Link Approach of Lattice SUSY for N=D=4 super Yang-Mills

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Why Lattice SUSY ?

In addition to the standard motivations:

- To establish a formulation to investigate non-perturbative aspects of supersymmetric theories numerically
- Fundamental origin of SUSY breaking mechanism by regularization point of view
- The role of lattice chiral fermion species doubler problem in establishing exact lattice SUSY
- Necessity of N=D=4 super Yang Mills and AdS/CFT correspondence

We define forward difference operator as:

$$\Delta_{+\mu}\phi(x) = \phi(x+n_\mu) - \phi(x) \qquad\qquad (a=1)$$

Operation of the difference operator to a product of fields:

$$\begin{aligned}\Delta_{+\mu}(\phi_1\phi_2)(x) &= \phi_1(x+n_\mu)\phi_2(x+n_\mu) - \phi_1(x)\phi_2(x) \\ &= (\underline{\phi_1(x+n_\mu)} - \phi_1(x))\underline{\phi_2(x)} + \underline{\phi_1(x+n_\mu)}(\phi_2(x+n_\mu) - \underline{\phi_2(x)}) \\ &= (\nabla_{+\mu}\phi_1)(x)\phi_2(x) + \phi_1(x+\underline{n_\mu})(\nabla_{+\mu}\phi_2)(x)\end{aligned}$$

Difference operator does not satisfy Leibniz rule.

simplest SUSY algebra in 1-dim.

$$\{Q,Q\} = 2H = 2i\partial \to \ \{Q,Q\} = 2i\Delta_{+\mu} \qquad (\Delta_{+\mu}\phi(x) = \phi(x+n_\mu) - \phi(x))$$

There are two main obstacles for exact lattice SUSY.

1) Leibniz rule breakdown of difference operator

2) chiral fermion doublers

We proposed two formulations:

A) Link approach

D'Adda, Kanamori, N.K., Nagata, (2005~2008)
N=2,4,D=2,3, Latt. super Y-M

$$\Delta_\mu(\phi_1(x)\phi_2(x)) = (\Delta_\mu\phi_1)(x)\phi_2(x) + \phi_1(x+n_\mu)(\Delta_\mu\phi_2)(x)$$
$$Q_A(\phi_1(x)\phi_2(x)) = (Q_A\phi_1)(x)\phi_2(x) + \phi_1(x+a_A)(Q_A\phi_2)(x)$$

Fields are on the lattice link

B) super doubler Approach

D'Adda, Kanamori, N.K., Saito, (2012~2018)

$$\delta(p_1+p_2+\cdots) \to \delta(f(p_1)+f(p_2)+\cdots)$$

exact Leibniz rule but non-local field theory

N=2 SUSY in two dimensions

$$\{Q_{i\alpha}, \overline{Q}_{j\beta}\} = 2i\delta_{ij}(\gamma^\mu)_{\alpha\beta}\partial_\mu \qquad (\overline{Q}_{i\alpha} = (C^{-1}Q^T C)_{i\alpha}) \quad (\to N = D = 4)$$

Dirac-Kaehler twist (N=2)

$$Q_{\alpha i} = (\mathbf{1}Q + \gamma^\mu Q_\mu + \gamma^5 \tilde{Q})_{\alpha i}$$

Twisted N=2 SUSY

conti: $\{Q, Q_\mu\} = i\partial_\mu \qquad \{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\partial_\mu$

Latt: $\{Q, Q_\mu\} = i\Delta_{+\mu} \qquad \{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_{-\mu}$

↓

$$(\Delta_{\pm\mu}\phi)(x) = \pm(\phi(x \pm n_\mu) - \phi(x))$$

super Yang-Mills extension

super connection plaquette
||
covariant derivative

$$\{\nabla, \nabla_\mu\} = +i\mathcal{U}_{+\mu}, \qquad \{\tilde{\nabla}, \nabla_\mu\} = +i\epsilon_{\mu\nu}\mathcal{U}_{-\mu}$$

Link Approach

We introduce a shift to the super charge operation!

$$\Delta_\mu(\phi_1(x)\phi_2(x)) = (\Delta_\mu\phi_1)(x)\phi_2(x) + \phi_1(x+n_\mu)(\Delta_\mu\phi_2)(x)$$
$$Q_A(\phi_1(x)\phi_2(x)) = (Q_A\phi_1)(x)\phi_2(x) + \phi_1(x+a_A)(Q_A\phi_2)(x)$$

$$Q_A = \frac{\partial}{\partial\theta_A} + \cdots \quad \rightarrow \quad \frac{\partial}{\partial\theta_A}x = (x+a_A)\frac{\partial}{\partial\theta_A}$$

$(\theta_A : \text{super coordinate})$ $\qquad$ $x\theta_A = \theta_A(x+a_A)$

There are two possible ways to interpret:

1) θ_A and x are non-commutative with a shift a_A

We take 2) $\frac{\partial}{\partial\theta_A}$ gets link nature connecting two neighboring sites: $(x+a_A, x)$

$$\{Q, Q_\mu\} = i\Delta_{+\mu} \rightarrow \{\nabla, \nabla_\mu\} = i\mathcal{U}_{+\mu}$$

$$(Q)_{x+a+a_\mu, x+a_\mu}(Q_\mu)_{x+a_\mu,x} + (Q_\mu)_{x+a+a_\mu,x+a}(Q)_{x+a,x} = i(\Delta_\mu)_{x+n_\mu,x},$$

$$(a + a_\mu = n_\mu)$$

shift nature

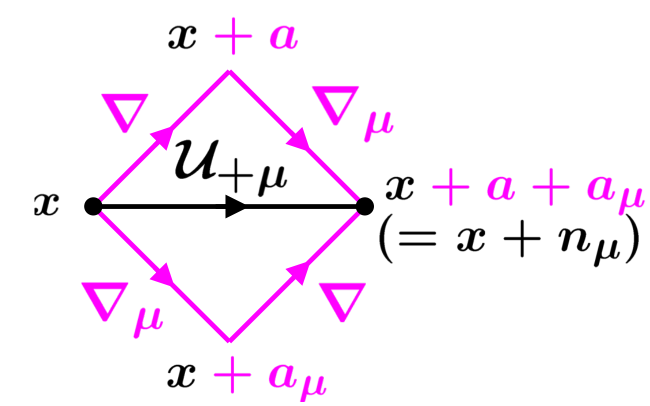
Fermionic gauged link

$$\nabla_A = \nabla_{x+a_A,x}$$

$$\nabla_A(\phi_1(x)\phi_2(x)) = (\nabla_A\phi_1)(x)\phi_2(x) + \phi_1(x+a_A)(\nabla_A\phi_2)(x)$$

additive shifts:

$$(\{\nabla_A, \nabla_B\}\phi)(x) = \{\nabla_A, \nabla_B\}\phi(x) - \phi(x+a_A+a_B)\{\nabla_A, \nabla_B\}$$



$$\{\nabla, \nabla_\mu\} = +i\mathcal{U}_{+\mu}, \qquad \{\tilde{\nabla}, \nabla_\mu\} = +i\epsilon_{\mu\nu}\mathcal{U}_{-\mu}$$

shift vectors:

$$a + a_\mu = n_\mu \qquad \tilde{a} + a_\mu = -|\epsilon_{\mu\nu}|n_\nu$$

$$a + \tilde{a} + a_1 + a_2 = 0$$

solutions:

$$a = (\text{arbitrary}) \qquad a_\mu = +n_\mu - a$$

$$\tilde{a} = -n_1 - n_2 + a$$

(one arbitrary shift vector)

