Black holes from anomalies

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High temperature limit of CFT on $S^{D-1} \times R$

- Equilibrium thermal partition functions of CFTs at high T.
- Euclidean CFT on $S_r^{D-1} \times S_\beta^1$, at $\beta \equiv T^{-1} \ll r^{-1}$.
- Seek for higher dim'l version of 2d Cardy formula: $Z(\tau) \sim \mathrm{Tr}\left[e^{2\pi i \tau L_0}\right] \sim \exp\left[\frac{\pi i c}{12\tau}\right]$ at $\tau \to i0^+$

- Two contributions to the Cardy "free energy" $\log Z(\beta, \ldots)$.
- 0-mode path integral on S^{D-1}:

  Absent in standard partition function (conformal mass, antiperiodic b.c. for fermion)

  May exist in SUSY indices, but cannot contribute to the divergent leading part at $\beta \to 0$.
- "Heavy" Kaluza-Klein (KK) modes on S^1:

  Divergent contributions to $\log Z$ come from these KK modes at $\beta \to 0$.

- KK modes → effective action of background fields (chemical potentials) on S^{D-1}.
- E.g. metric on (squashed) S^{D-1}, background vector fields, scalars, …
- ∃ special vector & scalar in KK reduction: gravi-photon & dilaton

$$ds_D^2 = ds_{D-1}^2 + e^{-2\Phi}(d\tau + a)^2 \qquad \tau \sim \tau + \beta \ , \quad \beta e^{-\Phi} \sim \text{radius of } S^1$$

Effective action & derivative expansion

- In CFT, small β expansion = r^{-1} expansion = derivative expansion on S^{D-1}.
- ∞-tower of derivative expansion. Most coefficients depend on coupling constants.

$$\frac{A_1}{\beta^3}\int d^3x e^{3\Phi}\sqrt{g} + \frac{A_2}{\beta}\int d^3x\sqrt{g}e^{\Phi}R + A_3\int d^3x\sqrt{g}(\beta e^{-\Phi})(\beta^{-1}da)^2 + \cdots$$

cosmological constant — Einstein-Hilbert — gravi-photon kinetic

- However, Chern-Simons terms on S^{D-1} (for even D) are coupling independent.
- 't Hooft anomalies of global symmetries determine them. (More later)
- They are sub-leading terms in the normal thermal free energy.

$$k\beta^{-2}\int a\wedge da + k_I\beta^{-1}\int \boxed{\mathcal{A}^I}\wedge da + k_{IJ}\int \mathcal{A}^I\wedge d\mathcal{A}^J + \cdots$$

background gauge fields (~ chemical potential) for global symmetries

- Strong-coupling large N CFT w/ AdS dual: The expansion should constrain large AdS black hole's thermodynamics (at large Hawking temperature).

Example: Kerr black holes in AdS$_5$

- Background fields on S^3: chemical potentials $\beta, \omega_1, \omega_2$ for E, J_1, J_2

$$ds^2 = r^2\left[d\theta^2 + \sum_{i=1}^{2} n_i^2\left(d\phi_i - \frac{i\omega_i}{\beta}d\tau\right)^2\right] + d\tau^2 \;=\; r^2\left[d\theta^2 + \sum_i n_i^2 d\phi_i^2 + \frac{r^2(\sum_i \omega_i n_i^2 d\phi_i)^2}{\beta^2(1 - r^2\sum_i \frac{n_i^2\omega_i^2}{\beta^2})}\right] + e^{-2\Phi}(d\tau + a)^2$$

$$e^{-2\Phi} \;=\; 1 - r^2\sum_i \frac{n_i^2\omega_i^2}{\beta^2} \;, \quad a = -i\frac{r^2\sum_i \omega_i n_i^2 d\phi_i}{\beta(1 - r^2\sum_i \frac{n_i^2\omega_i^2}{\beta^2})} \qquad (n_1, n_2) = (\cos\theta, \sin\theta)$$

- Leading terms in derivative expansion. (Assume $\Omega \equiv \omega_1 = \omega_2$ for simplicity.)

$$S = \log Z + \beta E + 2\Omega J = A_1\frac{\beta}{(\beta^2 - \Omega^2)^2} + A_2\frac{3\beta^3 - 4\beta\Omega^2}{(\beta^2 - \Omega^2)^2} + A_3\frac{\beta\Omega^2}{(\beta^2 - \Omega^2)^2} + \beta E + 2\Omega J + \mathcal{O}(\beta^0)$$

