⟷ **QCD**

Maximally transcendental parts are equal between the two theories

(conjecture)

Transcendental number / function

A root of a nonzero polynomial equation with integer coefficients?

Yes ↙ ↘ No

algebraic number
(代数数)

$\sqrt{2}$: $x^2 - 2 = 0$

transcendental number
(超越数)

e , π

Similarly, a transcendental function is an analytic function that does not satisfy a polynomial equation, in contrast to an algebraic function.

Transcendentality degree

Number	Function	Transcendental degree
2/3, $\sqrt{2}$	rational function	0
π	Log(x)	1
ζ_k	$\mathrm{Li}_k(x)$	k
$\epsilon \qquad (D = 4 - 2\epsilon)$		-1

Riemann zeta value:

$$\zeta_k = \sum_{n=1}^{\infty} \frac{1}{n^k}, \qquad k \geq 2$$

$$S_k(j) = \sum_{n=1}^{j} \frac{1}{n^k} \xrightarrow{j \to \infty} \zeta_k$$

Polylogarithms:

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} = \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t} dt$$

$$\mathrm{Li}_1(z) = -\log(1-z) \qquad \mathrm{Li}_k(1) = \zeta_k$$

Maximal Transcendentality Principle

N=4 SYM ⟷ **QCD**

- Anomalous dimension of twist-2 operators:

$$\boxed{\gamma^{\mathcal{N}=4}(j) = \gamma^{\mathrm{QCD}}(j)|_{\mathrm{max.\ trans}}}$$

[Kotikov, Lipatov, Onishchenko, Velizhanin]

Maximal Transcendentality Principle

N=4 SYM ⟷ QCD

- Anomalous dimension of twist-2 operators:

$$\gamma^{\mathcal{N}=4}(j) = \gamma^{\text{QCD}}(j)|_{\text{max. trans}}$$

[Kotikov, Lipatov, Onishchenko, Velizhanin]

- Higgs-3gluon two-loop amplitudes: [Brandhuber, Travaglini, GY 2012]

N=4 SYM form factor ⟷ Higgs plus 3-gluon amplitudes $m_t \to \infty$

$$\mathcal{O}_0 = \text{tr}(F_{\mu\nu}F^{\mu\nu})$$

Maximal Transcendentality Principle

Maximal transcendental part of
Higgs amplitudes: $m_t \to \infty$

[Gehrmann, Jaquier, Glover, Koukoutsakis 2011]

QCD

$$
\begin{aligned}
&-2G(0,0,1,0,u)+G(0,0,1-v,1-v,u)+2G(0,0,-v,1-v,u)-G(0,1,0,1-v,u)+4G(0,1,1,0,u)-G(0,1,1-v,0,u)+G(0,1-v,0,1-v,u)\\
&+G(0,1-v,1-v,0,u)-G(0,1-v,-v,1-v,u)+2G(0,-v,0,1-v,u)+2G(0,-v,1-v,0,u)-2G(0,-v,1-v,1-v,u)-2G(1,0,0,1-v,u)\\
&-2G(1,0,1-v,0,u)+4G(1,1,0,0,u)-4G(1,1,1,0,u)-2G(1,1-v,0,0,u)+G(1-v,0,0,1-v,u)-G(1-v,0,1,0,u)-2G(-v,1-v,1-v,u)H(0,v)\\
&-2G(1-v,1,0,0,u)+2G(1-v,1,0,1-v,u)+2G(1-v,1,1-v,0,u)+G(1-v,1-v,0,0,u)+2G(1-v,1-v,1,0,u)-2G(1-v,1-v,-v,1-v,u)\\
&-G(1-v,-v,1-v,0,u)+4G(1-v,-v,-v,1-v,u)-2G(-v,0,1-v,1-v,u)-2G(-v,1-v,0,1-v,u)-2G(-v,1-v,1-v,0,u)+4G(1,0,1,0,u)\\
&+4G(-v,-v,1-v,1-v,u)-4G(-v,-v,-v,1-v,u)-G(0,0,1-v,u)H(0,v)-G(0,1,0,u)H(0,v)-G(0,1-v,0,u)H(0,v)+G(0,1-v,1-v,u)H(0,v)\\
&-G(0,-v,1-v,u)H(0,v)-2G(1,0,0,u)H(0,v)+G(1,0,1-v,u)H(0,v)+G(1,1-v,0,u)H(0,v)+G(1-v,0,0,u)H(0,v)-G(1-v,0,1-v,u)H(0,v)\\
&-G(1-v,1,0,u)H(0,v)-G(1-v,1-v,0,u)H(0,v)-G(1-v,-v,1-v,u)H(0,v)+G(-v,0,1-v,u)H(0,v)+G(-v,1-v,0,u)H(0,v)+H(1,0,0,1,v)\\
&-G\ldots-G(0,0,-v,u)H(1,v)+G(0,1,0,u)H(1,v)-G(0,1-v,0,u)H(1,v)+G(0,1-v,-v,u)H(1,v)-2G(0,-v,0,u)H(1,v)\\
&+2\ldots v)+2G(1,0,0,u)H(1,v)-G(1-v,0,0,u)H(1,v)+G(1-v,0,-v,u)H(1,v)-2G(1-v,1,0,u)H(1,v)-G(1-v,0,-v,1-v,u)\\
&+G\ldots)-4G(1-v,-v,-v,u)H(1,v)+2G(-v,0,1-v,u)H(1,v)+2G(-v,1-v,0,u)H(1,v)-4G(-v,1-v,-v,u)H(1,v)\\
&-4\ldots,v)+4G(-v,-v,-v,u)H(1,v)+G(0,0,u)H(0,0,v)+G(0,1-v,u)H(0,0,v)+G(1-v,0,u)H(0,0,v)+H(1,0,1,0,v)\\
&-G(0,0,u)H(0,1,v)+G(0,-v,u)H(0,1,v)-G(1,0,u)H(0,1,v)+2G(1-v,0,u)H(0,1,v)+2G(1-v,1-v,u)H(0,1,v)-3G(1-v,-v,u)H(0,1,v)\\
&-G(-v,0,u)H(0,1,v)-2G(-v,1-v,u)H(0,1,v)+4G(-v,-v,u)H(0,1,v)-G(0,0,u)H(1,0,v)+G(0,-v,u)H(1,0,v)-G(1,0,u)H(1,0,v)\\
&+2G(1-v,0,u)H(1,0,v)-2G(1-v,1-v,u)H(1,0,v)+G(1-v,-v,u)H(1,0,v)-G(-v,0,u)H(1,0,v)+2G(-v,1-v,u)H(1,0,v)+G(0,0,u)H(1,1,v)\\
&-2G(0,-v,u)H(1,1,v)-2G(-v,0,u)H(1,1,v)+4G(-v,-v,u)H(1,1,v)+G(0,u)H(0,0,1,v)-3G(1-v,u)H(0,0,1,v)+4G(-v,u)H(0,0,1,v)\\
&+G(0,u)H(0,1,0,v)+G(1-v,u)H(0,1,0,v)-G(0,u)H(0,1,1,v)+2G(-v,u)H(0,1,1,v)+G(0,u)H(1,0,0,v)+G(1-v,u)H(1,0,0,v)+H(1,1,0,0,v)\\
&-G(0,u)H(1,0,1,v)+2G(-v,u)H(1,0,1,v)-G(0,u)H(1,1,0,v)+4G(1-v,u)H(1,1,0,v)-2G(-v,u)H(1,1,0,v)+H(0,0,1,1,v)+H(0,1,0,1,v)\\
&+G(1-v,1-v,u)H(0,0,v)+2G(1-v,1-v,-v,u)H(1,v)-G(1-v,-v,0,1-v,u)+H(0,1,1,0,v)+G(1-v,0,1-v,0,u)-G(0,1-v,1,0,u)\\
&+4G(-v,1-v,-v,1-v,u)
\end{aligned}
$$

$$G(1-v,1-v,v,1-v;u)$$

Multiple polyLogarithm

$\Updownarrow$

[Brandhuber, Travaglini, GY 2012]

N=4

$$
-2\left[J_4\left(-\frac{uv}{w}\right)+J_4\left(-\frac{vw}{u}\right)+J_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-\frac{1}{u_i}\right)+\frac{\log^4 u_i}{4!}\right]-2\left[\sum_{i=1}^{3}\mathrm{Li}_2\left(1-\frac{1}{u_i}\right)\right]^2
$$

$$
+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2+2(J_2^2-\zeta_2 J_2)-\frac{\log^4(uvw)}{4!}-\zeta_3\log(uvw)-\frac{123}{8}\zeta_4
$$

$$
J_4(x)=\mathrm{Li}_4(x)-\log(-x)\mathrm{Li}_3(x)+\frac{\log^2(-x)}{2!}\mathrm{Li}_2(x)-\frac{\log^3(-x)}{3!}\mathrm{Li}_1(x)-\frac{\log^4(-x)}{48},\qquad J_2=\sum_{i=1}^{3}\left(\mathrm{Li}_2(1-u_i)+\frac{1}{2}\log(u_i)\log(u_{i+1})\right),
$$

$u=\frac{s_{12}}{q^2}$, $v=\frac{s_{23}}{q^2}$, $w=\frac{s_{13}}{q^2}$, where $q^2=s_{123}$

Maximal Transcendentality Principle

N=4 SYM ⟷ **QCD**

Questions that we will address:

(1) To which extend is the correspondence correct?

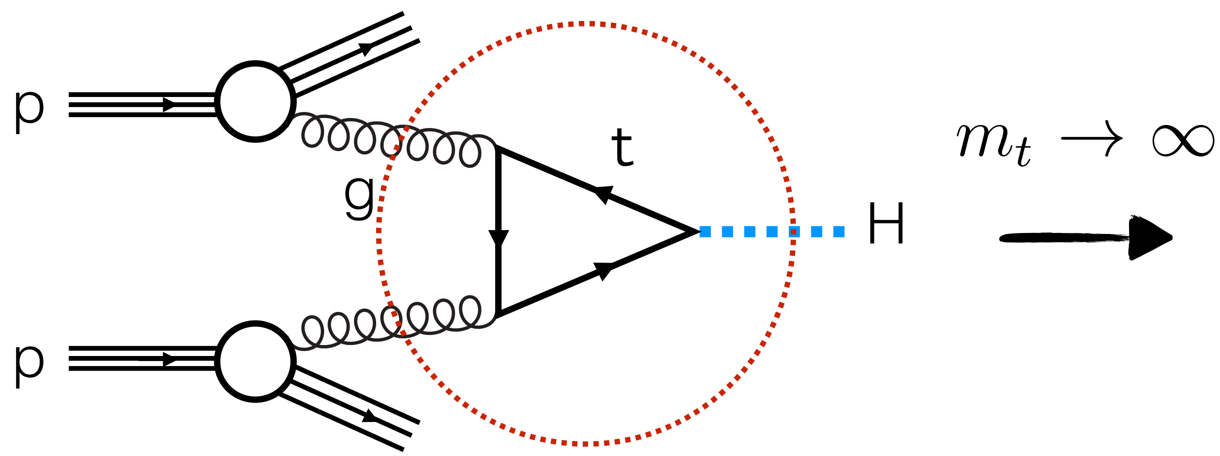
(2) What is the origin of the correspondence?

(3) Can we go beyond N=4? How about lower trans. parts?

Outline

- Motivations
- Higgs amplitudes
- Hidden analytic structure
- Explanation and implication

Higgs Effective Field Theory



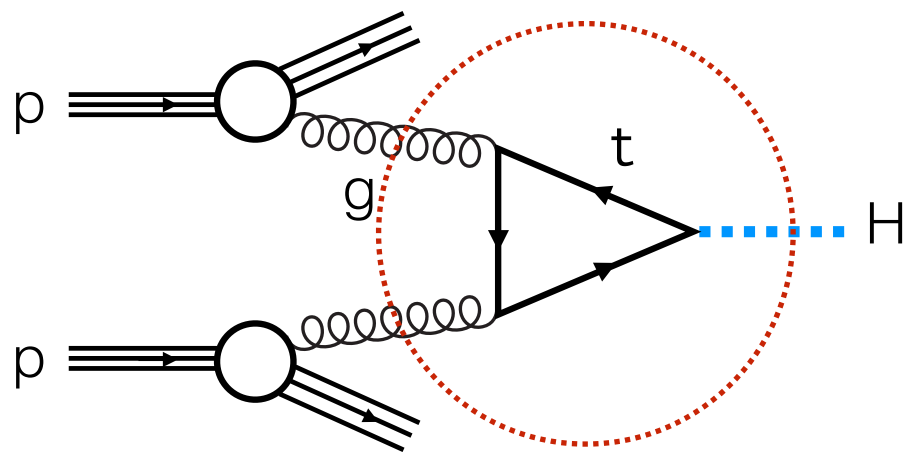
Effective Higgs-gluon vertices:

$$\mathscr{L}_{\text{eff}} = C_0 H \text{tr}(F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}(\frac{1}{m_t^2})$$

There have been computations for inclusive Higgs production to NNNLO orders in the heavy quark limit.