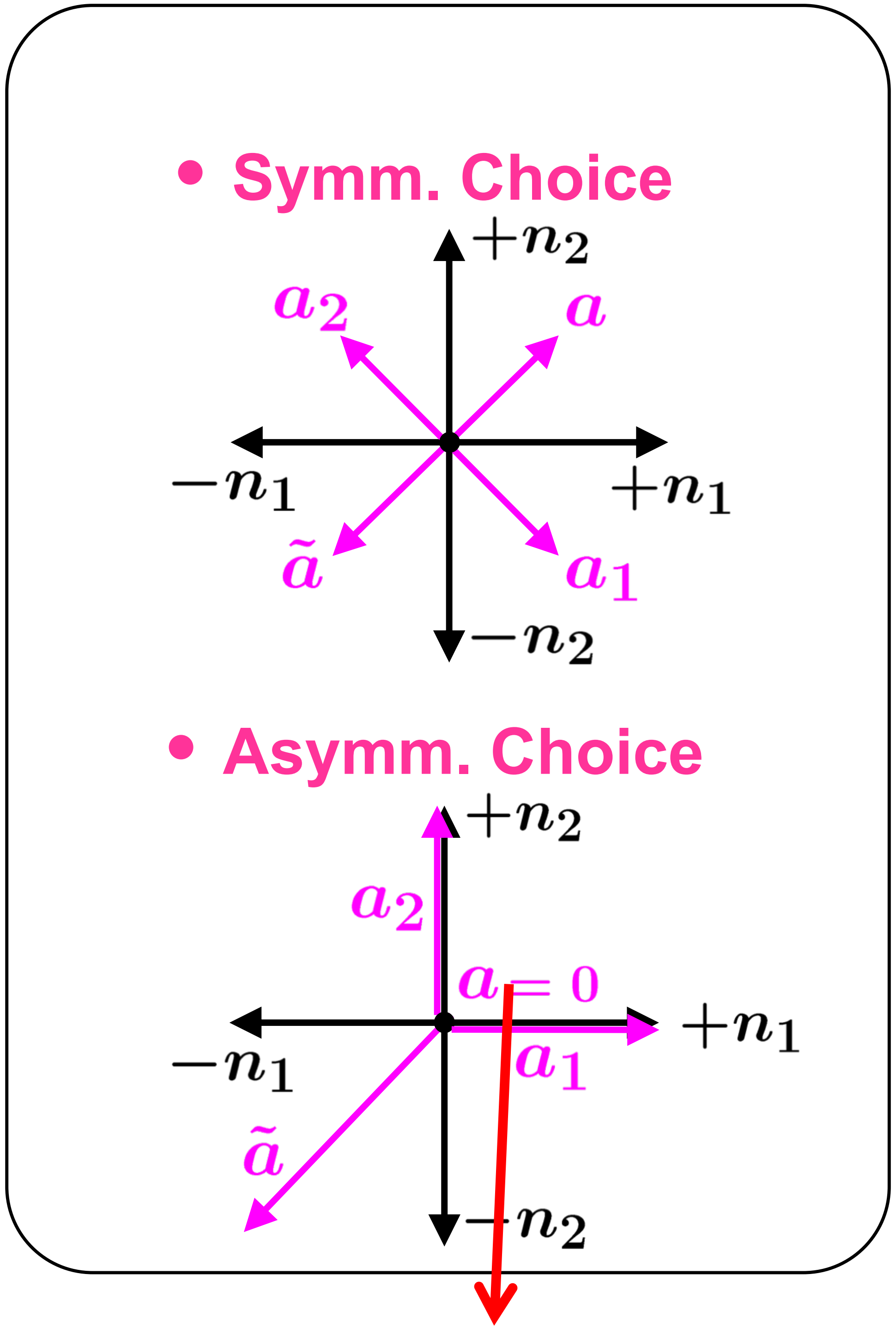
Shift vectors

half integers

dual site =fermionic site

$$a = (+\frac{1}{2},+\frac{1}{2}), \qquad a_1 = (+\frac{1}{2},-\frac{1}{2}),$$
$$\tilde{a} = (-\frac{1}{2},-\frac{1}{2}), \qquad a_2 = (-\frac{1}{2},+\frac{1}{2}),$$

$$a = (0,0), \qquad a_1 = (+1,0),$$
$$\tilde{a} = (-1,-1), \qquad a_2 = (0,+1).$$



Equivalent to orbifold construction: $Q^2 = 0$ by Kaplan et.al.

Ordering ambiguity

Critique by
Bruckmann, Kok & Catterall

$$\phi_1(x)\phi_2(x) = \phi_2(x)\phi_1(x)$$

$$Q_A(\phi_1(x)\phi_2(x)) = (Q_A\phi_1(x))\phi_2(x) + \phi_1(x+a_A)Q_A\phi_2(x)$$
$$\| \qquad\qquad\qquad\qquad \| \;??$$
$$Q_A(\phi_2(x)\phi_1(x)) = (Q_A\phi_2(x))\phi_1(x) + \phi_2(x+a_A)Q_A\phi_1(x)$$

Shiftless fields product is commutable.

This ambiguity is resolved by respecting shift nature of the product.

$$(Q_A\phi_i(x))\phi_j(x) = \phi_j(x+a_A)(Q_A\phi_i(x)) \quad i,j=1,2$$

$$(Q_A\phi_i)_{x+a_A,x}(\phi_j)_{x,x} = (\phi_j)_{x+a_A,x+a_A}(Q_A\phi_i)_{x+a_A,x} \qquad ((\phi_i)_{y,x} = \phi_i(x))$$

In general we change order of product of fields by respecting link nature:

$$(\phi_A)_{x+a_A+a_B,x+a_B}(\phi_B)_{x+a_B,x} \;\sim\; (-1)^{|\phi_A||\phi_B|}(\phi_B)_{x+a_A+a_B,x+a_A}(\phi_A)_{x+a_A,x}$$

Ordering equivalence class

Extension to non-Abelian gauge theory: non-Abelian nature remains

$$(\phi_A)^{ab}_{x+a_A+a_B,x+a_B}(\phi_B)^{bc}_{x+a_B,x} \;\sim\; (-1)^{|\phi_A||\phi_B|}(\phi_B)^{bc}_{x+a_A+a_B,x+a_A}(\phi_A)^{ab}_{x+a_A,x}$$