- Extremize in β, Ω & solve for E, J, S. Fitting coefficients, one reproduces all divergent parts of E, J, S carried by Kerr black holes. (Here, $\Omega = -a\beta$.) analysis by Nahmgoong (2018)

$$E = \frac{\pi l^2}{G}\left(\frac{3 + a^2}{8(1 - a^2)^3}\frac{l^4\pi^4}{\beta^4} - \frac{3 + 3a^2 - 2a^4}{8(1 - a^2)^3}\frac{l^2\pi^2}{\beta^2} - \frac{3 - 11a^2 + 6a^4 - 2a^6}{16(1 - a^2)^3} + \mathcal{O}(\beta^2)\right)$$

$$J = \frac{\pi l^2}{G}\left(\frac{a}{4(1 - a^2)^3}\frac{l^4\pi^4}{\beta^4} - \frac{2a - a^3}{4(1 - a^2)^3}\frac{l^2\pi^2}{\beta^2} + \frac{a}{8(1 - a^2)^3} + \mathcal{O}(\beta^2)\right)$$

$$S = \frac{\pi^2 l^3}{G}\left(\frac{1}{2(1 - a^2)}\frac{l^3\pi^3}{\beta^3} - \frac{3 - 2a^2}{4(1 - a^2)^2}\frac{l\pi}{\beta} + \mathcal{O}(\beta)\right)$$

- Related to fluid-gravity calculus of large BH's. [Bhattacharyya,Lahiri,Loganayagam,Minwalla]

High T expansion of indices: expectation

- So far, we constrained high T partition functions (with unknowns $A_1, A_2, A_3 \ldots$).

- Similar calculus for an index: This method is much more powerful.
- Naturally expect CS terms may determine the divergent part (e.g. coupling-independent).
- Index: For 4d SCFTs on $S^3 \times S^1$, index is defined by

$$Z(\omega_1, \omega_2) \equiv \mathrm{Tr}\left[e^{-\beta(E-\frac{3}{2}R-J_1-J_2)}e^{-\frac{1}{2}\Delta R}e^{-\omega_1 J_1 - \omega_2 J_2}\right]$$

$$\Delta = \omega_1 + \omega_2 + 2\pi i \qquad E - \frac{3}{2}R - J_1 - J_2 \sim \{\mathcal{Q}, \mathcal{Q}^\dagger\}$$

- β is merely a regulator of the index. E.g. easy to compute at $\beta \to 0^+$.
- Since leading term is $O(\beta^0)$, $\beta \to 0$ is a fake thermal circle parameter.
- True derivative expansion parameters are $|\omega_i| \ll 1$.

$$e^{-2\Phi} = 1 + \boxed{r^2 \sum_i \frac{n_i^2 \omega_i^2}{\beta^2}} \ , \quad a = -i\frac{r^2 \sum_i \omega_i n_i^2 d\phi_i}{\beta(1 - r^2 \sum_i \frac{n_i^2 \omega_i^2}{\beta^2})}$$

- Some reordering of naïve derivative expansion is inevitable.

Background field set-up

- The background fields for the index:

- same metric, dilaton, gravi-photon as before (up to small shifts of ω_i by β)
- Background $U(1)_R$ & other flavor symmetries' gauge fields:

$$A^I = -\frac{i\Delta_I}{\beta} d\tau \xrightarrow{\text{rearrange to 3d fields}} A^I = A_4^I(d\tau + a) + \mathcal{A}^I \qquad A_4^I = -\frac{i\Delta^I}{\beta} \qquad \mathcal{A}^I = -A_4^I a$$

- Non-CS terms:

Indeed, all vanish at $\beta \to 0$.

