Entanglement of Purification in QFTs

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Based on arXiv:1901.00330 and arXiv:1904.12124

Entanglement in quantum field theory

Entangled state: non-separable

♦ Simple example: two qubit system

• Separable states

$\{|0\rangle_A|0\rangle_B, |1\rangle_A|1\rangle_B, \dots\}$

• Entangled states, e.g., EPR/Bell state

$\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$

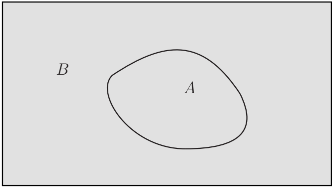
♦ States in QFT

• spacelike correlation

$$\mathcal{C}_{AB} := \langle \mathcal{O}_A \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle \neq 0$$

• Highly entangled state: vaccum state...

• Non-entangled state: boundary state in CFTs...



Entanglement measure

- Entanglement entropy (EE),
  Reduced density matrix of a subsystem A : $\rho_A := tr_{\bar{A}}\rho$

$$S_A = -tr \rho_A \log \rho_A$$

- Rényi entropy

$$S_A^n = \frac{\log tr \rho_A^n}{1-n}$$

- Entanglement of purification (the main topic in this talk)

Outline

■ Entanglement and Reeh-Schlieder theorem

■ Entanglement of purification (EoP)
- Definition and properties
- Purifications and Reeh-Schlieder theorem
- Constraints on calculation of EoP

A “baby” Reeh-Schlieder theorem

Alice and Bob

Case I: They share a product state, e.g.,$|0\rangle_A|0\rangle_B$

Alice/Bob **cannot** construct the total Hilbert space by local operations

Case II: They share an entangled state, $\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$

They can!

Redhead, More ado about nothing, Foundations of Physics, 25(1), 123-137.

A "baby" Reeh-Schlieder theorem

- Projection $|0\rangle_A\ {}_A\langle 0| \to |0\rangle_A|0\rangle_B$
- Controlled-NOT (CNOT) gate on $A \to \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)$:

$$\boxed{\text{CNOT gate: } |1\rangle_C|0\rangle_A \to |1\rangle_C|1\rangle_A, \quad |1\rangle_C|1\rangle_A \to |1\rangle_C|0\rangle_A}$$

The "baby" Reeh-Schlieder theorem:

Alice/Bob could construct their Hilbert space by only local operations if they share an entangled state.

Reeh-Schlieder theorem in QFTs

Framework of algebra QFT

A framework of algbraic QFT,

- Any open region $O \to \mathscr{R}(O)$

$$\phi(f) := \int_O dx f(x)\phi(x)$$

- Microcausality: $[\mathscr{R}(O_A),\mathscr{R}(O_B)]=0$ if O_A, O_B spacelike
- Hilbert space $\mathcal{H}$: $\mathscr{U}|0\rangle$, $\mathscr{U}$ is the algebra for entire spacetime
- Haag duality: $\mathscr{R}(O)' = \mathscr{R}(O')$

$$\boxed{\mathscr{R}' := \{\mathcal{O}, \quad [\mathcal{O},\mathscr{R}] = 0\}}$$

Reeh-Schlieder theorem in QFTs

Cyclic state

Cyclic state $|\Psi\rangle$ with respect to $\mathcal{H}$:

The set $\mathcal{H}_O := \{\mathcal{O}|\Psi\rangle, \mathcal{O} \in \mathscr{R}(O)\}$ is dense in $\mathcal{H}$

- $|0\rangle$ is a cyclic state for $\mathscr{U}$
- **Reeh-Schlieder theorem:**
  For any bound region O, the vacuum state $|0\rangle$ is cyclic for $\mathscr{R}(O)$
- The theorem can be generalized to some excited states
  Witten, arXiv:1803.04993