[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger 2016]

Higgs Effective Field Theory



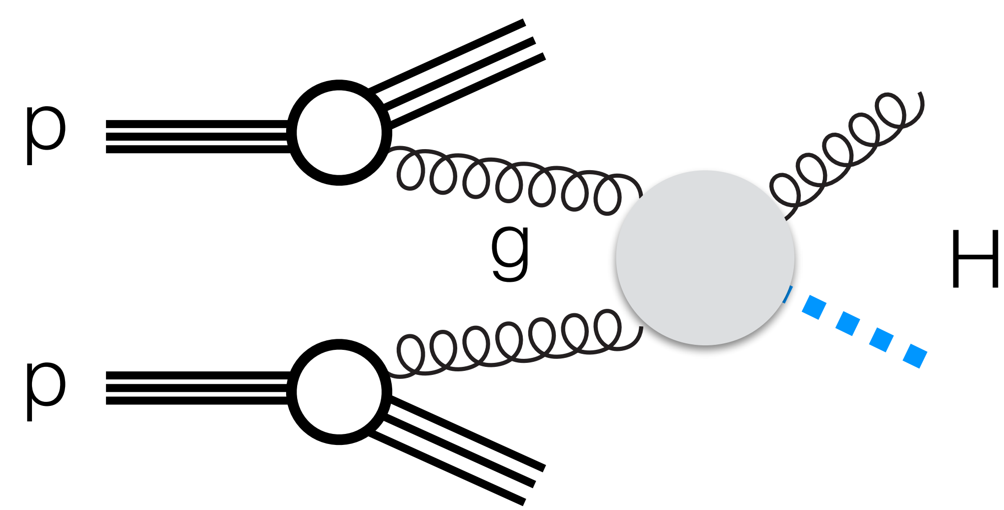
$m_t \to \infty$

Effective Higgs-gluon vertices:

$$\mathscr{L}_{\text{eff}} = C_0 H \text{tr}(F_{\mu\nu}F^{\mu\nu}) + \mathscr{O}(\frac{1}{m_t^2})$$

Higgs plus jet production
is sensitive to new physics.

EFT description is not good
when $p_T \sim 2m_t$



High dimension operators contributions become important.

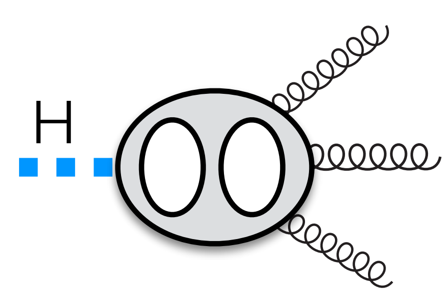
Higgs Effective Field Theory

$$\mathcal{L}_{\text{eff}} = C_0 O_0 + \frac{1}{m_{\text{t}}^2} \sum_{i=1}^{4} C_i O_i + \mathcal{O}\left(\frac{1}{m_{\text{t}}^4}\right)$$

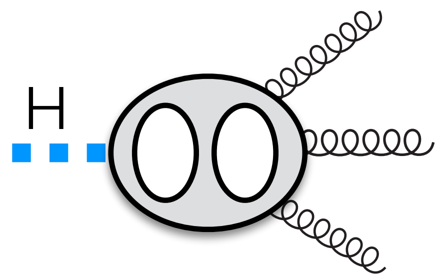
Dimension-7 operators

$$O_1 = H\text{tr}(F_{\mu}{}^{\nu} F_{\nu}{}^{\rho} F_{\rho}{}^{\mu})$$
$$O_2 = H\text{tr}(D_{\rho} F_{\mu\nu} D^{\rho} F^{\mu\nu})$$
$$O_3 = H\text{tr}(D^{\rho} F_{\rho\mu} D_{\sigma} F^{\sigma\mu})$$
$$O_4 = H\text{tr}(F_{\mu\rho} D^{\rho} D_{\sigma} F^{\sigma\mu})$$

We consider two-loop S-matrix of Higgs plus 3 partons with high dimensional operators.



Higgs amplitudes as form factors



$$\mathcal{O}_0 = \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

Higgs amplitudes are equivalent to form factors:

$$\mathcal{F}_{\mathcal{O}_i,n} = \int d^4x\, e^{-iq\cdot x}\langle p_1, p_2, p_3 \,|\, \mathcal{O}_i(x) \,|\, 0\rangle$$

$$\mathcal{O}_1 = \mathrm{tr}(F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu)$$

$$\mathcal{O}_2 = \mathrm{tr}(D_\rho F_{\mu\nu} D^\rho F^{\mu\nu})$$

$$\mathcal{O}_3 = \mathrm{tr}(D^\rho F_{\rho\mu} D_\sigma F^{\sigma\mu})$$

$$\mathcal{O}_4 = \mathrm{tr}(F_{\mu\rho} D^\rho D_\sigma F^{\sigma\mu})$$

Operator relation:

$$\mathcal{O}_2 = \frac{1}{2}\partial^2\mathcal{O}_0 - 4\, g_{\mathrm{YM}}\, \mathcal{O}_1 + 2\,\mathcal{O}_4 \longrightarrow \mathcal{F}_{\mathcal{O}_2} = \frac{1}{2}\, q^2\, \mathcal{F}_{\mathcal{O}_0} - 4\, g_{\mathrm{YM}}\, \mathcal{F}_{\mathcal{O}_1} + 2\, \mathcal{F}_{\mathcal{O}_4}$$

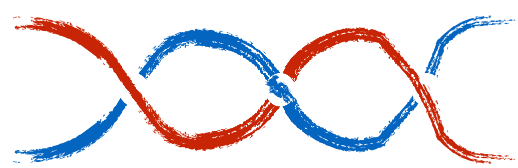
$$D_\rho F^{\rho\mu} = -g \sum_{i=1}^{n_f} \bar\psi_i \gamma^\mu \psi_i \longrightarrow \mathcal{O}_3 \to \mathcal{O}_3' = g^2 \sum_{i,j=1}^{n_f} (\bar\psi_i\gamma^\mu\psi_i)(\bar\psi_j\gamma_\mu\psi_j)$$

$$\mathcal{O}_4 \to \mathcal{O}_4' = g F_{\mu\nu} D^\mu \sum_{i,j=1}^{n_f} (\bar\psi_i \gamma^\nu T^A \psi_i)$$

Two-loop QCD computation

Our strategy:

On-shell unitarity



Integration by parts

On-shell unitarity method

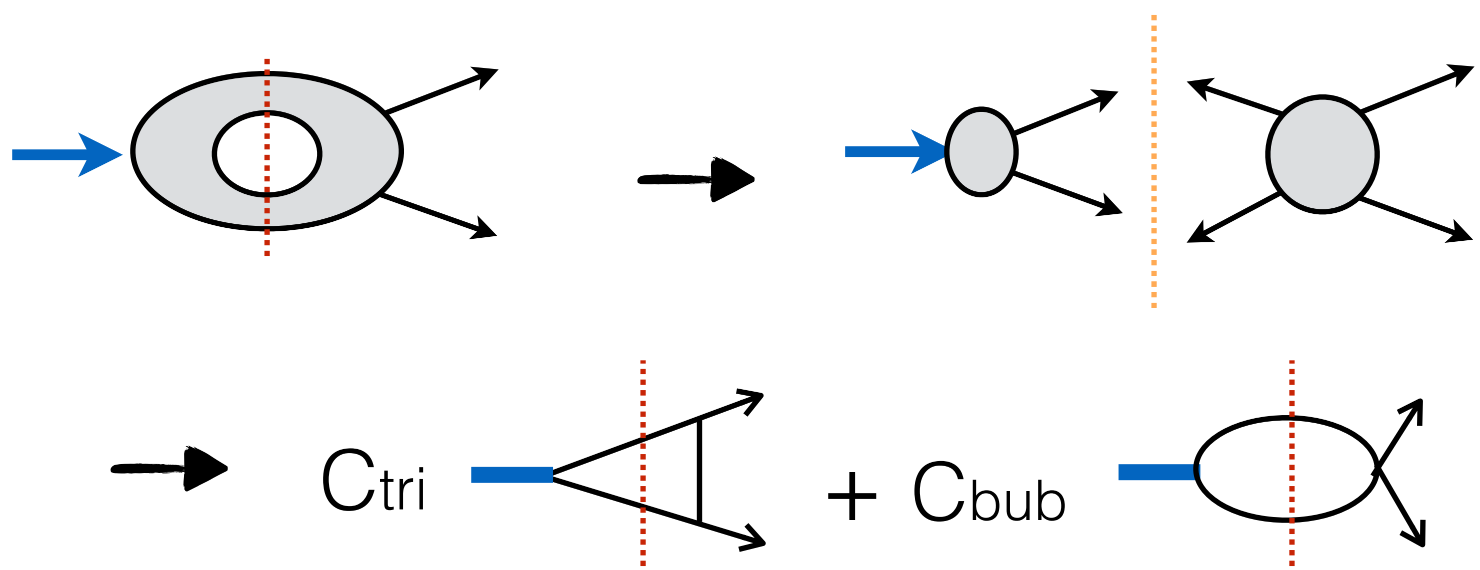
Unitarity requires that loop amplitudes/form factors have consistent discontinuities by cutting propagators.

[Bern, Dixon, Dunbar, Kosower 1994]

[Britto, Cachazo, Feng 2004]

On the cut, the loop quantity factorizes into a product of tree-level or lower-loop results.

One-Loop example:



The form factors are guarantee to be correct once they satisfy all cut constraints.

Integration by part reduction

[Chetyrkin, Tkachov 1981]

$$\int d^D l_1 \dots d^D l_L \; \frac{\partial}{\partial l_i^{\mu}} \,(\text{integrand}) = 0 \,.$$

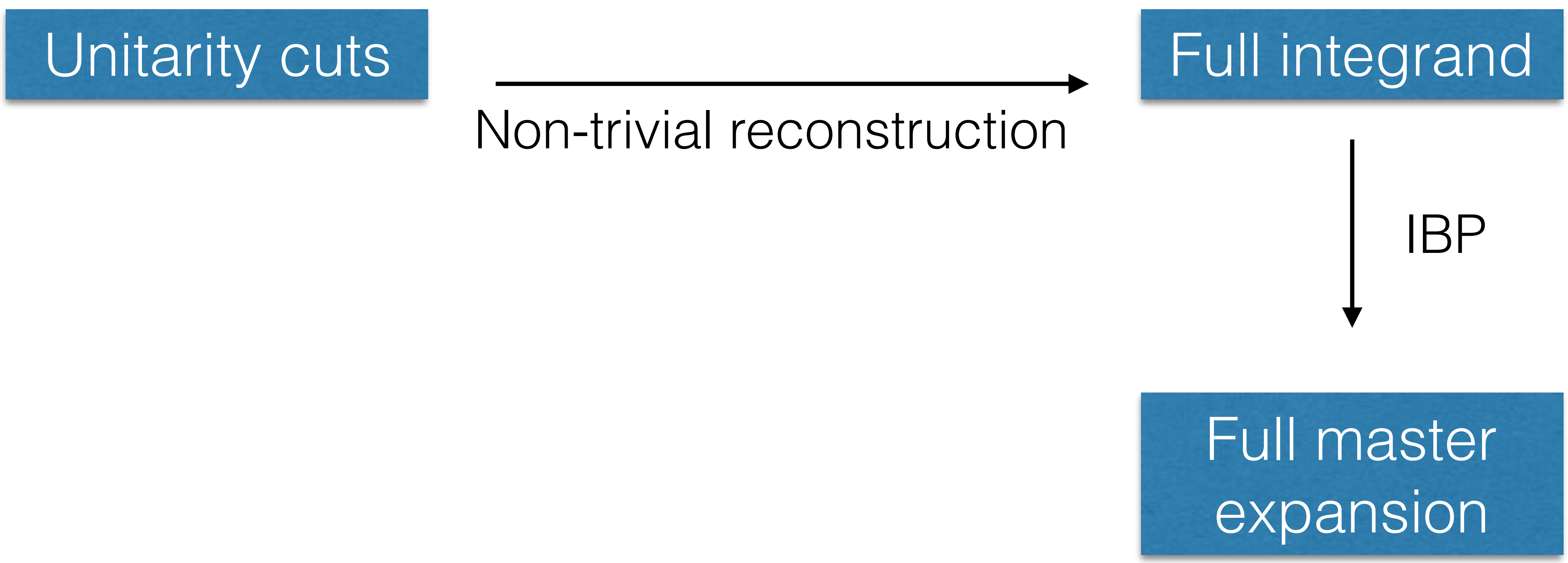
A set of linear relations between different integrals.