$$\begin{aligned}
\frac{1}{(2\pi)^2}\int \beta^{-3}e^{3\Phi}\sqrt{g} &= \frac{\beta r^3}{2(\beta^2 - r^2\omega^2)^2} = \frac{\beta}{2r\omega^4} + \mathcal{O}\left(\frac{\beta^3}{r^3\omega^6}\right) \qquad (2.62)\\
\frac{1}{(2\pi)^2}\int \beta^{-1}e^{\Phi}\sqrt{g}\,\mathcal{R}^{ab}{}_{ab} &= \frac{r(3\beta^3 - 4\beta r^2\omega^2)}{(\beta^2 - r^2\omega^2)^2} = -\frac{4\beta}{r\omega^2} + \mathcal{O}\left(\frac{\beta^3}{r^3\omega^4}\right)\\
\frac{1}{(2\pi)^2}\int \beta e^{-\Phi}\sqrt{g}\,\mathcal{F}^I_{ab}\mathcal{F}^J_{ab} &= \frac{\beta\Delta^I\Delta^J r^3\omega^2}{(\beta^2 - r^2\omega^2)^2} = \frac{\beta\Delta^I\Delta^J}{r\omega^2} + \mathcal{O}\left(\frac{\beta^3}{r^3\omega^4}\right)\\
\frac{1}{(2\pi)^2}\int \beta^3 e^{-3\Phi}\sqrt{g}\,(\nabla_c\mathcal{F}^I_{ab})(\nabla^c\mathcal{F}^{Jab}) &= \frac{2\beta^3 r\omega^2\Delta^I\Delta^J}{(\beta^2 - r^2\omega^2)^2} = \frac{2\beta^3\Delta^I\Delta^J}{r^3\omega^2} + \mathcal{O}\left(\frac{\beta^5}{r^5\omega^4}\right)\\
\frac{1}{(2\pi)^2}\int \beta^3 e^{-3\Phi}\sqrt{g}\,(\nabla_c\mathcal{F}^I_{ab})(\nabla^a\mathcal{F}^{Jcb}) &= \frac{\beta^3 r\omega^2\Delta^I\Delta^J}{(\beta^2 - r^2\omega^2)^2} = \frac{\beta^3\Delta^I\Delta^J}{r^3\omega^2} + \mathcal{O}\left(\frac{\beta^5}{r^5\omega^4}\right)\\
\frac{1}{(2\pi)^2}\int \beta e^{-\Phi}\sqrt{g}\,\mathcal{R}^{ab}{}_a{}^c\mathcal{R}_b{}^d{}_{cd} &= \frac{2\beta(8r^4\omega^4 - 8\beta^2r^2\omega^2 + 3\beta^4)}{r(\beta^2 - r^2\omega^2)^2} = \frac{16\beta}{r} + \mathcal{O}\left(\frac{\beta^3}{r^3\omega^2}\right)\\
\frac{1}{(2\pi)^2}\int \beta e^{-\Phi}\sqrt{g}\,\mathcal{R}_{abcd}\mathcal{R}^{abcd} &= \frac{32\beta r^4\omega^4 - 16\beta^3r^2\omega^2 + 6\beta^5}{r(\beta^2 - r^2\omega^2)^2} = \frac{32\beta}{r} + \mathcal{O}\left(\frac{\beta^3}{r^3\omega^2}\right)\\
\frac{1}{(2\pi)^2}\int \beta^3 e^{-3\Phi}\sqrt{g}\,\mathcal{F}^{Iab}\mathcal{F}^{Jc}_a\mathcal{R}_b{}^d{}_{cd} &= \frac{2\beta\Delta^I\Delta^J r\omega^2(\beta^2 - 2r^2\omega^2)}{(\beta^2 - r^2\omega^2)^2} = -\frac{4\beta\Delta^I\Delta^J}{r} + \mathcal{O}\left(\frac{\beta^3}{r^3\omega^2}\right)\\
\frac{1}{(2\pi)^2}\int \beta^3 e^{-3\Phi}\sqrt{g}\,\mathcal{F}^I_{ab}\mathcal{F}^J_{cd}\mathcal{R}^{abcd} &= \frac{2\beta\Delta^I\Delta^J r\omega^2(\beta^2 - 4r^2\omega^2)}{(\beta^2 - r^2\omega^2)^2} = -\frac{8\beta\Delta^I\Delta^J}{r} + \mathcal{O}\left(\frac{\beta^3}{r^3\omega^2}\right)\\
\frac{1}{(2\pi)^2}\int \beta^5 e^{-5\Phi}\sqrt{g}\,\mathcal{F}^{Iab}\mathcal{F}^{Jc}_a\mathcal{F}^{Kd}_b\mathcal{F}^L_{cd} &= \frac{\beta\Delta^I\Delta^J\Delta^K\Delta^L r^3\omega^4}{(\beta^2 - r^2\omega^2)^2} = \frac{\beta\Delta^I\Delta^J\Delta^K\Delta^L}{r} + \mathcal{O}\left(\frac{\beta^3}{r^3\omega^2}\right)\\
\frac{1}{(2\pi)^2}\int \beta^5 e^{-5\Phi}\sqrt{g}\,\mathcal{F}^{Iab}\mathcal{F}^J_{ab}\mathcal{F}^{Kcd}\mathcal{F}^L_{cd} &= \frac{2\beta\Delta^I\Delta^J\Delta^K\Delta^L r^3\omega^4}{(\beta^2 - r^2\omega^2)^2} = \frac{2\beta\Delta^I\Delta^J\Delta^K\Delta^L}{r} + \mathcal{O}\left(\frac{\beta^3}{r^3\omega^2}\right).
\end{aligned}$$

CS coefficients from anomalies

- Knowing CS coefficients, one can determine the Cardy free energy.
- There turn out to be two types of CS terms.