Entanglement of purifications

Definition

Given state ρ and two subsystems A and B

- $\rho = |0\rangle\langle 0|$, $\rho_{AB} := tr_{\overline{AB}}\rho$ is mixed state
- **Purifications:**
  A pure state $|\psi\rangle$ by introducing $\tilde{A}$ and $\tilde{B}$ with

$$\rho_{AB} = tr_{\tilde{A}\tilde{B}}|\psi\rangle\langle\psi|$$

  For example $|0\rangle$ is a purification with $\tilde{A}\tilde{B} = \overline{AB}$
- **EoP**

$$E_P(\rho_{AB}) = \min_{\rho_{AB} = tr_{\tilde{A}\tilde{B}}|\psi\rangle\langle\psi|} S(\rho_{A\tilde{A}})$$

with $\rho_{A\tilde{A}} = tr_{B\tilde{B}}|\psi\rangle\langle\psi|$

Entanglement of purifications

Motivation and properties

- EoP characterizes the correlation between A and B
  - Mutual information $I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$

$$\frac{1}{2} I(\rho_{AB}) \leq E_P(\rho_{AB}) \leq \min\{S(\rho_A), S(\rho_B)\}$$

- Holographic conjecture : Minimal cross of entanglement wedge.

A lot of concepts:

  - Entanglement wedge/subregion duality
  - Unitary and Disentangling operations (I will show later)
  - Surface/state correspondence conjecture

Takayanagi-Umemoto, arXiv:1708.09393;
Nguyen et.al., arXiv:1709.07424

Entanglement of purifications

The set of purifications

Application of Reeh-Schlieder theorem

- The dense set

$$\mathcal{H}' = \{\mathcal{O}_{\overline{AB}}|0\rangle, \mathcal{O}_{\overline{AB}} \in \mathscr{R}(\overline{AB})\}$$

  We can take $\tilde{A}\tilde{B} = \overline{AB}$

- The constraint $\rho_{AB} = tr_{\tilde{A}\tilde{B}}|\psi\rangle\langle\psi\rangle$ with $|\psi\rangle = \mathcal{O}_{\overline{AB}}(\psi)|0\rangle$

$$\mathcal{O}^{\dagger}_{\overline{AB}}(\psi)\mathcal{O}_{\overline{AB}}(\psi) = \mathcal{O}_{\overline{AB}}(\psi)\mathcal{O}^{\dagger}_{\overline{AB}}(\psi) = \mathbf{1}$$

$$\boxed{tr_{AB}(O_{AB}\, tr_{\tilde{A}\tilde{B}}|\psi\rangle\langle\psi|) = tr_{AB}(O_{AB}\rho_{AB})}$$

$$\boxed{\langle 0|(\mathcal{O}_{\overline{AB}}(\psi)\mathcal{O}^{\dagger}_{\overline{AB}}(\psi) - \mathbf{1})\mathcal{O}_{AB}|0\rangle = 0}$$

Entanglement of purifications

Unitary operations

First step: purifications

EoP $\leftrightarrow$ Unitary operations on $\overline{AB}$ or $\tilde{A}\tilde{B}$

$$\mathcal{H}_\psi = \{\mathcal{U}_{\overline{AB}}|0\rangle, \quad \text{unitary} \quad \mathcal{U}_{\overline{AB}} \in \mathscr{R}(\overline{AB})\}.$$

Second step: min $S(\rho_{A\tilde{A}})$

$$\rho_{A\tilde{A}} = tr_{B\tilde{B}} \mathcal{U}_{\overline{AB}}|0\rangle\langle 0|\mathcal{U}^\dagger_{\overline{AB}}$$

EoP in QFTs

EoP and disentanglement

Purifications:

$$|\psi\rangle = \mathcal{U}_{\tilde{A}\tilde{B}}(\psi)|0\rangle$$

- Consider A and B are adjacent,

- $\mathcal{U}_{\tilde{A}\tilde{B}}(\psi)$ has no effect on AB
- Disentangle $\tilde{A}$ from $\tilde{B} \to S_{A\tilde{A}}$ smaller or larger?