Any integrand = (coefficients) x (Master integrals)

Public packages:
Reduze 2, FIRE, LiteRed, etc

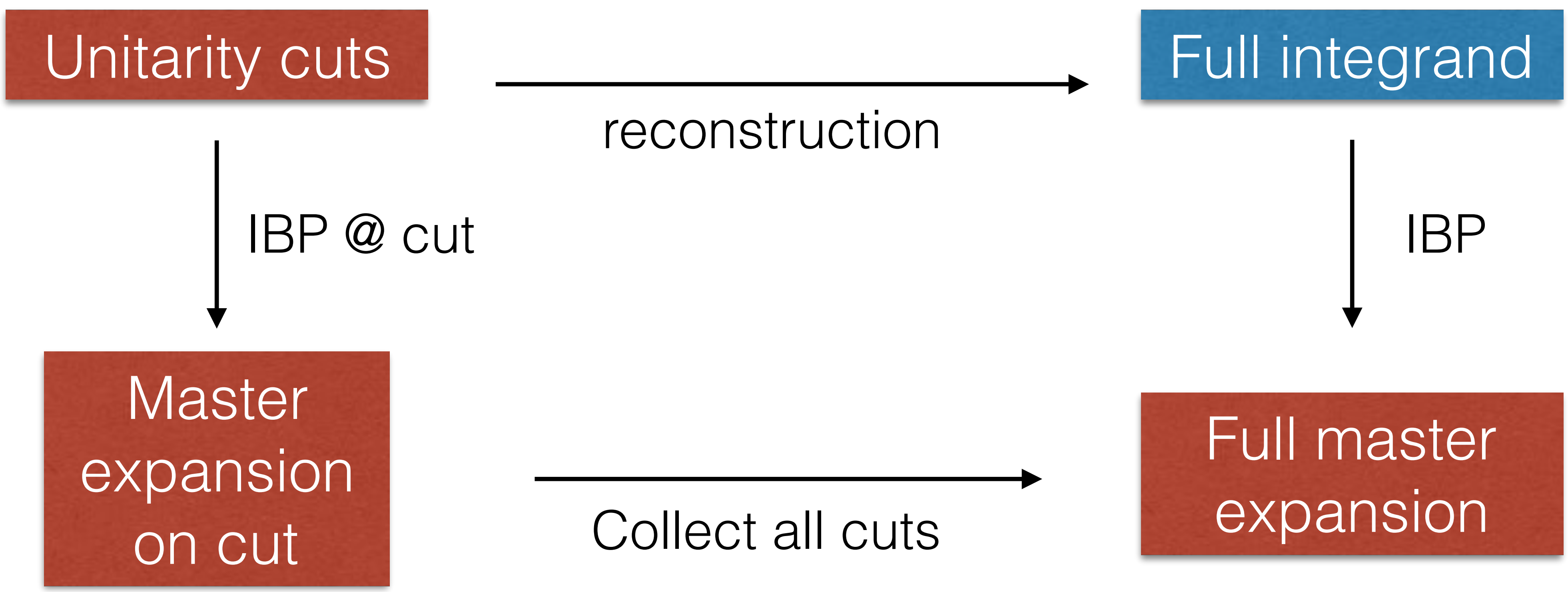
Unitarity + IBP

- Usual strategy (blue color):



Unitarity + IBP

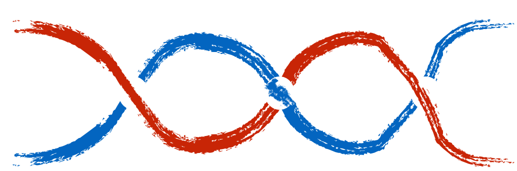
- Improved strategy (red color):



Improved strategy

$$F^{(l)}|_{\text{cut}} = \sum_{\text{helicities}} F^{\text{tree}} \prod_j A_j^{\text{tree}} = \sum_i c_i \, M_i|_{\text{cut}}$$

On-shell unitarity

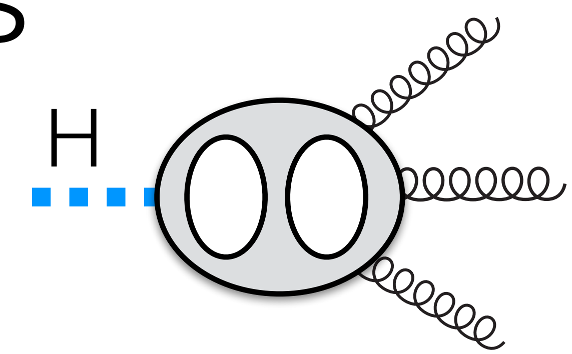


Integration by parts

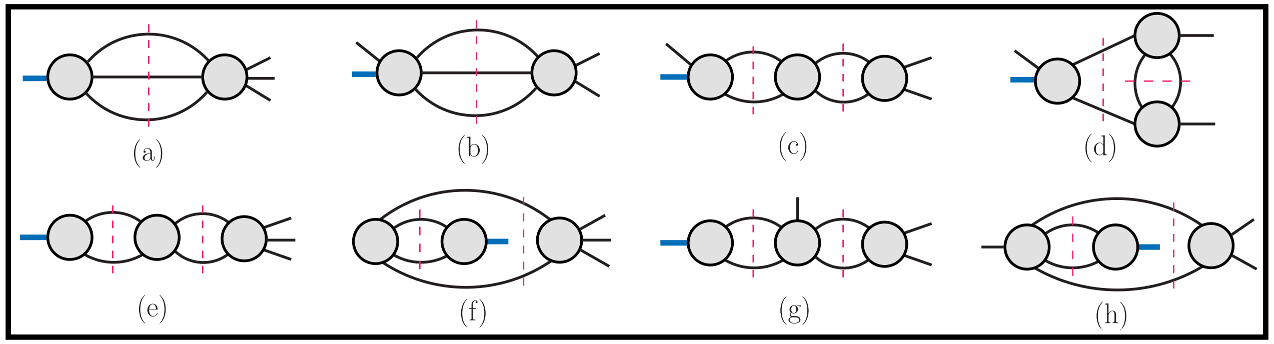
Advantages:

- no need to reconstruct full integrand
- IBP is simplified
- Different cuts provide self-consistency checks

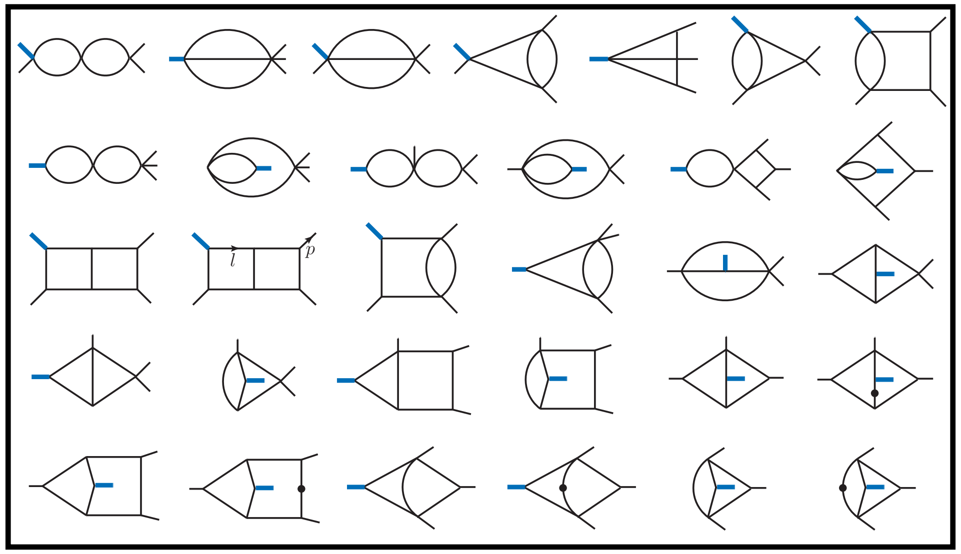
Higgs plus three gluons



All cuts that are needed:



Master integrals are known in terms of 2d Harmonic polylogarithms.



[Gehrmann, Remiddi 2001]

Loop structure of form factors

Form factors have divergences:

IR divergences	soft and collinear divergences
UV divergences	renormalization of coupling g and operators O

The IR and UV are mixed in general in a non-trivial way.

General structure of (bare) amplitudes/form factors:

full result = IR + UV + finite remainder

Loop structure of form factors

In the case of N=4 SYM:

$$\log F_{\text{bare}} = \underbrace{\sum_{\ell=1}^{\infty} g^{2\ell}\left(-\frac{\gamma_{\text{cusp}}^{(\ell)}}{(2\ell\epsilon)^2} - \frac{\mathcal{G}_0^{(\ell)}}{2\ell\epsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{ii+1}}\right)^{\ell\epsilon}}_{\text{Universal infrared divergences}} - \underbrace{(\log Z)}_{\text{UV divergences}} + \text{Fin} + \mathcal{O}(\epsilon)$$

Universal infrared divergences

UV divergences

Similar (but a little more complicated) structure is also known for QCD.

Operator mixing

$$\hat{\mathcal{O}}_I^{\mathrm{R}} = Z_I{}^J \hat{\mathcal{O}}_J^{\mathrm{B}}$$

Dim-6 operator basis:

$$\hat{\mathcal{O}}_I = \{\mathcal{O}_1, \hat{\mathcal{O}}_2, \mathcal{O}_3, \mathcal{O}_4\}$$

$$\mathcal{O}_1 = \mathrm{tr}(F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu)$$

$$\hat{\mathcal{O}}_2 \equiv \partial^2 \mathcal{O}_0$$

$$\mathcal{O}_3 = \mathrm{tr}(D^\rho F_{\rho\mu} D_\sigma F^{\sigma\mu})$$

$$\mathcal{O}_4 = \mathrm{tr}(F_{\mu\rho} D^\rho D_\sigma F^{\sigma\mu})$$

$$\mathcal{O}_2 = \frac{1}{2}\,\partial^2 \mathcal{O}_0 - 4\, g_{\mathrm{YM}}\, \mathcal{O}_1 + 2\, \mathcal{O}_4$$

$$\mathcal{O}_2 = \mathrm{tr}(D_\rho F_{\mu\nu} D^\rho F^{\mu\nu})$$

$$\mathcal{O}_0 = \mathrm{tr}(F_{\mu\nu} F^{\mu\nu})$$

No mixing at one-loop:

$$(Z^{(1)}_{\hat{\mathcal{O}}}) = \frac{1}{\epsilon}\begin{pmatrix} \frac{N_c}{2} + n_f & 0 & 0 & 0 \\ 0 & -\beta_0 & 0 & 0 \\ 0 & 0 & (Z^{(1)})_3{}^3 & 0 \\ 0 & 0 & 0 & \frac{8C_F}{3} + \frac{2n_f}{3} \end{pmatrix}$$

Starting to mix at two-loop: [Jin, GY 1910.09384]

$$(Z^{(2)})\big|_{\frac{1}{\epsilon^2}\text{-part.}} = \frac{1}{2}\big(Z^{(1)}\big)^2 - \frac{1}{2\epsilon} Z^{(1)} \beta_0$$

$$(Z^{(2)}_{\hat{\mathcal{O}}})\big|_{\frac{1}{\epsilon}\text{-part.}} = \frac{1}{\epsilon}\begin{pmatrix} \frac{25N_c^2}{12} + \frac{5N_c n_f}{12} - \frac{3n_f}{4N_c} & -N_c^2 & (Z^{(2)})_1{}^3 & \frac{5}{9} + \frac{5N_c^2}{12} \\ 0 & -\beta_1 & 0 & 0 \\ 0 & 0 & (Z^{(2)})_3{}^3 & n_f\left(\frac{5N_c}{72} + \frac{1}{18N_c}\right) + \frac{1}{36} + \frac{7}{72N_c^2} \\ 0 & \left(-\frac{5N_c}{6} + \frac{2}{9N_c}\right)n_f & (Z^{(2)})_4{}^3 & \frac{80N_c^2}{27} - \frac{20}{9} + \frac{7}{27N_c^2} + \left(\frac{25N_c}{27} + \frac{13}{18N_c}\right)n_f \end{pmatrix}$$

We also study one- and two-loop mixing up to operators of dim-14. [Jin, Ren, GY in prep.]

Loop structure of form factors

General structure of (bare) amplitudes/form factors:

full result = universal IR + UV + finite remainder

The non-trivial information is contained in the finite remainder:

$$F^{(1)} = I^{(1)}(\epsilon)F^{(0)} + F^{(1),\mathrm{fin}} + \mathcal{O}(\epsilon)\,,$$
$$F^{(2)} = I^{(2)}(\epsilon)F^{(0)} + I^{(1)}(\epsilon)F^{(1)} + F^{(2),\mathrm{fin}} + \mathcal{O}(\epsilon)$$

Outline

- Motivations
- Higgs amplitudes
- Hidden analytic structure
- Explanation and implication

Higgs and 3-parton results

external particles	$(1^-, 2^-, 3^-)$	$(1^-, 2^-, 3^+)$	$(1^q, 2^{\bar{q}}, 3^-)$
form factors	$\mathcal{F}^{(l)}_{\mathcal{O}_i,\alpha}$	$\mathcal{F}^{(l)}_{\mathcal{O}_i,\beta}$	$\mathcal{F}^{(l)}_{\mathcal{O}_i,\gamma}$

There are six different color factors: $\mathscr{R}^{(l)}_{\mathcal{O}} = \mathscr{F}^{(l),\text{fin}}_{\mathcal{O}} / \mathscr{F}^{(0)}_{\mathcal{O}}$

$$\mathscr{R}^{(2)}_{\mathcal{O}} = N_c^2 \mathscr{R}^{(2),N_c^2}_{\mathcal{O}} + N_c^0 \mathscr{R}^{(2),N_c^0}_{\mathcal{O}} + \frac{1}{N_c^2} \mathscr{R}^{(2),N_c^{-2}}_{\mathcal{O}} + n_f N_c \mathscr{R}^{(2),n_f N_c}_{\mathcal{O}} + \frac{n_f}{N_c} \mathscr{R}^{(2),n_f/N_c}_{\mathcal{O}} + n_f^2 \mathscr{R}^{(2),n_f^2}_{\mathcal{O}}$$