- Gauge non-invariant CS terms

- 4d effective action $S_{\text{eff}} = -\log Z$ respects ‘t Hooft anomaly: $\delta_\epsilon S_{\text{eff}} \sim \epsilon\, F \wedge F + \ldots$
- This should be reflected in the 3d background fields’ effective action.
- It demands the existence of certain gauge non-invariant CS terms.

[Banerjee, Bhattacharya, Bhattacharyya, Jain, Minwalla, Sharma] (2012)

$$S_{\text{non-inv}} = -i\frac{\beta(5a-3c)}{8\pi^2}\int_{S^3} C_{IJK}\left(A_4^I \mathcal{A}^J \wedge d\mathcal{A}^K + A_4^I A_4^J \mathcal{A}^K \wedge da + \frac{1}{3}A_4^I A_4^J A_4^K a \wedge da\right)$$

- Gauge invariant CS terms: More elaborate arguments by [Jensen, Loganayagam, Yarom] (2013) determine them all from anomalies. (See also [Di Pietro, Komargodski] (2014).)

$$S_{\text{inv}} \sim -i(a-c)\int_{S^3} k_I \mathcal{A}^I \wedge da$$

[C_{IJK}, k_I are coefficients of the cubic anomaly polynomial in a suitable normalization.]

Cardy free energy & BH's

- Plugging in our background fields, one obtains the Cardy free energy.

$$\log Z \sim (5a-3c)\frac{C_{IJK}\Delta^I\Delta^J\Delta^K}{6\omega_1\omega_2} + \frac{4\pi^2(a-c)k_I\Delta^I}{\omega_1\omega_2}$$

- If one only turns on the chemical potential for the U(1) superconformal R-symmetry, one obtains a universal formula in terms of two central charges.

$$\log Z \sim -\frac{16i}{27}\frac{3c-2a}{\omega_1\omega_2} \qquad C_{111}=6\ ,\quad k_1=1\ ,\quad \Delta^1=\frac{2\pi i}{3}$$

- It is known that $3c-2a>0$ is always met by interacting SCFTs. [Hofman, Maldacena]
- Legendre transformation of this free energy is a bit subtle. The proper way to understand it has been clarified only recently. [Choi, J. Kim, SK, Nahmgoong] (2018) [Choi, SK] (2019)
- Making this Legendre transformation at large $J_1, J_2 \gg a, c$, one always obtains a positive macroscopic entropy at $3c-2a>0$. [J. Kim, SK, Song] (2019)

- The entropy obtained this way precisely agrees with the Bekenstein-Hawking entropy of BPS black holes in AdS_5.

Conclusion & remarks

- I just used anomalies & some SUSY to derive the Cardy index.
- The method applies to non-Lagrangian SCFTs in even dimensions.

- 4d non-Lagrangian theories: Argyres-Douglas, Minahan-Nemeschansky, “class S”, …
- This method (effective action of 1d background fields) also reproduces the well-known 2d Cardy formula [Joonho Kim, Kimyeong Lee, Jaemo Park] (2018)

$$Z(\tau) \sim \mathrm{Tr}\left[e^{2\pi i\tau L_0}\right] \sim \exp\left[\frac{\pi i c}{12\tau}\right] \quad \text{at } \tau \to i0^+$$

- 6d SCFTs & AdS_7 black holes: → See the talk by **June Nahmgoong** tomorrow.

- Anomalies constrain other coupling-independent observables in CFT.
- Rather abstract “d.o.f.” measured by anomaly is related to the high T d.o.f.

- Today’s talk is only a small part of the recent advances in BPS AdS black holes.
- Counted BPS AdS_{D+1} black holes from $SCFT_D$ Cardy indices for all $D = 3,4,5,6$.

  [Choi, J. Kim, SK, Nahmgoong] [Choi, SK] [J. Kim, SK, Song] [Nahmgoong] [Choi, Hwang, SK] ……