N=D=4 Twisted SUSY (continuum)

Kato, N.K. Miyake (2005)
Marcus (1995)

$$\{Q_{\alpha i}\overline{Q}_{\dot{\beta}j}\} = 2\delta_{ij}(\sigma^{\mu})_{\alpha\dot{\beta}}P_{\mu} \rightarrow \ \{Q_{\alpha i},\overline{Q}_{j\beta}\} = 2\delta_{ij}(\gamma^{\mu})_{\alpha\beta}P_{\mu}$$

$$(\{\gamma^{\mu},\gamma^{\nu}\} = 2\delta_{\mu\nu})$$

$$\overline{Q}_{i\alpha} = (C^{-1}Q^{T}C)_{i\alpha}$$

Dirac-Kaehler twisting

$$J'_{\mu\nu} = J_{\mu\nu} + R_{\mu\nu}$$

$$Q_{\alpha i} = \frac{1}{\sqrt{2}}(\mathbf{1}Q + \gamma^{\mu}Q_{\mu} + \frac{1}{2}\gamma^{\mu\nu}Q_{\mu\nu} + \tilde{\gamma^{\mu}}\tilde{Q}_{\mu} + \gamma^{5}\tilde{Q})_{\alpha i}$$

$$\{Q_A, D_B\} = 0$$

super charge:

$$\begin{aligned} \{Q,Q_{\mu}\} &= +i\partial_{\mu}, \\ \{Q_{\rho\sigma},Q_{\mu}\} &= -i\delta_{\rho\sigma\mu\nu}\partial_{\nu}, \\ \{Q_{\rho\sigma},\tilde{Q}_{\mu}\} &= +i\epsilon_{\rho\sigma\mu\nu}\partial_{\nu}, \\ \{\tilde{Q},\tilde{Q}_{\mu}\} &= +i\partial_{\mu}, \\ \{others\} &= 0, \end{aligned}$$

super derivative:

$$\begin{aligned} \{D,D_{\mu}\} &= -i\partial_{\mu}, \\ \{D_{\rho\sigma},D_{\mu}\} &= +i\delta_{\rho\sigma\mu\nu}\partial_{\nu}, \\ \{D_{\rho\sigma},\tilde{D}_{\mu}\} &= -i\epsilon_{\rho\sigma\mu\nu}\partial_{\nu}, \\ \{\tilde{D},\tilde{D}_{\mu}\} &= -i\partial_{\mu}, \\ \{others\} &= 0, \end{aligned}$$

Meaning of twisting

4 dim. Euclidean space: $SO(4) \simeq SU_L(2) \times SU_R(2)$ $\leftarrow J_{\mu\nu}$

N=4 SUSY internal symmetry: $SU(4) \supset SO(4) \simeq SU(2)_1 \otimes SU(2)_2$ $\leftarrow R_{\mu\nu}$

$\mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$ Marcus, Dirac-Kaehler

N=4 on-shell gauge multiplet with SU(4) internal symmetry: (1,4,6,4,1)

$\omega_\mu(\mathbf{2},\mathbf{2},\mathbf{1})$ $\lambda_u^\alpha(\mathbf{2},\mathbf{1},\mathbf{4})$

$\bar\lambda_{\dot\alpha}^u(\mathbf{1},\mathbf{2},\bar{\mathbf{4}})$ $\phi_{uv}(\mathbf{1},\mathbf{1},\mathbf{6})$

$(SU_L(2), SU_R(2), SU_I(4))$

Twisting procedure for N=4 is to take $SU_L(2)'$ and $SU_R(2)'$ as diagonal subgroups of $SU_L(2) \otimes SU(2)_1$ and $SU_R(2) \otimes SU(2)_2$

$$\lambda_u^\alpha(\mathbf{2},\mathbf{1},\mathbf{4}): \ (\mathbf{2},\mathbf{1},\mathbf{4}) \to (\mathbf{2},\mathbf{1}) \otimes [(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})] \sim (\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2})$$

$$\rho^+(\mathbf{1},\mathbf{1}) \ \rho_{\mu\nu}^+(\mathbf{3},\mathbf{1}) \ \lambda_\mu^+(\mathbf{2},\mathbf{2})$$

$$\phi_{uv}(\mathbf{1},\mathbf{1},\mathbf{6}) \to A(\mathbf{1},\mathbf{1}), \ B(\mathbf{1},\mathbf{1}) \ V_\mu(\mathbf{2},\mathbf{2})$$

$$J'_{\mu\nu} = J_{\mu\nu} + R_{\mu\nu}$$

Dirac-Kaehler twisting:

$$Q_{\alpha i} = \frac{1}{\sqrt{2}}(\mathbf{1}Q + \gamma^{\mu}Q_{\mu} + \frac{1}{2}\gamma^{\mu\nu}Q_{\mu\nu} + \tilde{\gamma}^{\mu}\tilde{Q}_{\mu} + \gamma^{5}\tilde{Q})_{\alpha i}$$

Decomposition into chiral sectors:

$$\begin{aligned} Q^{\pm} &\equiv \frac{1}{\sqrt{2}}(Q \mp \tilde{Q}), \\ Q^{\pm}_{\rho} &\equiv \frac{1}{\sqrt{2}}(Q_{\rho} \mp \tilde{Q}_{\rho}), \\ Q^{\pm}_{\mu\nu} &\equiv \frac{1}{\sqrt{2}}(Q_{\mu\nu} \pm \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}Q_{\rho\sigma}). \end{aligned}$$

The same structure:

$$Q^{+}(\mathbf{1},\mathbf{1}),\quad Q^{+}_{\rho\sigma}(\mathbf{3},\mathbf{1}),\quad Q^{+}_{\mu}(\mathbf{2},\mathbf{2}), \;\longrightarrow\; \rho^{+}(\mathbf{1},\mathbf{1}),\ \rho^{+}_{\mu\nu}(\mathbf{3},\mathbf{1}),\ \lambda^{+}_{\mu}(\mathbf{2},\mathbf{2})$$

$$Q^{-}(\mathbf{1},\mathbf{1}),\quad Q^{-}_{\rho\sigma}(\mathbf{1},\mathbf{3}),\quad Q^{-}_{\mu}(\mathbf{2},\mathbf{2}), \;\longrightarrow\; \rho^{-}(\mathbf{1},\mathbf{1}),\ \rho^{-}_{\mu\nu}(\mathbf{1},\mathbf{3}),\ \lambda^{-}_{\mu}(\mathbf{2},\mathbf{2})$$