3-gluon case

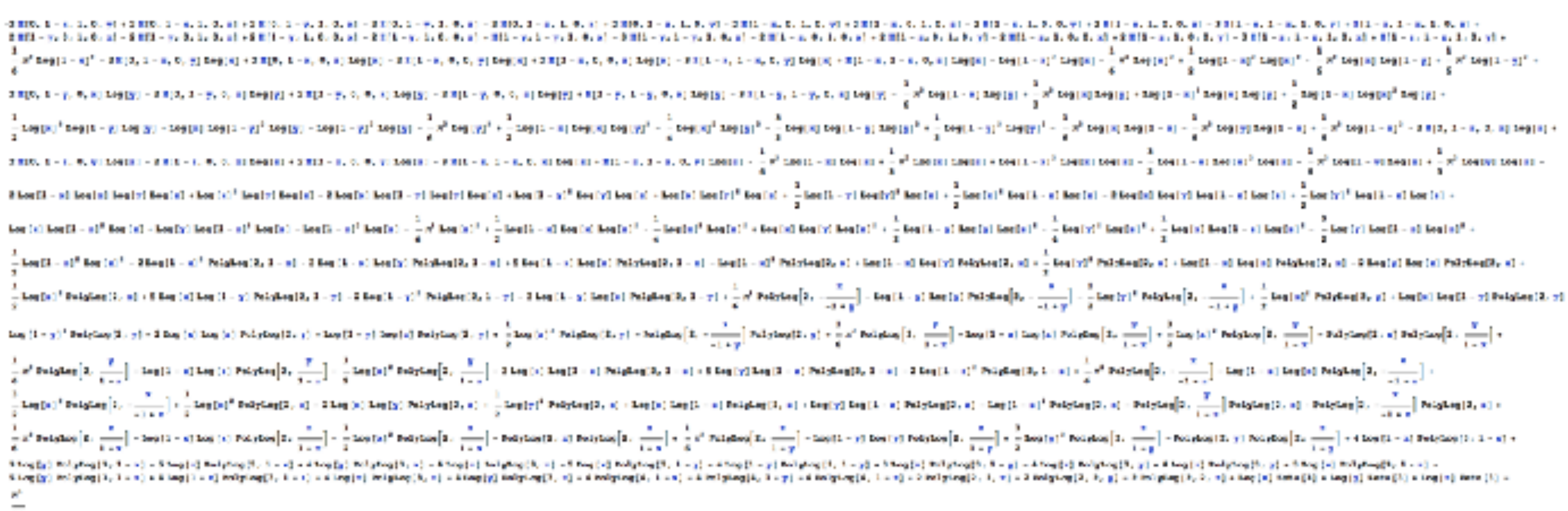
Finite remainder for $\mathcal{O}_1 \to ggg$ $\quad \mathcal{O}_1 = \mathrm{tr}(F^3)$

Organized according to **transcendentality degree**:

$$\mathcal{R}^{(2),N_c^2}_{\mathcal{O}_1,\alpha} = \sum_{i=0}^{4} \Omega^{(2)}_{\mathcal{O}_1;i}$$

Degree-4 part:

$$\Omega^{(2)}_{\mathcal{O}_2;4} =$$



Finite remainder

Simplified results: (using e.g. “symbol”)

$$\Omega^{(2)}_{\mathcal{O}_1;4} = -\frac{3}{2}\mathrm{Li}_4(u) + \frac{3}{4}\mathrm{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{2}\log(w)\mathrm{Li}_3\left(-\frac{u}{v}\right) + \frac{\zeta_2}{8}\left[5\log^2(u) - 2\log(v)\log(w)\right]$$
$$+ \frac{\log^2(u)}{32}\left[\log^2(u) + 2\log^2(v) - 4\log(v)\log(w)\right] - \frac{1}{4}\zeta_4 - \frac{1}{2}\zeta_3\log(-q^2) + \mathrm{perms}(u,v,w)$$

$$\rightarrow \quad \boxed{\Omega^{(2)}_{\mathcal{O}_1;4} = \Omega^{(2),\mathcal{N}=4}_{\mathcal{O}_1;4}}$$

[Q.Jin and GY, PRL 121 101603 (2018)]

[Brandhuber, Kostacinska, Penante, Travaglini 2017]

Higgs plus 3-parton results

external particles	$(1^-, 2^-, 3^-)$	$(1^-, 2^-, 3^+)$	$(1^q, 2^{\bar{q}}, 3^-)$
form factors	$\mathcal{F}^{(l)}_{\mathcal{O}_i,\alpha}$	$\mathcal{F}^{(l)}_{\mathcal{O}_i,\beta}$	$\mathcal{F}^{(l)}_{\mathcal{O}_i,\gamma}$

There are six different color factors: $\mathscr{R}^{(l)}_{\mathcal{O}} = \mathscr{F}^{(l),\mathrm{fin}}_{\mathcal{O}} / \mathscr{F}^{(0)}_{\mathcal{O}}$

$$\mathscr{R}^{(2)}_{\mathcal{O}} = N_c^2 \mathscr{R}^{(2),N_c^2}_{\mathcal{O}} + N_c^0 \mathscr{R}^{(2),N_c^0}_{\mathcal{O}} + \frac{1}{N_c^2} \mathscr{R}^{(2),N_c^{-2}}_{\mathcal{O}} + n_f N_c \mathscr{R}^{(2),n_f N_c}_{\mathcal{O}} + \frac{n_f}{N_c} \mathscr{R}^{(2),n_f/N_c}_{\mathcal{O}} + n_f^2 \mathscr{R}^{(2),n_f^2}_{\mathcal{O}}$$

A different expansion:

$$\mathscr{R}^{(2)}_{\mathcal{O}} = C_A^2 \mathscr{R}^{(2),C_A^2}_{\mathcal{O}} + C_A C_F \mathscr{R}^{(2),C_A C_F}_{\mathcal{O}} + C_F^2 \mathscr{R}^{(2),C_F^2}_{\mathcal{O}} + n_f C_A \mathscr{R}^{(2),n_f C_A}_{\mathcal{O}} + n_f C_F \mathscr{R}^{(2),n_f C_F}_{\mathcal{O}} + n_f^2 \mathscr{R}^{(2),n_f^2}_{\mathcal{O}}$$

$$C_A = N_c\,, \qquad C_F = \frac{N_c^2 - 1}{2N_c}$$

$$\mathscr{R}^{(2)}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar{q}},3^\pm) = C_A^2\mathscr{R}^{(2),C_A^2}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar{q}},3^\pm) + C_AC_F\mathscr{R}^{(2),C_AC_F}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar{q}},3^\pm) + C_F^2\mathscr{R}^{(2),C_F^2}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar{q}},3^\pm)$$