EoP in QFTs

EoP and disentanglement

- Extreme case

$$|\psi_\infty\rangle := \mathcal{U}_{\tilde{A}\tilde{B}}(\psi_\infty)|0\rangle = |\chi_{\tilde{B}}\rangle \otimes |\chi_{\overline{\tilde{B}}}\rangle$$

This is called *holographic compression* by Wilming and Eisert [arXiv:1809.10156] in the lattice models

By Lie-Araki inequalities,

$$S_{A\tilde{A}} \geq S_B - S_{\tilde{B}}$$

Near the state $|\psi_\infty\rangle$,

$$S_{A\tilde{A}} \gtrsim S_B \gg S_A$$

EoP in QFTs

EoP and disentanglement

- Estimate the disentanglement, the min $S_{A\tilde{A}} \to |\psi\rangle_M$

$$S_A \geq S_{A\tilde{A}}(|\psi\rangle_M) \geq \frac{1}{2} I(\rho_{AB}) = \frac{1}{2} S_A$$

in the limit $L_B \to \infty$, $S_{A\tilde{A}} := \lambda S_A$ with $1 \geq \lambda \geq \frac{1}{2}$

• Strong subadditivity

$$\begin{aligned} S_{\tilde{A}}(|\psi\rangle_M) &\leq S_{A\tilde{A}}(|\psi\rangle_M) + S_{\tilde{A}\tilde{B}}(|\psi\rangle_M) - S_{A\tilde{A}\tilde{B}}(|\psi\rangle_M) \\ &= S_{A\tilde{A}}(|\psi\rangle_M) + \cancel{S_{AB} - S_B} \end{aligned}$$

• Lie-Araki inequality

$$S_{\tilde{A}}(|\psi\rangle_M) \geq S_A - S_{A\tilde{A}}((|\psi\rangle_M))$$

- Estimation:

$$S_A \geq S_{\tilde{A}}((|\psi\rangle_M)) \geq (1-\lambda)S_A$$

  - The correlations between $\tilde{A}$ and $\tilde{B}$, A and $\tilde{A}$ are still large
  - If the hEoP conjecture is right, $\lambda \simeq \frac{1}{2}$

EoP in QFTs

Constraint of modular Hamiltonian

- Consider a state near $|\psi\rangle_M$

$$|\psi(\delta)\rangle := e^{i\delta\mathcal{H}_{\tilde{A}\tilde{B}}}|\psi\rangle_M$$

$\mathcal{H}_{\tilde{A}\tilde{B}}$ hermitian, δ is small dimensionless parameter

- Trivial case:

$$\mathcal{H}_{\tilde{A}\tilde{B}} = \mathcal{H}_{\tilde{A}} + \mathcal{H}_{\tilde{B}} \to S_{A\tilde{A}}(|\psi(\delta)\rangle) = S_{A\tilde{A}}(|\psi\rangle_M)$$

- Non-trivial case:

$$\mathcal{H}_{\tilde{A}\tilde{B}} = \sum_i \mathcal{H}_{\tilde{A},i}\mathcal{H}_{\tilde{B},i}$$

EoP in QFTs

Constraint of modular Hamiltonian

The variantion of $S_{A\tilde{A}}$

$$\Delta S_{A\tilde{A}} := S_{A\tilde{A}}(|\psi(\delta)\rangle) - S_{A\tilde{A}}(|\psi\rangle_M)$$

Perturbative calculation:

$$\Delta S_{A\tilde{A}} = \delta S_1 + \delta^2 S_2 + O(\delta^3)$$

with

$$S_1 = i\ {}_M\langle\psi|[K_{A\tilde{A},M}, H_{\tilde{A}\tilde{B}}]|\psi\rangle_M$$

where $K_{A\tilde{A},M} = -\log \rho_{A\tilde{A},M}$.

EoP in QFTs

Constraint of modular Hamiltonian

- $S_{A\tilde{A}}$ is minimal $\to$ $S_1 = 0$,

$$_M\langle\psi|[K_{A\tilde{A},M}, H_{\tilde{A}\tilde{B}}]|\psi\rangle_M = 0$$

- Take $H_{\tilde{A}\tilde{B}} = H_{\tilde{A}}H_{\tilde{B}}$

$$_M\langle\psi|[K_{A\tilde{A},M}, H_{\tilde{A}}]H_{\tilde{B}}|\psi\rangle_M = 0$$