Extension to gauge theory

super covariant derivative (continuum):

$$
\begin{aligned}
\nabla &\equiv D - i\Gamma(x,\theta_A), \\
\nabla_\mu &\equiv D_\mu - i\Gamma_\mu(x,\theta_A), \\
\nabla_{\rho\sigma} &\equiv D_{\rho\sigma} - i\Gamma_{\rho\sigma}(x,\theta_A), \\
\tilde{\nabla}_\mu &\equiv \tilde{D}_\mu - i\tilde{\Gamma}_\mu(x,\theta_A), \\
\tilde{\nabla} &\equiv \tilde{D} - i\tilde{\Gamma}(x,\theta_A).
\end{aligned}
$$

$\Gamma_A(x,\theta)$:super connection

Dirac-Kaehler twisted super covariante derivative ansatz

$$
\nabla_{\pm\mu} \equiv \nabla_{\underline{\mu}} \pm \mathcal{V}_\mu,
$$

$$
\begin{aligned}
\nabla_{\underline{\mu}}| &= \mathcal{D}_\mu \equiv \partial_\mu - i\omega_\mu, \qquad (|_{\theta=0}) \\
\mathcal{V}_\mu| &= V_\mu.
\end{aligned}
$$

$$
\begin{aligned}
\{\nabla, \nabla_\mu\} &= -i\nabla_{+\mu}, \\
\{\nabla_{\rho\sigma}, \nabla_\mu\} &= +i\delta_{\rho\sigma\mu\nu}\nabla_{-\nu}, \\
\{\nabla_{\rho\sigma}, \tilde{\nabla}_\mu\} &= -i\epsilon_{\rho\sigma\mu\nu}\nabla_{+\nu}, \\
\{\tilde{\nabla}, \tilde{\nabla}_\mu\} &= -i\nabla_{-\mu},
\end{aligned}
$$

←

$$
\begin{aligned}
\{D, D_\mu\} &= -i\partial_\mu, \\
\{D_{\rho\sigma}, D_\mu\} &= +i\delta_{\rho\sigma\mu\nu}\partial_\nu, \\
\{D_{\rho\sigma}, \tilde{D}_\mu\} &= -i\epsilon_{\rho\sigma\mu\nu}\partial_\nu, \\
\{\tilde{D}, \tilde{D}_\mu\} &= -i\partial_\mu, \\
\{others\} &= 0,
\end{aligned}
$$

$$
\{\nabla_A, \nabla_B\} = \nabla_{\pm\mu}, W, F
$$

$$
\begin{aligned}
\{\nabla, \tilde{\nabla}\} &= -iW, & \{\nabla_{\mu\nu}, \nabla_{\rho\sigma}\} &= +i\epsilon_{\mu\nu\rho\sigma}W, \\
\{\nabla_\mu, \tilde{\nabla}_\nu\} &= -i\delta_{\mu\nu}F, & \{others\} &= 0,
\end{aligned}
$$

$$
\begin{aligned}
W| &= A, \\
F| &= B.
\end{aligned}
$$

$$\{\nabla_A, \nabla_B\} = \nabla_{\pm\mu}, W, F$$

def. of fermions and SUSY trans. of W, F, $\nabla_{\pm\mu}$

graded Jacobi id.

$$[\tilde{\nabla}, F] = -\rho, \quad \cdots$$

$$[\nabla_A, \{\nabla_B, \nabla_C\}] = [\nabla_A, (\nabla_{\pm\mu}, W, F)] = \lambda, \tilde{\lambda}, \rho, \tilde{\rho}$$

SUSY trans. of fermions

$$\{\nabla, \tilde{\lambda}_\mu\} = +i[\nabla_{+\mu}, W], \quad \cdots$$

$$\{\nabla_A, [\nabla_B, \{\nabla_C, \nabla_D\}]\} = \{\nabla_A, (\lambda, \tilde{\lambda}\rho, \tilde{\rho})\} = \cdots$$

N=D=4 twisted SUSY trans.:

$$s_A\varphi \equiv [\nabla_A, \varphi\}|_{\theta' s=0}$$

equations of motion:

$$\begin{aligned}
[\mathcal{D}_{+\mu}, \lambda_\mu] + [A, \tilde{\rho}] &= 0, \\
[\mathcal{D}_{-\mu}, \tilde{\lambda}_\mu] - [A, \rho] &= 0, \\
[\mathcal{D}_{+\mu}, \rho] + [\mathcal{D}_{-\nu}, \rho_{\mu\nu}] - [B, \tilde{\lambda}_\mu] &= 0, \\
[\mathcal{D}_{-\mu}, \tilde{\rho}] - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[\mathcal{D}_{+\nu}, \rho_{\rho\sigma}] + [B, \lambda_\mu] &= 0, \\
\delta_{\mu\nu\rho\sigma}[\mathcal{D}_{-\rho}, \lambda_{-\sigma}] + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[A, \rho_{\rho\sigma}] + \epsilon_{\mu\nu\rho\sigma}[\mathcal{D}_{+\rho}, \tilde{\lambda}_\sigma] &= 0.
\end{aligned}$$

on-shell closure of algebra:

$$\{s_C, s_D\}\varphi = (\mathcal{D}_{\pm\mu}, A, B,)\varphi$$

N=D=4 Dirac-Kaehler twisted super Yang-Mills action (continuum)

$$
\begin{aligned}
S_{TSYM}^{N=D=4} &= S_B + S_F \\
&= \int d^4x \ \mathrm{tr} \left[-\frac{1}{2}[\mathcal{D}_{+\mu}, \mathcal{D}_{+\nu}][\mathcal{D}_{-\mu}, \mathcal{D}_{-\nu}] + \frac{1}{4}[\mathcal{D}_{+\mu}, \mathcal{D}_{-\mu}][\mathcal{D}_{+\nu}, \mathcal{D}_{-\nu}] \right. \\
&\quad + \frac{1}{2}[\mathcal{D}_{+\mu}, A][\mathcal{D}_{-\mu}, B] + \frac{1}{2}[\mathcal{D}_{-\mu}, A][\mathcal{D}_{+\mu}, B] + \frac{1}{4}[A, B][A, B] \\
&\quad - i\lambda_\mu[\mathcal{D}_{+\mu}, \rho] - i\tilde{\rho}[A, \rho] - i\lambda_\mu[\mathcal{D}_{-\nu}, \rho_{\mu\nu}] + i\tilde{\rho}[\mathcal{D}_{-\mu}, \tilde{\lambda}_\mu] + i\lambda_\mu[B, \tilde{\lambda}_\mu] \\
&\quad \left. - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{\lambda}_\mu[\mathcal{D}_{+\nu}, \rho_{\rho\sigma}] + \frac{i}{8}\epsilon_{\mu\nu\rho\sigma}\rho_{\mu\nu}[A, \rho_{\rho\sigma}] \right]
\end{aligned}
$$

on-shell exact N=D=4 SUSY invariance
for all super charges

$$
\begin{aligned}
s \ S_{TSYM}^{N=D=4} &= 0, \\
s_\mu \ S_{TSYM}^{N=D=4} &= 0, \\
s_{\mu\nu} \ S_{TSYM}^{N=D=4} &= 0, \\
\tilde{s}_\mu \ S_{TSYM}^{N=D=4} &= 0, \\
\tilde{s} \ S_{TSYM}^{N=D=4} &= 0,
\end{aligned}
$$