$$\begin{aligned}
\mathcal{R}^{(2),C_A^2}_{\mathcal{O}_0;4} = {}& \frac{4}{3}G(0,z)G(1,z)^3 - \frac{3}{2}G(0,z)^2G(1,z)^2 + 2G(-z,y)^2G(1,z)^2 - 2G(0,z)G(-z,y)G(1,z)^2 - \frac{4}{3}G(-z,y)^3G(1,z) \\
& - 4G(0,0,-z,y)G(1,z) + 2G(-z,0,1-z,y)G(1,z) - 3G(0,1,z)^2 + G(0,y)^2\left(-\frac{3}{2}G(0,z)^2 + G(1,z)G(0,z)\right. \\
& \left. + \frac{1}{2}G(1,z)^2 + \frac{1}{2}G(1-z,y)^2 + G(1,y)(2G(0,z) - 2G(1,z) - 2G(1-z,y)) + (G(0,z)+G(1,z))G(1-z,y)\right) \\
& + G(1-z,y)^2\left(\frac{1}{2}G(0,z)^2 + 2G(1,z)G(0,z) - 2G(1,z)G(-z,y)\right) \\
& + G(1-z,y)\left(-G(1,z)G(0,z)^2 + 4G(1,z)^2G(0,z) - 2G(1,z)G(-z,y)G(0,z) + 4G(1,z)G(-z,y)^2\right) \\
& + G(0,y)\left(3G(1,z)G(0,z)^2 - 2G(1,z)^2G(0,z) - G(1-z,y)^2G(0,z) + \left(2G(0,z)G(1,z) - 2G(1,z)^2\right)G(-z,y)\right. \\
& \left. + G(1-z,y)\left(G(0,z)^2 - 4G(1,z)G(0,z) - 2G(1,z)G(-z,y)\right)\right) \\
& + G(1,y)\left(G(1-z,y)(2G(1,z)G(-z,y) - 2G(0,z)G(1,z)) - G(0,z)G(1,z)^2\right) \\
& + \left(-3G(1,z)^2 + 8G(-z,y)G(1,z) - 4G(1-z,y)^2 - 4G(-z,y)^2 + G(0,y)(-4G(0,z) - 2G(1,z))\right. \\
& \left. + G(1,y)(2G(0,z) + 4G(1,z) + 4G(1-z,y)) + G(1-z,y)(8G(-z,y) - 8G(1,z))\right)G(0,1,z) \\
& + (-2G(0,y)G(0,z) - 2G(1,y)G(0,z) + 4G(1,z)G(0,z) - 2G(1,z)G(-z,y) - 2G(0,1,z))G(0,1-z,y) \\
& + (2G(0,y)G(1,z) - 2G(0,z)G(1,z) + 2G(1,y)G(1,z))G(0,-z,y) + G(0,1,y)(2G(0,z)G(1-z,y) \\
& + G(0,y)(-4G(0,z) + 4G(1,z) + 4G(1-z,y)) - 2G(1,z)G(-z,y) - 2G(-z,1-z,y)) \\
& + G(1-z,1,y)(2G(0,z)G(1,z) - 2G(-z,y)G(1,z) + G(0,y)(-2G(0,z) + 4G(1,z) + 4G(1-z,y)) \\
& - 4G(0,1,z) - 2G(-z,1-z,y)) + \left(-2G(1-z,y)^2 + (4G(1,z) - 2G(0,z))G(1-z,y) - 2G(0,z)G(1,z)\right. \\
& \left. + G(0,y)(2G(0,z) - 2G(1,z) - 2G(1-z,y)) + G(1,y)(2G(1-z,y) - 2G(1,z))\right)G(-z,1-z,y) \\
& + (4G(0,z) - 4G(1,z) - 4G(1-z,y))G(0,0,1,y) \\
& + (6G(0,y) + 2G(0,z) - 6G(1,y) + 10G(1,z) + 8G(1-z,y) - 8G(-z,y))G(0,0,1,z) \\
& + 4G(0,z)G(0,0,1-z,y) + (6G(0,y) + 4G(0,z) - 6G(1,y) + 4G(1,z) + 8G(1-z,y) - 8G(-z,y))G(0,1,1,z) \\
& + (2G(0,z) - 4G(1,z) - 4G(1-z,y))G(0,1-z,1,y) + 4G(0,z)G(0,1-z,1-z,y) \\
& + (2G(0,y) - 2G(0,z) + 2G(1,y) + 2G(1,z))G(0,-z,1-z,y) \\
& + (-2G(0,y) - 4G(1,z) - 4G(1-z,y))G(1-z,0,1,y) - 4G(0,y)G(1-z,1-z,1,y) \\
& + (2G(1,z) + 2G(1-z,y))G(-z,0,1,y) + (2G(1,z) + 2G(1-z,y))G(-z,1-z,1,y) \\
& + (-4G(1,y) - 4G(1,z) + 8G(1-z,y))G(-z,1-z,1-z,y) + 8G(1-z,y)G(-z,-z,1-z,y) \\
& - 12G(0,0,0,1,z) - 8G(0,0,1,1,z) - 4G(0,0,-z,1-z,y) - 2G(0,1,1,1,z) + 2G(0,1-z,0,1,y) \\
& + 4G(0,1-z,1-z,1,y) - 2G(0,1-z,-z,1-z,y) + 4G(1-z,0,0,1,y) + 4G(1-z,0,1-z,1,y) \\
& + 4G(1-z,1-z,0,1,y) - 2G(1-z,-z,0,1,y) - 2G(1-z,-z,1-z,1,y) - 12G(-z,1-z,1-z,1-z,y) \\
& - 8G(-z,-z,1-z,1-z,y) - 8G(-z,-z,-z,1-z,y) + \left(-3G(1,z)^2 + 3G(0,z)G(1,z) + 2G(-z,y)G(1,z)\right. \\
& - 4G(1-z,y)^2 + (3G(0,z) - 8G(1,z))G(1-z,y) + G(1,y)(-2G(0,z) + 2G(1,z) + 2G(1-z,y)) \\
& \left. + G(0,y)(-G(0,z) + G(1,z) + 3G(1-z,y)) - 2G(0,1-z,y) - 2G(1-z,1,y) + 2G(-z,1-z,y)\right)\zeta(2) \\
& + \frac{119\zeta(4)}{8} + (-7G(0,y) - 7G(0,z) + 6G(1,y) + 7G(1,z) + G(1-z,y))\zeta(3)\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}^{(2),C_AC_F}_{\mathcal{O}_0;4} = {}& -\frac{4}{3}G(0,z)G(1,z)^3 + 3G(0,z)^2G(1,z)^2 - 8G(0,0,-z,y)G(1,z) - 10G(-z,0,1-z,y)G(1,z) \\
& - 2G(0,1,y)^2 + 11G(0,1,z)^2 + \left(2G(0,z)^2G(1,z) - 2G(0,z)G(1,z)^2\right)G(1-z,y) \\
& + G(0,y)^2\left(3G(0,z)^2 - 2G(1,z)G(0,z) - 2G(1-z,y)G(0,z) + G(1,y)(-4G(0,z) + 4G(1,z) + 4G(1-z,y))\right) \\
& + G(0,y)\left(-6G(1,z)G(0,z)^2 + 3G(1,z)^2G(0,z) + \left(4G(0,z)G(1,z) - 2G(0,z)^2\right)G(1-z,y)\right. \\
& \left. + G(1,y)(2G(0,z)G(1,z) + 2G(0,z)G(1-z,y))\right) \\
& + G(1,y)\left(3G(0,z)G(1,z)^2 + G(1-z,y)(6G(0,z)G(1,z) - 10G(1,z)G(-z,y))\right) \\
& + \left(-2G(0,y)^2 + (8G(0,z) - 4G(1,y) + 10G(1,z) + 4G(1-z,y))G(0,y) - 5G(1,z)^2 - 8G(0,z)G(1,z)\right. \\
& \left. + G(1,y)(-6G(0,z) - 16G(1,z) - 16G(1-z,y)) + (-4G(0,z) - 4G(1,z))G(1-z,y)\right)G(0,1,z) \\
& + \left(-2G(0,z)^2 + 2G(0,y)G(0,z) + 6G(1,y)G(0,z) + 8G(1-z,y)G(0,z) + 10G(1,z)G(-z,y)\right. \\
& \left. + 2G(0,1,z)\right)G(0,1-z,y) + (2G(0,y)G(1,z) - 2G(0,z)G(1,z) - 10G(1,y)G(1,z))G(0,-z,y) \\
& + G(1,y)(10G(1,z) - 10G(1-z,y))G(-z,1-z,y) \\
& + G(1-z,1,y)\left(-2G(0,y)^2 + (2G(0,z) - 4G(1,y) - 8G(1,z) - 8G(1-z,y))G(0,y) - 6G(0,z)G(1,z)\right. \\
& \left. + 10G(1,z)G(-z,y) + 16G(0,1,z) + 10G(-z,1-z,y)\right) \\
& + G(0,1,y)(-2G(0,z)G(1,z) + 10G(-z,y)G(1,z) + G(0,y)(6G(0,z) - 4G(1,y) - 6G(1,z) - 6G(1-z,y)) \\
& - 8G(0,z)G(1-z,y) + 4G(0,1,z) - 8G(1-z,1,y) + 10G(-z,1-z,y)) \\
& + (4G(0,y) - 4G(0,z) + 8G(1,y) + 4G(1,z) + 4G(1-z,y))G(0,0,1,y) \\
& + (-10G(0,y) + 6G(0,z) + 22G(1,y) + 2G(1,z) + 4G(1-z,y))G(0,0,1,z) - 4G(0,z)G(0,0,1-z,y) \\
& + 8G(0,y)G(0,1,1,y) + (-26G(0,y) + 26G(1,y) + 28G(1,z) + 12G(1-z,y))G(0,1,1,z) \\
& + (-4G(0,y) - 2G(0,z) + 4G(1,y) + 8G(1,z) + 8G(1-z,y))G(0,1-z,1,y) \\
& - 16G(0,z)G(0,1-z,1-z,y) + (2G(0,y) - 2G(0,z) - 10G(1,y) - 10G(1,z))G(0,-z,1-z,y) \\
& + (2G(0,y) + 4G(0,z) + 4G(1,y) + 8G(1,z) + 8G(1-z,y))G(1-z,0,1,y) + 12G(0,y)G(1-z,1,1,y) \\
& + 16G(0,y)G(1-z,1-z,1,y) + (-10G(1,z) - 10G(1-z,y))G(-z,0,1,y) \\
& + (-10G(1,z) - 10G(1-z,y))G(-z,1-z,1,y) + 20G(1,y)G(-z,1-z,1-z,y) \\
& - 12G(0,0,0,1,y) - 20G(0,0,0,1,z) - 8G(0,0,1,1,y) - 28G(0,0,1,1,z) + 12G(0,0,1-z,1,y) \\
& - 8G(0,0,-z,1-z,y) - 46G(0,1,1,1,z) + 8G(0,1,1-z,1,y) + 6G(0,1-z,0,1,y) \\
& + 4G(0,1-z,1,1,y) - 16G(0,1-z,1-z,1,y) + 10G(0,1-z,-z,1-z,y) + 4G(1-z,0,1,1,y) \\
& - 16G(1-z,0,1-z,1,y) - 16G(1-z,1-z,0,1,y) + 10G(1-z,-z,0,1,y) + 10G(1-z,-z,1-z,1,y) \\
& + \left(-2G(0,y)^2 + (-8G(0,z) + 2G(1,y) + 10G(1,z) + 4G(1-z,y))G(0,y) - 2G(0,z)^2 - 5G(1,z)^2\right. \\
& + 6G(0,z)G(1,z) + G(1,y)(6G(0,z) - 6G(1,z) - 6G(1-z,y)) + (4G(0,z) - 4G(1,z))G(1-z,y) \\
& \left. - 2G(0,1,y) + 2G(0,1-z,y) + 2G(1-z,1,y)\right)\zeta(2) \\
& - \frac{311\zeta(4)}{4} + (22G(0,y) + 22G(0,z) - 18G(1,y) - 52G(1,z) - 34G(1-z,y))\zeta(3)\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}^{(2),C_F^2}_{\mathcal{O}_0;4} = {}& \left(-G(0,z)^2 + 2G(1,z)G(0,z) + 2G(1-z,y)G(0,z) + 2G(1,y)^2 + G(1,y)(-4G(1,z) - 4G(1-z,y))\right)G(0,y)^2 \\
& + \left(-\frac{4}{3}G(1,y)^3 - 2G(0,z)G(1,z)^2 + 2G(0,z)^2G(1,z) + \left(2G(0,z)^2 - 4G(0,z)G(1,z)\right)G(1-z,y)\right)G(0,y) \\
& - 12G(1-z,1,1,y)G(0,y) - 16G(1-z,1-z,1,y)G(0,y) - G(0,z)^2G(1,z)^2 \\
& + 2G(0,1,y)^2 - 8G(0,1,z)^2 + \left(2G(0,z)G(1,z)^2 - 2G(0,z)^2G(1,z)\right)G(1-z,y) \\
& + G(1,y)\left(G(1-z,y)(8G(1,z)G(-z,y) - 4G(0,z)G(1,z)) - 2G(0,z)G(1,z)^2\right) \\
& + \left(2G(0,y)^2 + (-4G(0,z) + 4G(1,y) - 8G(1,z) - 4G(1-z,y))G(0,y) + 8G(1,z)^2 + 8G(0,z)G(1,z)\right. \\
& \left. + (4G(0,z) + 4G(1,z))G(1-z,y) + G(1,y)(4G(0,z) + 12G(1,z) + 12G(1-z,y))\right)G(0,1,z) \\
& + \left(2G(0,z)^2 - 4G(0,y)G(0,z) - 4G(1,y)G(0,z) - 8G(1-z,y)G(0,z) - 8G(1,z)G(-z,y)\right)G(0,1-z,y) \\
& + 8G(1,y)G(1,z)G(0,-z,y) \\
& + G(1-z,1,y)\left(2G(0,y)^2 + (-4G(0,z) + 4G(1,y) + 8G(1,z) + 8G(1-z,y))G(0,y) + 4G(0,z)G(1,z)\right. \\
& \left. - 8G(1,z)G(-z,y) - 12G(0,1,z) - 8G(-z,1-z,y)\right) \\
& + G(0,1,y)\left(4G(1,y)^2 + 4G(0,z)G(1-z,y) + G(0,y)(4G(1,y) + 4G(1,z) + 4G(1-z,y)) - 8G(1,z)G(-z,y)\right. \\
& \left. - 4G(0,1,z) + 8G(1-z,1,y) - 8G(-z,1-z,y)\right) + G(1,y)(8G(1-z,y) - 8G(1,z))G(-z,1-z,y) \\
& + (-8G(0,y) - 16G(1,y))G(0,0,1,y) + (4G(0,y) - 8G(0,z) - 16G(1,y) - 12G(1,z) - 4G(1-z,y))G(0,0,1,z) \\
& + 4G(0,z)G(0,0,1-z,y) + 8G(1,z)G(0,0,-z,y) + (-8G(0,y) - 8G(1,y))G(0,1,1,y) \\
& + (20G(0,y) - 4G(0,z) - 20G(1,y) - 32G(1,z) - 12G(1-z,y))G(0,1,1,z) \\
& + (4G(0,y) + 4G(0,z) - 4G(1,y) - 8G(1,z) - 8G(1-z,y))G(0,1-z,1,y) + 16G(0,z)G(0,1-z,1-z,y) \\
& + (8G(1,y) + 8G(1,z))G(0,-z,1-z,y) + (-4G(1,y) - 8G(1,z) - 8G(1-z,y))G(1-z,0,1,y) \\
& + (8G(1,z) + 8G(1-z,y))G(-z,0,1,y) + 8G(1,z)G(-z,0,1-z,y) \\
& + (8G(1,z) + 8G(1-z,y))G(-z,1-z,1,y) - 16G(1,y)G(-z,1-z,1-z,y) + 24G(0,0,0,1,y) \\
& + 32G(0,0,0,1,z) + 16G(0,0,1,1,y) + 36G(0,0,1,1,z) - 12G(0,0,1-z,1,y) + 8G(0,0,-z,1-z,y) \\
& + 8G(0,1,1,1,y) + 48G(0,1,1,1,z) - 8G(0,1,1-z,1,y) - 8G(0,1-z,0,1,y) - 4G(0,1-z,1,1,y) \\
& + 16G(0,1-z,1-z,1,y) - 8G(0,1-z,-z,1-z,y) - 4G(1-z,0,0,1,y) - 4G(1-z,0,1,1,y) \\
& + 16G(1-z,0,1-z,1,y) + 16G(1-z,1-z,0,1,y) - 8G(1-z,-z,0,1,y) - 8G(1-z,-z,1-z,1,y) \\
& + \left(2G(0,y)^2 + (12G(0,z) - 8G(1,y) - 8G(1,z) - 4G(1-z,y))G(0,y) + 2G(0,z)^2 + 4G(1,y)^2 + 8G(1,z)^2\right. \\
& \left. - 12G(0,z)G(1,z) + (4G(1,z) - 4G(0,z))G(1-z,y) + G(1,y)(-4G(0,z) + 4G(1,z) + 4G(1-z,y))\right)\zeta(2) \\
& + 69\zeta(4) + (-16G(0,y) - 16G(0,z) + 12G(1,y) + 44G(1,z) + 32G(1-z,y))\zeta(3)\,.
\end{aligned}$$

$$G(1-z,-z,1-z,1,y)$$

Multiple polyLogarithm

$$\mathscr{R}^{(2)}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar q},3^\pm) = C_A^2\mathscr{R}^{(2),C_A^2}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar q},3^\pm) + C_AC_F\mathscr{R}^{(2),C_AC_F}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar q},3^\pm) + C_F^2\mathscr{R}^{(2),C_F^2}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar q},3^\pm)$$