- For any operators $O_{\tilde{A}}$ and $O_{\tilde{B}}$,

$$_M\langle\psi|[K_{A\tilde{A},M}, O_{\tilde{A}}]O_{\tilde{B}}|\psi\rangle_M = 0$$

$$\boxed{O = H_1 + iH_2 \text{ with } H_1 = \frac{O+O^\dagger}{2} \text{ and } H_2 = \frac{O-O^\dagger}{2i}}$$

EoP in QFTs

Constraint of modular Hamiltonian

$$
{}_M\langle\psi|[K_{A\tilde{A},M}, O_{\tilde{A}}]O_{\tilde{B}}|\psi\rangle_M = 0
$$

- Non-separating property$\rightarrow$ $O_{\tilde{B}}|\psi\rangle_M \neq 0$ for any $O_{\tilde{B}} \neq 0$
- $K_{A\tilde{A},M} \in \mathscr{R}(A\tilde{A}) \rightarrow [K_{A\tilde{A},M}, O_{\tilde{A}}] \in \mathscr{R}(A\tilde{A})$
- Note the correlation between $A\tilde{A}$ and $\tilde{B}$ is still very large
- Much stronger condition:

$$
[K_{A\tilde{A},M}, O_{\tilde{A}}] = 0
$$

for any $O_{\tilde{A}} \in \mathscr{R}(\tilde{A})$

$$\boxed{\Delta S_{A\tilde{A}} = \delta S_1 + \delta^2 S_2 + O(\delta^3)}$$

More constraint: $S_2 \geq 0$

$$\begin{aligned} S_2 = -\frac{1}{2}\Big(& {}_M\langle\psi|H^2_{\tilde{A}\tilde{B}} K_{A\tilde{A},M}|\psi\rangle_M + {}_M\langle\psi|K_{A\tilde{A},M} H^2_{\tilde{A}\tilde{B}}|\psi\rangle_M \\ & + {}_M\langle\psi|H_{\tilde{A}\tilde{B}} K_{A\tilde{A},M} H_{\tilde{A}\tilde{B}}|\psi\rangle_M \Big) \\ & - {}_M\langle\psi|H_{\tilde{A}\tilde{B}}|\psi\rangle_M \Big({}_M\langle\psi|\rho^{-1}_{A\tilde{A},M} H_{\tilde{A}\tilde{B}}|\psi\rangle_M \\ & + {}_M\langle\psi|H_{\tilde{A}\tilde{B}} \rho^{-1}_{A\tilde{A},M}|\psi\rangle_M \Big) \\ & + {}_M\langle\psi|H_{\tilde{A}\tilde{B}} \rho^{-1}_{A\tilde{A},M} H_{\tilde{A}\tilde{B}}|\psi\rangle_M + {}_M\langle\psi|H^2_{\tilde{A}\tilde{B}}|\psi\rangle_M + \ldots \end{aligned}$$

Conclusion and future work

- Two step for the calculation of EoP in QFTs:

1. Construct the set of purifications,

$$\mathcal{H}_\psi = \{\mathcal{U}_{\overline{AB}}|0\rangle, \text{unitary} \quad \mathcal{U}_{\overline{AB}} \in \mathscr{R}(\overline{AB})\}$$

2. Find the minimal value of $S_{A\tilde{A}}$,

• 2D CFTs, $\tilde{A}$ is connnectd with A, $\tilde{A}\tilde{B} = \overline{AB}$

$$\langle 0|\mathcal{U}^\dagger_{\tilde{A}\tilde{B}}\sigma(x_1)\sigma(x_2)\mathcal{U}_{\tilde{A}\tilde{B}}|0\rangle$$

with

$$\mathcal{U}_{\tilde{A}\tilde{B}} = e^{i\sum_j \mathcal{H}_{j,\tilde{A}}\mathcal{H}_{j,\tilde{B}}} \quad \text{e.g.} \quad \mathcal{H}_{\tilde{A}} = \int_{\tilde{A}} dx f_j(x) T(x) + \text{h.c.}$$

Direct calculation seems not possible

Conclusion and future work

- Starting point $[K_{A\tilde{A},M}, O_{\tilde{A}}] = 0$ or $K_{A\tilde{A},M} \in \mathscr{R}(A)$
  - Similar to the modular zero mode condition
  - Holographic duality of this condition?

Thank you!