Link Approach for Lattice super Yang-Mills N=D=4

$$\begin{aligned}
\{Q, Q_\mu\} &= +i\Delta_{\pm\mu} \\
\{Q_{\rho\sigma}, Q_\mu\} &= -i\delta_{\rho\sigma\mu\nu}\Delta_{\pm\nu} \\
\{Q_{\rho\sigma}, \tilde{Q}_\mu\} &= +i\epsilon_{\rho\sigma\mu\nu}\Delta_{\pm\nu} \\
\{\tilde{Q}, \tilde{Q}_\mu\} &= +i\Delta_{\pm\mu}
\end{aligned}$$

$\rightarrow$

$$\{Q_A, Q_B\}_{x+a_A+a_B,x} = (\Delta_{\pm\mu})_{x\pm n_\mu,x}$$

$$\begin{aligned}
a_A + a_B &= +n_\mu \quad for \quad \Delta_{+\mu}, \\
a_A + a_B &= -n_\mu \quad for \quad \Delta_{-\mu},
\end{aligned}$$

$$\begin{aligned}
a + a_\mu &= \pm n_\mu, \\
a_{\rho\sigma} + a_\mu &= \pm|\delta_{\rho\sigma\mu\nu}|n_\nu, \quad for \quad \rho=\mu \quad or \quad \sigma=\mu \\
a_{\rho\sigma} + \tilde{a}_\mu &= \pm|\epsilon_{\rho\sigma\mu\nu}|n_\nu, \quad for \quad \rho\neq\sigma\neq\mu, \\
\tilde{a} + \tilde{a}_\mu &= \pm n_\mu.
\end{aligned}$$

symmetric shift vectors: a_A

$$\begin{aligned}
a &= (+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}), & a_1 &= (+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}), \\
a_{12} &= (-\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}), & a_2 &= (-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}), \\
a_{13} &= (-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}), & a_3 &= (-\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2}), \\
a_{14} &= (-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2}), & a_4 &= (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}), \\
a_{23} &= (+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}), & \tilde{a}_4 &= (+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2}), \\
a_{24} &= (+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2}), & \tilde{a}_3 &= (+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}), \\
a_{34} &= (+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}), & \tilde{a}_2 &= (+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}), \\
\tilde{a} &= (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}), & \tilde{a}_1 &= (-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}),
\end{aligned}$$

consistent solution

(Any one of shift vectors is arbitrary, can be taken to be 0.)

super connection and gauge link field

$$\partial_\mu - i\omega_\mu \pm V_\mu = \mathcal{D}_\mu \pm V_\mu \quad \rightarrow \quad \mp (\mathcal{U}_{\pm\mu})_{x\pm n_\mu, x} \quad = \quad \mp (e^{\pm i(A_\mu \pm i V_\mu)})_{x\pm n_\mu, x}$$

$$\mathcal{U}_{+\mu}\mathcal{U}_{-\mu} \neq 1$$

$$(\mathcal{U}_{\pm\mu})_{x\pm n_\mu, x} \rightarrow G^{-1}(x \pm n_\mu)(\mathcal{U}_{\pm\mu})_{x\pm n_\mu, x} G(x)$$

$$(\nabla_A)_{x+a_A, x} \rightarrow G^{-1}(x + a_A)(\nabla_A)_{x+a_A, x} G(x)$$

$$\begin{aligned} \{Q, Q_\mu\} &= +i\Delta_{\pm\mu} \\ \{Q_{\rho\sigma}, Q_\mu\} &= -i\delta_{\rho\sigma\mu\nu}\Delta_{\pm\nu} \\ \{Q_{\rho\sigma}, \tilde{Q}_\mu\} &= +i\epsilon_{\rho\sigma\mu\nu}\Delta_{\pm\nu} \\ \{\tilde{Q}, \tilde{Q}_\mu\} &= +i\Delta_{\pm\mu} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \{\nabla, \nabla_\mu\}_{x+a+a_\mu, x} &= +i(\mathcal{U}_{+\mu})_{x+n_\mu, x}, \\ \{\nabla_{\rho\sigma}, \nabla_\mu\}_{x+a_{\rho\sigma}+a_\mu, x} &= +i\delta_{\rho\sigma\mu\nu}(\mathcal{U}_{-\nu})_{x-n_\nu, x}, \\ \{\nabla_{\rho\sigma}, \tilde{\nabla}_\mu\}_{x+a_{\rho\sigma}+\tilde{a}_\mu, x} &= +i\epsilon_{\rho\sigma\mu\nu}(\mathcal{U}_{+\nu})_{x+n_\nu, x}, \\ \{\tilde{\nabla}, \tilde{\nabla}_\mu\}_{x+\tilde{a}+\tilde{a}_\mu, x} &= -i(\mathcal{U}_{-\mu})_{x-n_\mu, x}, \end{aligned}$$

$$\begin{aligned} \{\nabla, \tilde{\nabla}\}_{x+a+\tilde{a}, x} &= -i(W)_{x+a+\tilde{a}, x}, \\ \{\nabla_{\mu\nu}, \nabla_{\rho\sigma}\}_{x+a_{\mu\nu}+a_{\rho\sigma}, x} &= +i\epsilon_{\mu\nu\rho\sigma}(W)_{x+a_{\mu\nu}+a_{\rho\sigma}, x}, \\ \{\nabla_\mu, \tilde{\nabla}_\nu\}_{x+a_\mu+\tilde{a}_\nu, x} &= -i\delta_{\mu\nu}(F)_{x+a_\mu+\tilde{a}_\nu, x}, \\ \{others\} &= 0, \end{aligned}$$

Super connection ansatz and Jacobi Identities

$$\{\nabla_A, \nabla_B\} = \mathcal{U}_{\pm\mu}, W, F$$

def. of fermions and SUSY trans. of W, F, $\mathcal{U}_{\pm\mu}$

$$[\nabla_A\{\nabla_B, \nabla_C\}] = [\nabla_A, (\mathcal{U}_{\pm\mu}, W, F)] = \lambda, \tilde{\lambda}, \rho, \tilde{\rho}, s_A\mathcal{U}_{\pm\mu}, \cdots$$

graded Jacobi id.