$$\begin{aligned}
\mathcal{R}^{(2),C_A^2}_{\mathcal{O}_0;4} ={}& \frac{4}{3}G(0,z)G(1,z)^3 - \frac{3}{2}G(0,z)^2G(1,z)^2 + 2G(-z,y)^2G(1,z)^2 - 2G(0,z)G(-z,y)G(1,z)^2 - \frac{4}{3}G(-z,y)^3G(1,z) \\
&- 4G(0,0,-z,y)G(1,z) + 2G(-z,0,1-z,y)G(1,z) - 3G(0,1,z)^2 + G(0,y)^2\left(-\frac{3}{2}G(0,z)^2 + G(1,z)G(0,z)\right. \\
&\left.+\frac{1}{2}G(1,z)^2 + \frac{1}{2}G(1-z,y)^2 + G(1,y)(2G(0,z) - 2G(1,z) - 2G(1-z,y)) + (G(0,z)+G(1,z))G(1-z,y)\right) \\
&+ G(1-z,y)^2\left(\frac{1}{2}G(0,z)^2 + 2G(1,z)G(0,z) - 2G(1,z)G(-z,y)\right) \\
&+ G(1-z,y)\left(-G(1,z)G(0,z)^2 + 4G(1,z)^2G(0,z) - 2G(1,z)G(-z,y)G(0,z) + 4G(1,z)G(-z,y)^2\right) \\
&+ G(0,y)\left(3G(1,z)G(0,z)^2 - 2G(1,z)^2G(0,z) - G(1-z,y)^2G(0,z) + \left(2G(0,z)G(1,z) - 2G(1,z)^2\right)G(-z,y)\right. \\
&\left.\qquad + G(1-z,y)\left(G(0,z)^2 - 4G(1,z)G(0,z) - 2G(1,z)G(-z,y)\right)\right) \\
&+ G(1,y)\left(G(1-z,y)(2G(1,z)G(-z,y) - 2G(0,z)G(1,z)) - G(0,z)G(1,z)^2\right) \\
&+ \left(-3G(1,z)^2 + 8G(-z,y)G(1,z) - 4G(1-z,y)^2 - 4G(-z,y)^2 + G(0,y)(-4G(0,z) - 2G(1,z))\right. \\
&\left.\qquad + G(1,y)(2G(0,z) + 4G(1,z) + 4G(1-z,y)) + G(1-z,y)(8G(-z,y) - 8G(1,z))\right)G(0,1,z) \\
&+ (-2G(0,y)G(0,z) - 2G(1,y)G(0,z) + 4G(1,z)G(0,z) - 2G(1,z)G(-z,y) - 2G(0,1,z))G(0,1-z,y) \\
&+ (2G(0,y)G(1,z) - 2G(0,z)G(1,z) + 2G(1,y)G(1,z))G(0,-z,y) + G(0,1,y)(2G(0,z)G(1-z,y) \\
&\qquad + G(0,y)(-4G(0,z) + 4G(1,z) + 4G(1-z,y)) - 2G(1,z)G(-z,y) - 2G(-z,1-z,y)) \\
&+ G(1-z,1,y)(2G(0,z)G(1,z) - 2G(-z,y)G(1,z) + G(0,y)(-2G(0,z) + 4G(1,z) + 4G(1-z,y)) \\
&\qquad - 4G(0,1,z) - 2G(-z,1-z,y)) + \left(-2G(1-z,y)^2 + (4G(1,z) - 2G(0,z))G(1-z,y) - 2G(0,z)G(1,z)\right. \\
&\left.\qquad + G(0,y)(2G(0,z) - 2G(1,z) - 2G(1-z,y)) + G(1,y)(2G(1-z,y) - 2G(1,z))\right)G(-z,1-z,y) \\
&+ (4G(0,z) - 4G(1,z) - 4G(1-z,y))G(0,0,1,y) \\
&+ (6G(0,y) + 2G(0,z) - 6G(1,y) + 10G(1,z) + 8G(1-z,y) - 8G(-z,y))G(0,0,1,z) \\
&+ 4G(0,z)G(0,0,1-z,y) + (6G(0,y) + 4G(0,z) - 6G(1,y) + 4G(1,z) + 8G(1-z,y) - 8G(-z,y))G(0,1,1,z) \\
&+ (2G(0,z) - 4G(1,z) - 4G(1-z,y))G(0,1-z,1,y) + 4G(0,z)G(0,1-z,1-z,y) \\
&+ (2G(0,y) - 2G(0,z) + 2G(1,y) + 2G(1,z))G(0,-z,1-z,y) \\
&+ (-2G(0,y) - 4G(1,z) - 4G(1-z,y))G(1-z,0,1,y) - 4G(0,y)G(1-z,1-z,1,y) \\
&+ (2G(1,z) + 2G(1-z,y))G(-z,0,1,y) + (2G(1,z) + 2G(1-z,y))G(-z,1-z,1,y) \\
&+ (-4G(1,y) - 4G(1,z) + 8G(1-z,y))G(-z,1-z,1-z,y) + 8G(1-z,y)G(-z,-z,1-z,y) \\
&- 12G(0,0,0,1,z) - 8G(0,0,1,1,z) - 4G(0,0,-z,1-z,y) - 2G(0,1,1,1,z) + 2G(0,1-z,0,1,y) \\
&+ 4G(0,1-z,1-z,1,y) - 2G(0,1-z,-z,1-z,y) + 4G(1-z,0,0,1,y) + 4G(1-z,0,1-z,1,y) \\
&+ 4G(1-z,1-z,0,1,y) - 2G(1-z,-z,0,1,y) - 2G(1-z,-z,1-z,1,y) - 12G(-z,1-z,1-z,1-z,y) \\
&- 8G(-z,-z,1-z,1-z,y) - 8G(-z,-z,-z,1-z,y) + \left(-3G(1,z)^2 + 3G(0,z)G(1,z) + 2G(-z,y)G(1,z)\right. \\
&\qquad - 4G(1-z,y)^2 + (3G(0,z) - 8G(1,z))G(1-z,y) + G(1,y)(-2G(0,z) + 2G(1,z) + 2G(1-z,y)) \\
&\left.\qquad + G(0,y)(-G(0,z) + G(1,z) + 3G(1-z,y)) - 2G(0,1-z,y) - 2G(1-z,1,y) + 2G(-z,1-z,y)\right)\zeta(2) \\
&+ \frac{119\zeta(4)}{8} + (-7G(0,y) - 7G(0,z) + 6G(1,y) + 7G(1,z) + G(1-z,y))\zeta(3)\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}^{(2),C_AC_F}_{\mathcal{O}_0;4} ={}& -\frac{4}{3}G(0,z)G(1,z)^3 + 3G(0,z)^2G(1,z)^2 - 8G(0,0,-z,y)G(1,z) - 10G(-z,0,1-z,y)G(1,z) \\
&- 2G(0,1,y)^2 + 11G(0,1,z)^2 + \left(2G(0,z)^2G(1,z) - 2G(0,z)G(1,z)^2\right)G(1-z,y) \\
&+ G(0,y)^2\left(3G(0,z)^2 - 2G(1,z)G(0,z) - 2G(1-z,y)G(0,z) + G(1,y)(-4G(0,z) + 4G(1,z) + 4G(1-z,y))\right) \\
&+ G(0,y)\left(-6G(1,z)G(0,z)^2 + 3G(1,z)^2G(0,z) + \left(4G(0,z)G(1,z) - 2G(0,z)^2\right)G(1-z,y)\right. \\
&\left.\qquad + G(1,y)(2G(0,z)G(1,z) + 2G(0,z)G(1-z,y))\right) \\
&+ G(1,y)\left(3G(0,z)G(1,z)^2 + G(1-z,y)(6G(0,z)G(1,z) - 10G(1,z)G(-z,y))\right) \\
&+ \left(-2G(0,y)^2 + (8G(0,z) - 4G(1,y) + 10G(1,z) + 4G(1-z,y))G(0,y) - 5G(1,z)^2 - 8G(0,z)G(1,z)\right. \\
&\left.\qquad + G(1,y)(-6G(0,z) - 16G(1,z) - 16G(1-z,y)) + (-4G(0,z) - 4G(1,z))G(1-z,y)\right)G(0,1,z) \\
&+ \left(-2G(0,z)^2 + 2G(0,y)G(0,z) + 6G(1,y)G(0,z) + 8G(1-z,y)G(0,z) + 10G(1,z)G(-z,y)\right. \\
&\left.\qquad + 2G(0,1,z)\right)G(0,1-z,y) + (2G(0,y)G(1,z) - 2G(0,z)G(1,z) - 10G(1,y)G(1,z))G(0,-z,y) \\
&+ G(1,y)(10G(1,z) - 10G(1-z,y))G(-z,1-z,y) \\
&+ G(1-z,1,y)\left(-2G(0,y)^2 + (2G(0,z) - 4G(1,y) - 8G(1,z) - 8G(1-z,y))G(0,y) - 6G(0,z)G(1,z)\right. \\
&\left.\qquad + 10G(1,z)G(-z,y) + 16G(0,1,z) + 10G(-z,1-z,y)\right) \\
&+ G(0,1,y)(-2G(0,z)G(1,z) + 10G(-z,y)G(1,z) + G(0,y)(6G(0,z) - 4G(1,y) - 6G(1,z) - 6G(1-z,y)) \\
&\qquad - 8G(0,z)G(1-z,y) + 4G(0,1,z) - 8G(1-z,1,y) + 10G(-z,1-z,y)) \\
&+ (4G(0,y) - 4G(0,z) + 8G(1,y) + 4G(1,z) + 4G(1-z,y))G(0,0,1,y) \\
&+ (-10G(0,y) + 6G(0,z) + 22G(1,y) + 2G(1,z) + 4G(1-z,y))G(0,0,1,z) - 4G(0,z)G(0,0,1-z,y) \\
&+ 8G(0,y)G(0,1,1,y) + (-26G(0,y) + 26G(1,y) + 28G(1,z) + 12G(1-z,y))G(0,1,1,z) \\
&+ (-4G(0,y) - 2G(0,z) + 4G(1,y) + 8G(1,z) + 8G(1-z,y))G(0,1-z,1,y) \\
&- 16G(0,z)G(0,1-z,1-z,y) + (2G(0,y) - 2G(0,z) - 10G(1,y) - 10G(1,z))G(0,-z,1-z,y) \\
&+ (2G(0,y) + 4G(0,z) + 4G(1,y) + 8G(1,z) + 8G(1-z,y))G(1-z,0,1,y) + 12G(0,y)G(1-z,1,1,y) \\
&+ 16G(0,y)G(1-z,1-z,1,y) + (-10G(1,z) - 10G(1-z,y))G(-z,0,1,y) \\
&+ (-10G(1,z) - 10G(1-z,y))G(-z,1-z,1,y) + 20G(1,y)G(-z,1-z,1-z,y) \\
&- 12G(0,0,0,1,y) - 20G(0,0,0,1,z) - 8G(0,0,1,1,y) - 28G(0,0,1,1,z) + 12G(0,0,1-z,1,y) \\
&- 8G(0,0,-z,1-z,y) - 46G(0,1,1,1,z) + 8G(0,1,1-z,1,y) + 6G(0,1-z,0,1,y) \\
&+ 4G(0,1-z,1,1,y) - 16G(0,1-z,1-z,1,y) + 10G(0,1-z,-z,1-z,y) + 4G(1-z,0,1,1,y) \\
&- 16G(1-z,0,1-z,1,y) - 16G(1-z,1-z,0,1,y) + 10G(1-z,-z,0,1,y) + 10G(1-z,-z,1-z,1,y) \\
&+ \left(-2G(0,y)^2 + (-8G(0,z) + 2G(1,y) + 10G(1,z) + 4G(1-z,y))G(0,y) - 2G(0,z)^2 - 5G(1,z)^2\right. \\
&\qquad + 6G(0,z)G(1,z) + G(1,y)(6G(0,z) - 6G(1,z) - 6G(1-z,y)) + (4G(0,z) - 4G(1,z))G(1-z,y) \\
&\left.\qquad - 2G(0,1,y) + 2G(0,1-z,y) + 2G(1-z,1,y)\right)\zeta(2) \\
&- \frac{311\zeta(4)}{4} + (22G(0,y) + 22G(0,z) - 18G(1,y) - 52G(1,z) - 34G(1-z,y))\zeta(3)\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}^{(2),C_F^2}_{\mathcal{O}_0;4} ={}& \left(-G(0,z)^2 + 2G(1,z)G(0,z) + 2G(1-z,y)G(0,z) + 2G(1,y)^2 + G(1,y)(-4G(1,z) - 4G(1-z,y))\right)G(0,y)^2 \\
&+ \left(-\frac{4}{3}G(1,y)^3 - 2G(0,z)G(1,z)^2 + 2G(0,z)^2G(1,z) + \left(2G(0,z)^2 - 4G(0,z)G(1,z)\right)G(1-z,y)\right)G(0,y) \\
&- 12G(1-z,1,1,y)G(0,y) - 16G(1-z,1-z,1,y)G(0,y) - G(0,z)^2G(1,z)^2 \\
&+ 2G(0,1,y)^2 - 8G(0,1,z)^2 + \left(2G(0,z)G(1,z)^2 - 2G(0,z)^2G(1,z)\right)G(1-z,y) \\
&+ G(1,y)\left(G(1-z,y)(8G(1,z)G(-z,y) - 4G(0,z)G(1,z)) - 2G(0,z)G(1,z)^2\right) \\
&+ \left(2G(0,y)^2 + (-4G(0,z) + 4G(1,y) - 8G(1,z) - 4G(1-z,y))G(0,y) + 8G(1,z)^2 + 8G(0,z)G(1,z)\right. \\
&\left.\qquad + (4G(0,z) + 4G(1,z))G(1-z,y) + G(1,y)(4G(0,z) + 12G(1,z) + 12G(1-z,y))\right)G(0,1,z) \\
&+ \left(2G(0,z)^2 - 4G(0,y)G(0,z) - 4G(1,y)G(0,z) - 8G(1-z,y)G(0,z) - 8G(1,z)G(-z,y)\right)G(0,1-z,y) \\
&+ 8G(1,y)G(1,z)G(0,-z,y) \\
&+ G(1-z,1,y)\left(2G(0,y)^2 + (-4G(0,z) + 4G(1,y) + 8G(1,z) + 8G(1-z,y))G(0,y) + 4G(0,z)G(1,z)\right. \\
&\left.\qquad - 8G(1,z)G(-z,y) - 12G(0,1,z) - 8G(-z,1-z,y)\right) \\
&+ G(0,1,y)\left(4G(1,y)^2 + 4G(0,z)G(1-z,y) + G(0,y)(4G(1,y) + 4G(1,z) + 4G(1-z,y)) - 8G(1,z)G(-z,y)\right. \\
&\left.\qquad - 4G(0,1,z) + 8G(1-z,1,y) - 8G(-z,1-z,y)\right) + G(1,y)(8G(1-z,y) - 8G(1,z))G(-z,1-z,y) \\
&+ (-8G(0,y) - 16G(1,y))G(0,0,1,y) + (4G(0,y) - 8G(0,z) - 16G(1,y) - 12G(1,z) - 4G(1-z,y))G(0,0,1,z) \\
&+ 4G(0,z)G(0,0,1-z,y) + 8G(1,z)G(0,0,-z,y) + (-8G(0,y) - 8G(1,y))G(0,1,1,y) \\
&+ (20G(0,y) - 4G(0,z) - 20G(1,y) - 32G(1,z) - 12G(1-z,y))G(0,1,1,z) \\
&+ (4G(0,y) + 4G(0,z) - 4G(1,y) - 8G(1,z) - 8G(1-z,y))G(0,1-z,1,y) + 16G(0,z)G(0,1-z,1-z,y) \\
&+ (8G(1,y) + 8G(1,z))G(0,-z,1-z,y) + (-4G(1,y) - 8G(1,z) - 8G(1-z,y))G(1-z,0,1,y) \\
&+ (8G(1,z) + 8G(1-z,y))G(-z,0,1,y) + 8G(1,z)G(-z,0,1-z,y) \\
&+ (8G(1,z) + 8G(1-z,y))G(-z,1-z,1,y) - 16G(1,y)G(-z,1-z,1-z,y) + 24G(0,0,0,1,y) \\
&+ 32G(0,0,0,1,z) + 16G(0,0,1,1,y) + 36G(0,0,1,1,z) - 12G(0,0,1-z,1,y) + 8G(0,0,-z,1-z,y) \\
&+ 8G(0,1,1,1,y) + 48G(0,1,1,1,z) - 8G(0,1,1-z,1,y) - 8G(0,1-z,0,1,y) - 4G(0,1-z,1,1,y) \\
&+ 16G(0,1-z,1-z,1,y) - 8G(0,1-z,-z,1-z,y) - 4G(1-z,0,0,1,y) - 4G(1-z,0,1,1,y) \\
&+ 16G(1-z,0,1-z,1,y) + 16G(1-z,1-z,0,1,y) - 8G(1-z,-z,0,1,y) - 8G(1-z,-z,1-z,1,y) \\
&+ \left(2G(0,y)^2 + (12G(0,z) - 8G(1,y) - 8G(1,z) - 4G(1-z,y))G(0,y) + 2G(0,z)^2 + 4G(1,y)^2 + 8G(1,z)^2\right. \\
&\left.\qquad - 12G(0,z)G(1,z) + (4G(1,z) - 4G(0,z))G(1-z,y) + G(1,y)(-4G(0,z) + 4G(1,z) + 4G(1-z,y))\right)\zeta(2) \\
&+ 69\zeta(4) + (-16G(0,y) - 16G(0,z) + 12G(1,y) + 44G(1,z) + 32G(1-z,y))\zeta(3)\,.
\end{aligned}$$

$$G(1-z,-z,1-z,1,y)$$

Multiple polyLogarithm

Sum three terms together we get the 3-gluon result !

(not obvious in Nc expansion!)

$$\boxed{\mathscr{R}^{(2),C_A^2}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar q},3^\pm) + \mathscr{R}^{(2),C_AC_F}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar q},3^\pm) + \mathscr{R}^{(2),C_F^2}_{\mathrm{tr}(F^2);4}(1^q,2^{\bar q},3^\pm) = \mathscr{R}^{(2)}_{\mathrm{tr}(F^2);4}(1^-,2^-,3^\pm)}$$

$$\mathscr{R}^{(2)}_{\mathrm{tr}(F^2);4}(1^-,2^-,3^\pm) = -2\left[J_4\left(-\frac{uv}{w}\right) + J_4\left(-\frac{vw}{u}\right) + J_4\left(-\frac{wu}{v}\right)\right] - 8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-\frac{1}{u_i}\right) + \frac{\log^4 u_i}{4!}\right] - 2\left[\sum_{i=1}^{3}\mathrm{Li}_2\left(1-\frac{1}{u_i}\right)\right]^2$$

$$+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2 + 2(J_2^2 - \zeta_2 J_2) - \frac{\log^4(uvw)}{4!} - \zeta_3\log(uvw) - \frac{123}{8}\zeta_4$$

For length-3 operator, the correspondence is the same !

$$\mathcal{O}_4 = \mathrm{tr}(F_{\mu\rho} D^\rho D_\sigma F^{\sigma\mu})$$

$$\to \ \mathcal{O}'_4 = g F_{\mu\nu} D^\mu \sum_{i,j=1}^{n_f} (\bar\psi_i \gamma^\nu T^A \psi_i)$$

$$\mathcal{R}^{(2)}_{\mathcal{O}_4;4}(1^q,2^{\bar q},3^\pm) = C_A^2 \mathcal{R}^{(2),C_A^2}_{\mathcal{O}_4;4}(1^q,2^{\bar q},3^\pm) + C_A C_F \mathcal{R}^{(2),C_A C_F}_{\mathcal{O}_4;4}(1^q,2^{\bar q},3^\pm) + C_F^2 \mathcal{R}^{(2),C_F^2}_{\mathcal{O}_4;4}(1^q,2^{\bar q},3^\pm)$$

$$\begin{aligned}
\mathcal{R}^{(2),C_A^2}_{\mathcal{O}_4;4} = {}& \zeta(2)\Big(G(0,y)(3G(1-z,y)-2G(0,z)+3G(1,z)) - G(1-z,y)^2 + (3G(0,z)-4G(1,z))G(1-z,y) \\
& - 2G(0,1-z,y) - 2G(1-z,1,y) - \tfrac{1}{2}G(0,y)^2 - \tfrac{1}{2}G(0,z)^2 - 2G(1,z)^2 + 3G(0,z)G(1,z) - 2G(0,1,z)\Big) \\
& + \zeta(3)(11G(1-z,y) - 6G(0,y) - 6G(0,z) + 11G(1,z)) - 3G(1,z)G(0,0,-z,y) \\
& + \left(\tfrac{1}{4}G(0,z)^2 + \tfrac{1}{2}G(1,z)G(0,z)\right)G(1-z,y)^2 + \left(\tfrac{3}{2}G(0,z)G(1,z)^2 - \tfrac{1}{2}G(0,z)^2G(1,z)\right)G(1-z,y) \\
& + G(0,y)^2\left(\tfrac{1}{4}G(1-z,y)^2 + \left(\tfrac{1}{2}G(0,z)+\tfrac{1}{2}G(1,z)\right)G(1-z,y) - \tfrac{1}{4}G(0,z)^2 + \tfrac{1}{2}G(1,z)G(0,z) + \tfrac{1}{4}G(1,z)^2\right) \\
& + G(0,y)\Big(-G(0,z)G(1-z,y)^2 + \left(\tfrac{1}{2}G(0,z)^2 - 2G(0,z)G(1,z)\right)G(1-z,y) + \tfrac{1}{2}G(1,z)G(0,z)^2 \\
& \qquad - G(1,z)^2G(0,z)\Big) \\
& + G(0,1,z)\left(G(0,y)(G(1-z,y)+G(1,z)) - \tfrac{1}{2}G(1-z,y)^2 - 3G(1,z)G(1-z,y) - G(0,y)^2 - \tfrac{3}{2}G(1,z)^2\right) \\
& + \left(G(0,z)G(1-z,y) - G(0,z)^2 + 2G(1,z)G(0,z) - 2G(0,1,z)\right)G(0,1-z,y) \\
& + \left(-G(0,y)^2 - 2G(0,1,y)\right)G(1-z,1,y) + G(0,0,1,z)(2G(1-z,y)+G(0,y)+2G(1,z)) \\
& + G(0,z)G(0,0,1-z,y) + G(0,1,1,z)(3G(1-z,y) - G(0,y) + 3G(1,z)) - G(0,z)G(0,1-z,1-z,y) \\
& + 2G(0,y)G(1-z,1,1,y) + G(0,y)G(1-z,1-z,1,y) + 2G(0,0,1-z,1,y) - 3G(0,0,-z,1-z,y) \\
& + 2G(0,1,1-z,1,y) + 2G(0,1-z,0,1,y) + 2G(0,1-z,1,1,y) - G(0,1-z,1-z,1,y) \\
& + 2G(1-z,0,0,1,y) + 2G(1-z,0,1,1,y) - G(1-z,0,1-z,1,y) - G(1-z,1-z,0,1,y) \\
& + \tfrac{1}{2}G(0,z)G(1,z)^3 - \tfrac{1}{4}G(0,z)^2G(1,z)^2 - 3G(0,0,0,1,z) - 2G(0,0,1,1,z) - 3G(0,1,1,1,z) - \frac{27\zeta(4)}{8},
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}^{(2),C_A C_F}_{\mathcal{O}_4;4} = {}& \zeta(2)\big(G(0,y)(-2G(1-z,y) - 2G(1,z)) + (2G(1,z) - 2G(0,z))G(1-z,y) + 2G(0,1-z,y) \\
& \qquad + 2G(1-z,1,y) + G(0,y)^2 + G(0,z)^2 + G(1,z)^2 - 2G(0,z)G(1,z) + 2G(0,1,z)\big) \\
& + \zeta(3)(-38G(1-z,y) + 6G(0,y) + 6G(0,z) - 38G(1,z)) \\
& + G(0,y)^2\left(-G(1,z)G(1-z,y) - \tfrac{1}{2}G(1-z,y)^2 - \tfrac{1}{2}G(1,z)^2\right) - 2G(0,y)G(0,1,1,z) \\
& - 2G(0,y)G(1-z,1,1,y) + 2G(0,y)G(1-z,1-z,1,y) + \left(G(0,z)G(1,z) - \tfrac{1}{2}G(0,z)^2\right)G(1-z,y)^2 \\
& + G(0,1,z)\left(G(0,y)(2G(1-z,y) + 2G(1,z)) - G(1-z,y)^2 + G(0,y)^2\right) \\
& + \left(2G(0,z)G(1-z,y) + G(0,z)^2 - 2G(1,z)G(0,z) + 2G(0,1,z)\right)G(0,1-z,y) \\
& + \left(G(0,y)^2 + 2G(0,1,y)\right)G(1-z,1,y) + G(0,0,1,z)(-2G(1-z,y) - 4G(0,y) - 2G(1,z)) \\
& - 4G(0,z)G(0,0,1-z,y) + 6G(1,z)G(0,0,-z,y) - 2G(0,z)G(0,1-z,1-z,y) \\
& - 2G(0,0,1-z,1,y) + 6G(0,0,-z,1-z,y) - 2G(0,1,1-z,1,y) - 2G(0,1-z,0,1,y) \\
& - 2G(0,1-z,1,1,y) - 2G(0,1-z,1-z,1,y) - 2G(1-z,0,0,1,y) - 2G(1-z,0,1,1,y) \\
& - 2G(1-z,0,1-z,1,y) - 2G(1-z,1-z,0,1,y) + 6G(0,0,0,1,z) + 2G(0,0,1,1,z) + \frac{73\zeta(4)}{4},
\end{aligned}$$

$$\mathcal{R}^{(2),C_F^2}_{\mathcal{O}_4;4} = \zeta(3)(24G(1-z,y) + 24G(1,z)) - 22\zeta(4)\,.$$

Sum of the three terms is same as the 3-gluon case:

$$\mathcal{R}^{(2),C_A^2}_{\mathcal{O}_4;4}(1^q,2^{\bar q},3^\pm) + \mathcal{R}^{(2),C_A C_F}_{\mathcal{O}_4;4}(1^q,2^{\bar q},3^\pm) + \mathcal{R}^{(2),C_F^2}_{\mathcal{O}_4;4}(1^q,2^{\bar q},3^\pm) = \mathcal{R}^{(2)}_{\mathrm{tr}(F^3);4}(1^-,2^-,3^\pm)$$

$$\begin{aligned}
\mathcal{R}^{(2)}_{\mathrm{tr}(F^3);4} = {}& -\frac{3}{2}\mathrm{Li}_4(u) + \frac{3}{4}\mathrm{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{2}\log(w)\mathrm{Li}_3\left(-\frac{u}{v}\right) + \frac{\log^2(u)}{32}\left[\log^2(u) + 2\log^2(v) - 4\log(v)\log(w)\right] \\
& + \frac{\zeta_2}{8}\left[5\log^2(u) - 2\log(v)\log(w)\right] - \frac{1}{4}\zeta_4 + \text{perms}(u,v,w),
\end{aligned}$$

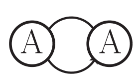
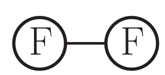
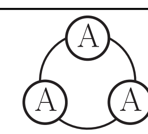
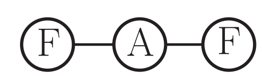
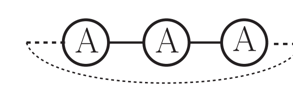
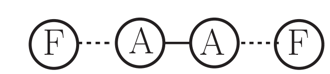
General correspondence

[Jin and GY, 1904.07260]

$$\begin{aligned} & \text{Max. Tran. of } (H \to q\bar{q}g)|_{C_F \to C_A} \\ = \ & \text{Max. Tran. of } (H \to 3g) \\ = \ & \text{Max. Tran. of } \mathcal{N} = 4 \text{ form factors} \end{aligned}$$

$C_A = N_c\,, \qquad C_F = \frac{N_c^2 - 1}{2N_c}$

It applies to form factors with more general operators:

	Length-2			**Length-3**		**Higher length**	
Operators							
Examples	$\mathrm{tr}(F^2)$	$\bar{\psi}\psi$	$\bar{\phi}\phi$	$\mathrm{tr}(F^3)$, $\mathrm{tr}(F_\mu{}^\nu D_\sigma F_\nu{}^\rho D^\sigma F_\rho{}^\mu)$	$F_{\mu\nu} D^\mu (\bar{\psi}\gamma^\nu \psi)$, $F_{\mu\nu}(\bar{\psi}\gamma^{\mu\nu}\psi)$	$\mathrm{tr}(F^L), L \geq 4$	$\bar{\psi}(F^L)\psi, L \geq 2$
External Partons	$(g,g,g), (\bar{\psi},\psi,g)$	$(\bar{\psi},\psi,g)$	$(\bar{\phi},\phi,g)$	(g,g,g)	$(\bar{\psi},g,\psi)$	$(g_1,\ldots,g_L)$	$(\bar{\psi},g_1,\ldots,g_L,\psi)$
Max. Trans. Remainder (with $C_F \to C_A$)	$R_{\mathrm{L2;4}}(u,v,w)$			$R_{\mathrm{L3;4}}(u,v,w)$		$\sum_i \mathcal{R}^{(2)}_{\mathrm{density;4}}(u_i,v_i,w_i)$	