SUSY trans. of fermions

$$\{\nabla_A, [\nabla_B, \{\nabla_C, \nabla_D\}]\} = \{\nabla_A, (\lambda, \tilde{\lambda}\rho, \tilde{\rho})\} = \cdots, s_A\lambda, \cdots$$

SUSY transformation

$$(s_A\varphi)_{x+a_\varphi+a_A,x} = s_A(\varphi)_{x+a_\varphi,x} \equiv [\nabla_A, \varphi\}_{x+a_\varphi+a_A,x}$$

$\cdots$

	s	$s_{\rho\sigma}$	$\tilde{s}$
$\mathcal{U}_{+\mu}$	0	$-\delta_{\rho\sigma\mu\nu}\lambda_\nu$	$+\tilde{\lambda}_\mu$
$\mathcal{U}_{-\mu}$	$+\lambda_\mu$	$+\epsilon_{\rho\sigma\mu\nu}\tilde{\lambda}_\nu$	0
W	0	0	0
F	$-\tilde{\rho}$	$-\frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}\rho_{\alpha\beta}$	$-\rho$
ρ	$+\frac{i}{2}([\mathcal{U}_{+\lambda},\mathcal{U}_{-\lambda}]+[W,F])$	$-i[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]$	0
λ_μ	0	$-i\epsilon_{\rho\sigma\mu\nu}[\mathcal{U}_{+\nu},W]$	$+i[\mathcal{U}_{-\mu},W]$
$\rho_{\mu\nu}$	$+i[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]$	$+i\delta_{\rho\sigma\mu\lambda}[\mathcal{U}_{+\nu},\mathcal{U}_{-\lambda}]-i\delta_{\rho\sigma\nu\lambda}[\mathcal{U}_{+\mu},\mathcal{U}_{-\lambda}]$ $-\frac{i}{2}\delta_{\rho\sigma\mu\nu}([\mathcal{U}_{+\lambda},\mathcal{U}_{-\lambda}]+[W,F])$	$+\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}[\mathcal{U}_{-\alpha},\mathcal{U}_{-\beta}]$
$\tilde{\lambda}_\mu$	$-i[\mathcal{U}_{+\mu},W]$	$+i\delta_{\rho\sigma\mu\nu}[\mathcal{U}_{-\nu},W]$	0
$\tilde{\rho}$	0	$+\frac{i}{2}\epsilon_{\rho\sigma\alpha\beta}[\mathcal{U}_{+\alpha},\mathcal{U}_{+\beta}]$	$-\frac{i}{2}([\mathcal{U}_{+\lambda},\mathcal{U}_{-\lambda}]-[W,F])$

SUSY transformations of N=D=4 Lattice super Y-M

	s_ρ	$\tilde{s}_\rho$
$\mathcal{U}_{+\mu}$	$-\rho_{\rho\mu}$	$-\delta_{\rho\mu}\tilde{\rho}$
$\mathcal{U}_{-\mu}$	$+\delta_{\rho\mu}\rho$	$-\frac{1}{2}\epsilon_{\rho\mu\alpha\beta}\rho_{\alpha\beta}$
W	$+\tilde{\lambda}_\rho$	$-\lambda_\rho$
F	0	0
ρ	0	$+i[\mathcal{U}_{-\rho},F]$
λ_μ	$+i[\mathcal{U}_{+\rho},\mathcal{U}_{-\mu}]$ $-\frac{i}{2}\delta_{\rho\mu}([\mathcal{U}_{+\lambda},\mathcal{U}_{-\lambda}]+[W,F])$	$-\frac{i}{2}\epsilon_{\rho\mu\alpha\beta}[\mathcal{U}_{+\alpha},\mathcal{U}_{+\beta}]$
$\rho_{\mu\nu}$	$-i\epsilon_{\rho\sigma\mu\nu}[\mathcal{U}_{-\sigma},F]$	$-i\delta_{\rho\sigma\mu\nu}[\mathcal{U}_{+\sigma},F]$
$\tilde{\lambda}_\mu$	$+\frac{i}{2}\epsilon_{\rho\mu\alpha\beta}[\mathcal{U}_{-\alpha},\mathcal{U}_{-\beta}]$	$+i[\mathcal{U}_{+\mu},\mathcal{U}_{-\rho}]$ $-\frac{i}{2}\delta_{\rho\mu}([\mathcal{U}_{+\lambda},\mathcal{U}_{-\lambda}]-[W,F])$
$\tilde{\rho}$	$-i[\mathcal{U}_{+\rho},F]$	0

On shell closure of SUSY algebra:

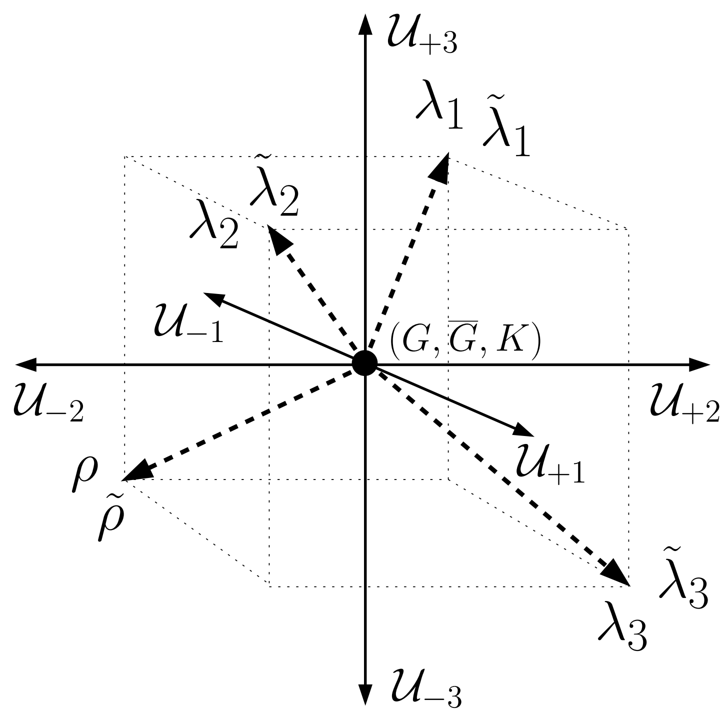
For the closure on fermonic fields, equations of motion are used.

$$\begin{aligned}
\{s, s_\mu\}(\varphi)_{x+a_\varphi,x} &\doteq +i[\mathcal{U}_{+\mu}, \varphi]_{x+n_\mu+a_\varphi,x}, \\
\{s_{\rho\sigma}, s_\mu\}(\varphi)_{x+a_\varphi,x} &\doteq +i\delta_{\rho\sigma\mu\nu}[\mathcal{U}_{-\nu}, \varphi]_{x-n_\nu+a_\varphi,x}, \\
\{s_{\rho\sigma}, \tilde{s}_\mu\}(\varphi)_{x+a_\varphi,x} &\doteq +i\epsilon_{\rho\sigma\mu\nu}[\mathcal{U}_{+\nu}, \varphi]_{x+n_\nu+a_\varphi,x}, \\
\{\tilde{s}, \tilde{s}_\mu\}(\varphi)_{x+a_\varphi,x} &\doteq -i[\mathcal{U}_{-\mu}, \varphi]_{x-n_\mu+a_\varphi,x},
\end{aligned}$$

$$\begin{aligned}
\{s, \tilde{s}\}(\varphi)_{x+a_\varphi,x} &\doteq -i[W, \varphi]_{x+a+\tilde{a}+a_\varphi,x}, \\
\{s_{\mu\nu}, s_{\rho\sigma}\}(\varphi)_{x+a_\varphi,x} &\doteq +i\epsilon_{\mu\nu\rho\sigma}[W, \varphi]_{x+a+\tilde{a}+a_\varphi,x}, \\
\{s_\mu, \tilde{s}_\nu\}(\varphi)_{x+a_\varphi,x} &\doteq -i\delta_{\mu\nu}[F, \varphi]_{x-a-\tilde{a}+a_\varphi,x}, \\
\{others\}(\varphi)_{x+a_\varphi,x} &\doteq 0,
\end{aligned}$$

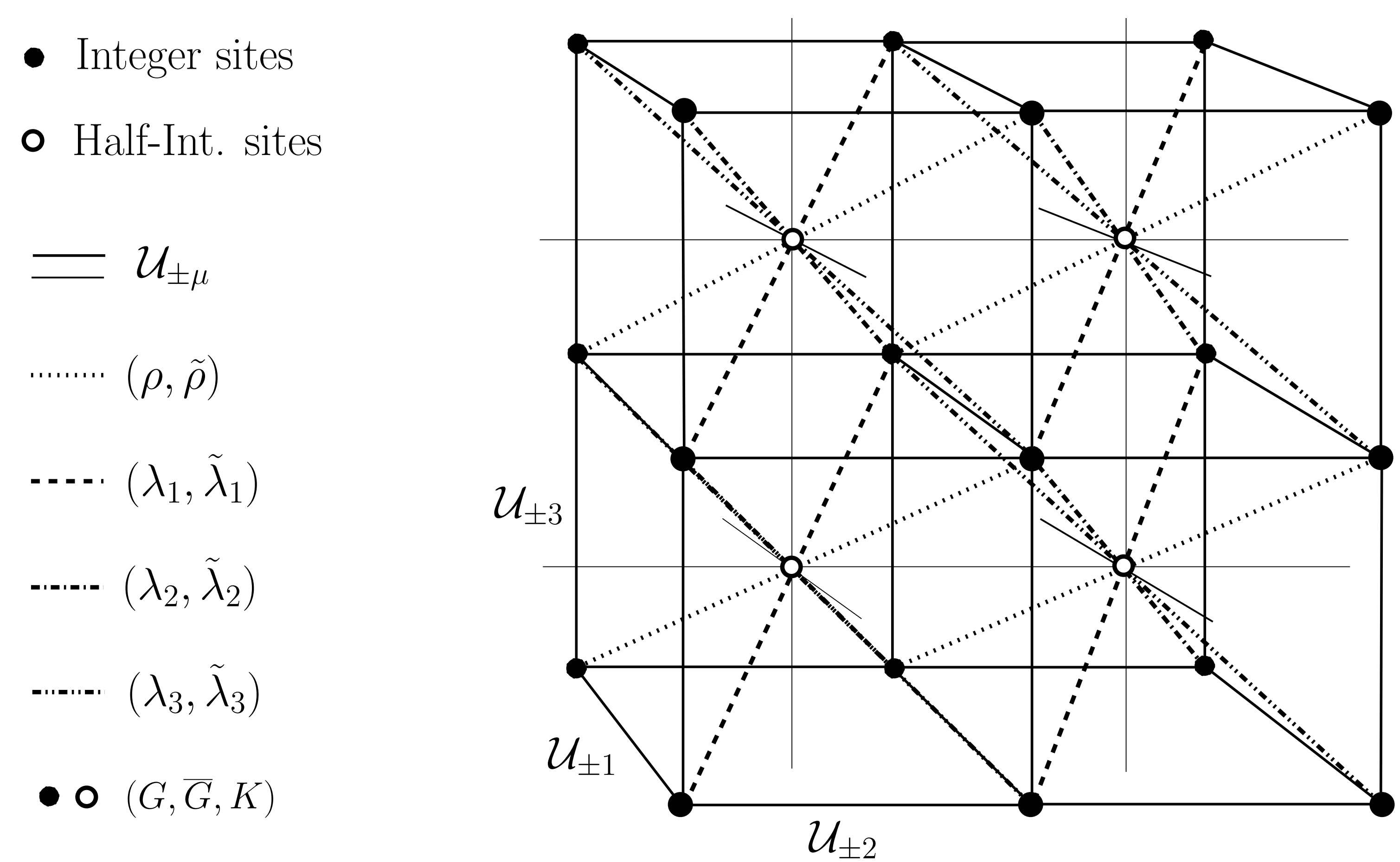
Equations of motion

$$\begin{aligned}
[\mathcal{U}_{+\mu}, \lambda_\mu] - [W, \tilde{\rho}] &= 0, \\
[\mathcal{U}_{-\mu}, \tilde{\lambda}_\mu] - [W, \rho] &= 0, \\
[\mathcal{U}_{+\mu}, \rho] - [\mathcal{U}_{-\nu}, \rho_{\mu\nu}] + [F, \tilde{\lambda}_\mu] &= 0, \\
[\mathcal{U}_{-\mu}, \tilde{\rho}] + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[\mathcal{U}_{+\nu}, \rho_{\rho\sigma}] + [F, \lambda_\mu] &= 0, \\
\delta_{\mu\nu\rho\sigma}[\mathcal{U}_{-\rho}, \lambda_{-\sigma}] + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[W, \rho_{\rho\sigma}] - \epsilon_{\mu\nu\rho\sigma}[\mathcal{U}_{+\rho}, \tilde{\lambda}_\sigma] &= 0,
\end{aligned}$$



3-dim. N=4 Lattice super Yang-Mills

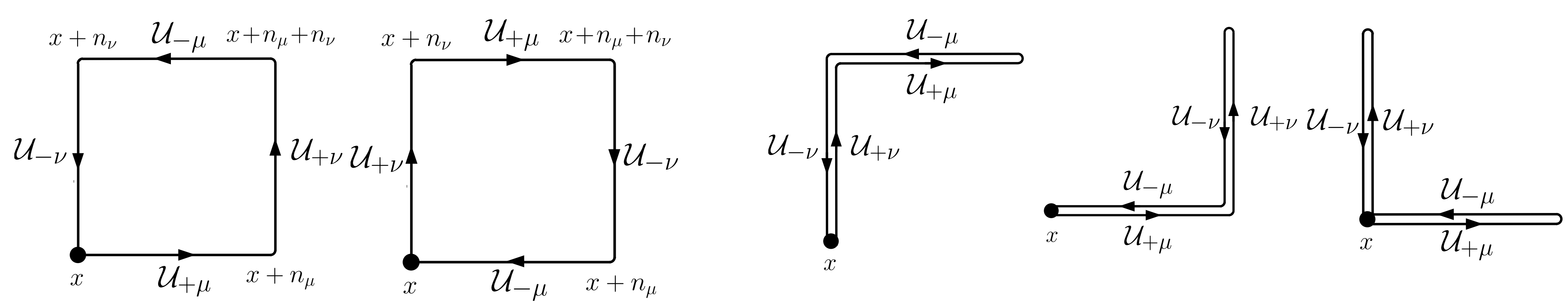
Geometrical locations of fields are determined automatically by graded Jacobi identities.