Simplicity at lower transcendental parts

$\mathcal{O}_1 \to ggg \qquad \mathcal{O}_1 = \mathrm{tr}(F^3)$

Degree-3 part:

$$\mathcal{R}^{(2),N_c^2}_{\mathcal{O}_1,\alpha;3} = \left(1 + \frac{u}{w}\right) T_3(u,v,w) + \frac{143}{72}\zeta_3 - \frac{11}{24}\zeta_2 \log(u) + \text{perms}(u,v,w)$$

$$\mathcal{R}^{(2),N_c^2}_{\mathcal{N}=4;3} = \left(1 + \frac{u}{w}\right) T_3(u,v,w) + \text{perms}(u,v,w)$$

$$\boxed{\begin{aligned} T_3 :=& \Big[-\mathrm{Li}_3\left(-\frac{u}{w}\right) + \log(u)\mathrm{Li}_2\left(\frac{v}{1-u}\right) - \frac{1}{2}\log(1-u)\log(u)\log\left(\frac{w^2}{1-u}\right) + \frac{1}{2}\mathrm{Li}_3\left(-\frac{uv}{w}\right) + \frac{1}{12}\log^3(w) \\ &+ \frac{1}{2}\log(u)\log(v)\log(w) + (u \leftrightarrow v)\Big] + \mathrm{Li}_3(1-v) - \mathrm{Li}_3(u) + \frac{1}{2}\log^2(v)\log\left(\frac{1-v}{u}\right) - \zeta_2\log\left(\frac{uv}{w}\right). \end{aligned}}$$

Simplicity at lower transcendental parts

$\mathcal{O}_1 \to ggg \qquad \mathcal{O}_1 = \mathrm{tr}(F^3)$

Degree-2 to 0 parts:

$$\mathcal{R}^{(2),N_c^2}_{\hat{\mathcal{O}}_1,\alpha;2} = \left(\frac{u^2}{w^2} - \frac{1}{2}\right) T_2(u,v) - \frac{55}{48}\log^2(u) - \frac{73}{72}\log(u)\log(v) + \frac{23}{6}\zeta_2 + \mathrm{perms}(u,v,w)$$

$$\boxed{T_2(u,v) := \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \log(u)\log(v) - \zeta_2}$$

$$\mathcal{R}^{(2),N_c^2}_{\mathcal{O}_1,\alpha;1} = \left(\frac{119}{18} + \frac{v}{w} + \frac{u^2}{2vw}\right)\log(u) + \mathrm{perms}(u,v,w)$$

$$\mathcal{R}^{(2),N_c^2}_{\mathcal{O}_1,\alpha;0} = \frac{487}{72}\frac{1}{uvw} - \frac{14075}{216}$$

Simplicity at lower transcendental parts

[Jin, GY 1910.09384]

$\mathcal{O}_4 \to q\bar{q}g$ $\qquad \mathcal{O}_4 \to \mathcal{O}'_4 = gF_{\mu\nu}D^\mu \sum_{i,j=1}^{n_f} (\bar{\psi}_i \gamma^\nu T^A \psi_i)$

Degree-3 part:

$$\mathcal{R}^{(2),N_c^2}_{\mathcal{O}_4;\gamma;3} = -\left(1+\frac{w}{u}\right) T_3(v,w,u) - \frac{1}{3}\left(1+\frac{v}{u}\right) T_3(w,v,u) - \pi^2\left[\frac{7\log(v)}{24}+\frac{\log(w)}{72}\right] - \frac{257\zeta_3}{18}.$$

$$\mathcal{R}^{(2),N_c^0}_{\mathcal{O}_4;\gamma;3} = \left[\frac{1}{2}\left(\frac{w}{v}+1\right)^2 + \frac{w}{v}+1\right] T_3(u,w,v) - \left[\frac{1}{6}\left(\frac{u}{v}+1\right)^2 - \frac{4}{3}\frac{u}{v} - 1\right] T_3(w,u,v)$$
$$+ T_3(v,w,u) + \frac{1}{3} T_3(w,v,u) - T_3(v,u,w)$$
$$+ \pi^2\left[\frac{31\log(u)}{72} - \frac{5\log(v)}{18} - \frac{7\log(w)}{36}\right] + \frac{137\zeta_3}{36}.$$

$$\mathcal{R}^{(2),N_c^{-2}}_{\mathcal{O}_4;\gamma;3} = -\frac{w}{u} T_3(v,w,u) - \frac{1}{3}\frac{v}{u} T_3(w,v,u) - T_3(v,u,w) - \frac{1}{3} T_3(w,u,v)$$
$$+ \frac{1}{18}\pi^2(5\log(u)+3\log(v)+\log(w)) + \frac{15\zeta_3}{2}.$$

$$\mathcal{R}^{(2),N_c n_f}_{\mathcal{O}_4;\gamma;3} = -\frac{2}{3} T_3(u,v,w) - \frac{2}{3} T_3(u,w,v) + \frac{1}{36}\pi^2(8\log(u) - 3\log(v) - 3\log(w)) + \frac{43\zeta(3)}{9}.$$

$$T_3 := \left[-\mathrm{Li}_3\left(-\frac{u}{w}\right) + \log(u)\mathrm{Li}_2\left(\frac{v}{1-u}\right) - \frac{1}{2}\log(1-u)\log(u)\log\left(\frac{w^2}{1-u}\right) + \frac{1}{2}\mathrm{Li}_3\left(-\frac{uv}{w}\right) + \frac{1}{12}\log^3(w)\right.$$
$$\left. + \frac{1}{2}\log(u)\log(v)\log(w) + (u \leftrightarrow v)\right] + \mathrm{Li}_3(1-v) - \mathrm{Li}_3(u) + \frac{1}{2}\log^2(v)\log\left(\frac{1-v}{u}\right) - \zeta_2\log\left(\frac{uv}{w}\right).$$

Outline

- Motivations
- Higgs amplitudes
- Hidden analytic structure
- Explanation and implication

What is the origin of the correspondence?

HUGE intermediate expressions $\xrightarrow{\text{Why?}}$ Much simpler analytic form

The idea is to apply “physical constraints”:
For example, the universal IR divergences can be used to fix the finite part.

What is the origin of the correspondence?

Use a set of "Uniform transcendental integrals" $I_{i,\mathrm{UT}}$ as integral basis, e.g.: $I_{\mathrm{tri}}^{(1)} = -\frac{1}{\epsilon^2} + \frac{\pi^2}{12} + \frac{7}{3}\zeta_3\epsilon + \frac{47}{1440}\pi^2\epsilon^2 + \mathcal{O}(\epsilon^3)$

$$\mathcal{F}^{(L)} = \sum_a \sum_i c_{i,(a)}\, \epsilon^a\, I_{i,\mathrm{UT}}^{(L)}$$

$c_{i,(a)}$ → Physical information

The maximal transcendentality part for minimal two-loop form factors can be uniquely fixed by the infrared divergences:

$$\mathcal{F}^{(2)}_{\text{deg-4}} = \sum_i c_{i,(0)}\, I_{i,\mathrm{UT}}^{(2)} = (\mathrm{IR\ div})\big|_{\text{deg-4}} + \mathcal{O}(\epsilon^0)$$

(For N=4 BPS case, this fixes the full result !)

What is the origin of the correspondence?

Use a set of "Uniform transcendental integrals" $I_{i,\mathrm{UT}}$ as integral basis, e.g.: $I^{(1)}_{\mathrm{tri}} = -\frac{1}{\epsilon^2} + \frac{\pi^2}{12} + \frac{7}{3}\zeta_3\epsilon + \frac{47}{1440}\pi^2\epsilon^2 + \mathcal{O}(\epsilon^3)$

$$\mathscr{F}^{(L)} = \sum_a \sum_i c_{i,(a)}\, \epsilon^a\, I^{(L)}_{i,\mathrm{UT}}$$

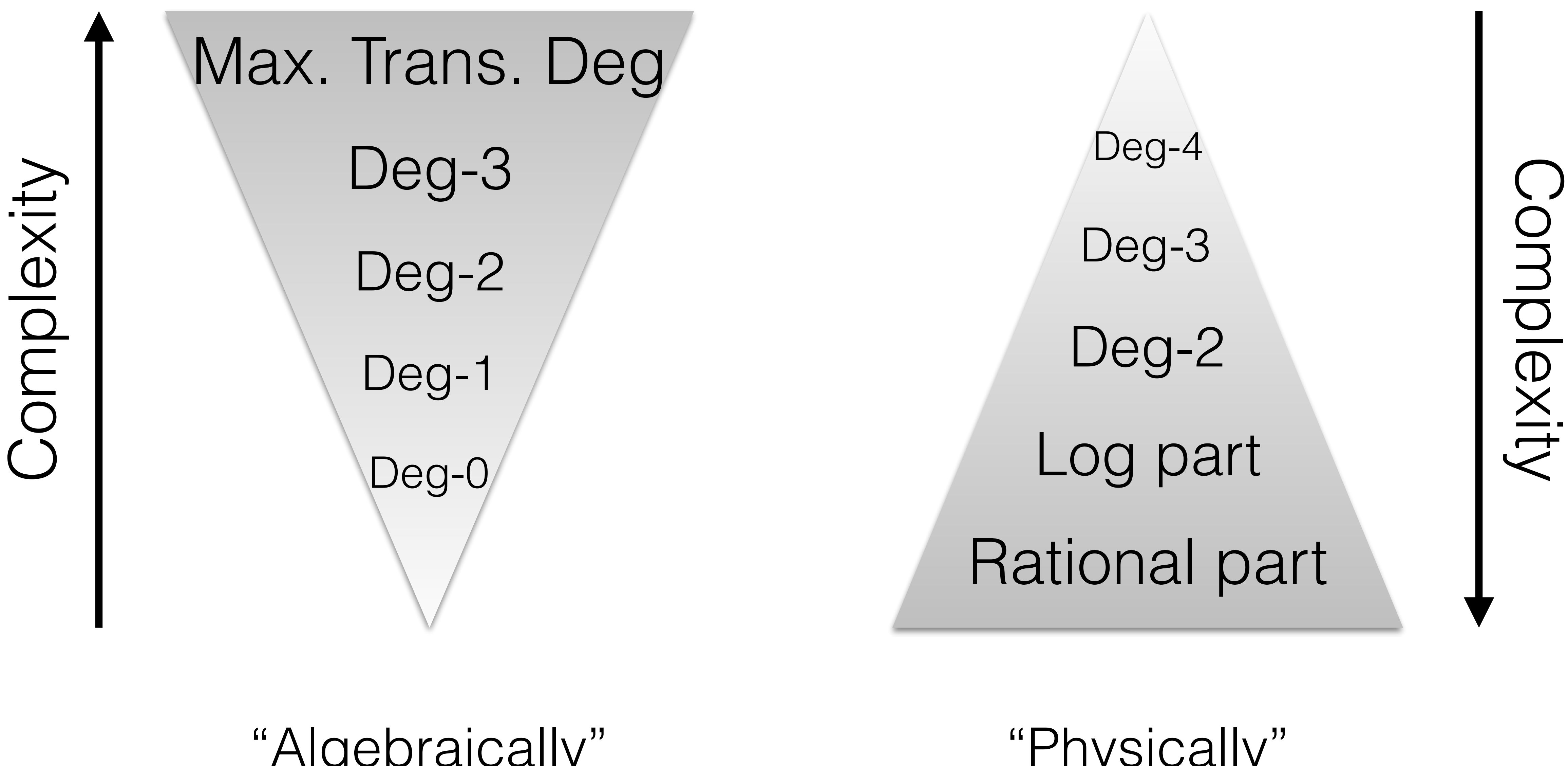
$c_{i,(a)}$ → Physical information

Lower transcendentality parts can be partially fixed by infrared divergences, plus additional physical constraints (e.g. unitarity cuts, collinear limit etc).

$$\mathscr{F}^{(2)}_{\mathrm{deg\text{-}(4-a)}} = \epsilon^a \sum_i c_{i,(a)}\, I^{(2)}_{i,\mathrm{UT}} = (\mathrm{IR/UV\ div})\big|_{\mathrm{deg\text{-}(4-a)}} + \mathcal{O}(\epsilon^0)$$

The higher the degree, the simpler the constraints.

Hierarchy of “complexity”



Similarity at one-loop: the most challenging part is the rational part.

Summary and outlook

N=4 SYM ⟷ **QCD**

(1) To which extend is the correspondence correct?

Higgs amplitudes of high dim operators, and external quarks

(2) What is the origin of the correspondence?

It can be proved using physical constraints.

(3) Does it help in practice? How about lower trans. parts?

Simple structure also exists and may be fixed in similar ways.

The most challenging part is the rational parts, which hopefully may be computed using alternative techniques.

Summary and outlook

The study of analytic structures and the maximal transcendentality correspondence can be very useful.

A new strategy:

Divide and conquer (according transcendentality degree) and bootstrap results (using physical constraints) without intermediate complicated steps.

Towards a better way of computation (in realistic QCD)?!

Summary and outlook

The study of analytic structures and the maximal transcendentality correspondence can be very useful.

A new strategy:

Divide and conquer (according transcendentality degree) and bootstrap results (using physical constraints) without intermediate complicated steps.

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Thank you for your attention