	$\mathcal{U}_{\pm\mu}$	W	F	ρ	λ_μ	$\rho_{\mu\nu}$	$\tilde{\lambda}_\mu$	$\tilde{\rho}$	∇	∇_μ	$\nabla_{\mu\nu}$	$\tilde{\nabla}_\mu$	$\tilde{\nabla}$
shift	$\pm n_\mu$	$a+\tilde{a}$	$-a-\tilde{a}$	$-a$	$-a_\mu$	$-a_{\mu\nu}$	$-\tilde{a}_\mu$	$-\tilde{a}$	$+a$	$+a_\mu$	$+a_{\mu\nu}$	$+\tilde{a}_\mu$	$+\tilde{a}$

$$
\begin{aligned}
S^{N=D=4}_{lat.\ TSYM} \;=\; \sum_x \mathrm{tr}\Bigg[& \frac{1}{2}[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]_{x,x-n_\mu-n_\nu}[\mathcal{U}_{-\mu},\mathcal{U}_{-\nu}]_{x-n_\mu-n_\nu,x} \\
& -\frac{1}{4}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}]_{x,x}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}]_{x,x} - \frac{1}{4}[W,F]_{x,x}[W,F]_{x,x} \\
& +\frac{1}{2}[\mathcal{U}_{+\mu},W]_{x,x-n_\mu-a-\tilde{a}}[\mathcal{U}_{-\mu},F]_{x-n_\mu-a-\tilde{a},x} \\
& +\frac{1}{2}[\mathcal{U}_{-\mu},W]_{x,x+n_\mu-a-\tilde{a}}[\mathcal{U}_{+\mu},F]_{x+n_\mu-a-\tilde{a},x} \\
& -i(\lambda_\mu)_{x,x+a_\mu}[\mathcal{U}_{+\mu},\rho]_{x+a_\mu,x} + i\tilde{\rho}_{x,x+\tilde{a}}[W,\rho]_{x+\tilde{a},x} - i(\lambda_\mu)_{x,x+a_\mu}[F,\tilde{\lambda}_\mu]_{x+a_\mu,x} \\
& +i(\lambda_\mu)_{x,x+a_\mu}[\mathcal{U}_{-\nu},\rho_{\mu\nu}]_{x+a_\mu,x} - i(\tilde{\rho})_{x,x+\tilde{a}}[\mathcal{U}_{-\mu},\tilde{\lambda}_\mu]_{x+\tilde{a},x} \\
& -\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}(\tilde{\lambda}_\mu)_{x,x+\tilde{a}_\mu}[\mathcal{U}_{+\nu},\rho_{\rho\sigma}]_{x+\tilde{a}_\mu,x} - \frac{i}{8}\epsilon_{\mu\nu\rho\sigma}(\rho_{\mu\nu})_{x,x+a_{\mu\nu}}[W,\rho_{\rho\sigma}]_{x+a_{\mu\nu},x}\Bigg],
\end{aligned}
$$

sum of link loops



SUSY transformation of fields ??

$$(s_A\varphi)_{x+a_\varphi+a_A,x} = s_A(\varphi)_{x+a_\varphi,x} \equiv [\nabla_A, \varphi\}_{x+a_\varphi+a_A,x}$$

We can show that s_A operation to lattice action vanishes for all s_A :

$$s_A S^{N=D=4}_{lat.\ TSYM} = 0$$

This sounds like the exact lattice SUSY invariance for all super charges.

However s_A or equivalently ∇_A carries a shift a_A , thus s_A operation to the action generates a link hole $(x, x+a_A)$.
This is because the action is the summation of closed link loops.
Furthermore gauge invariance is lost after s_A operation due to the link holes.

How do we cure this problem ?

We introduce local SUSY link parameter $(\epsilon^A)^{ab}_{x,x+a_A}$ which has an opposite shift of $(\nabla^A)^{ab}_{x+a_A,x}$ and carry gauge suffices.

correct SUSY transformation can be defined:

$$\delta\phi_\alpha = \epsilon^A[\nabla_A, \phi_\alpha\} = \epsilon^A s_A \phi_\alpha$$

Let us suppose for simplicity that the action is a product of 3 fields (generalization trivial):

$$S = \sum_x Tr(\phi_1\phi_2\phi_3)_{x,x}$$

Lattice SUSY operation on the action is

$$\delta S = \sum_x Tr(\epsilon^A(s_A\phi_1)\phi_2\phi_3 + \phi_1\epsilon^A(s_A\phi_2)\phi_3 + \phi_1\phi_2\epsilon^A(s_A\phi_3))$$

$$= \sum_x Tr(\underset{\substack{\parallel \\ 0}}{\epsilon^A s_A(\phi_1\phi_2\phi_3)} + \underbrace{[\phi_1, \epsilon_A\}}_{\neq 0} s_A(\phi_2\phi_3) + \phi_1 \underbrace{[\phi_2, \epsilon^A\}}_{\neq 0} s_A\phi_3)$$

non-Abelian gauge group

Since ϵ^A should be local and gauge variant link super parameter $[\phi_i, \epsilon^A\} \neq 0$ and thus exact SUSY invariance for s_A is lost on the lattice.

When one of shift vectors is zero: $a_A = 0$, ϵ_A is not a link parameter and can be taken as constant super parameter and thus: $[\phi_i, \epsilon^A\} = 0$.

One of 16 super charges can be kept exact when the corresponding shift vector vanishes.

Abelian gauge group

$$[\phi_i, \epsilon^A\} = (\phi_i)_{x+a_i-a_A, x-a_A}(\epsilon^A)_{x-a_A,x} - (\epsilon^A)_{x+a_i-a_A, x+a_i}(\phi_i)_{x+a_i,x} = 0$$

Here ordering equivalence class identification is imposed.

$$[\phi_A, \phi_B\} = 0 \qquad (\phi_A \neq \text{differential operator}) \qquad [\mathcal{U}_{\pm\mu}, \phi_A] \neq 0$$

Abelian action is invariant for all super charges with the ordering equivalence class identification.

Conclusions and Discussions

- We have formulated N=D=4 twisted super Yang-Mills theory on the lattice by the link approach.
- Anyone of N=4 super charges can be made exactly supersymmetric.
- Ordering equivalence class identification is defined to solve ordering ambiguity of a product of fields.
- Local super parameters having link nature are introduced to keep gauge invariance of the action before and after the SUSY transformation.

In order to cancel the terms including the local super parameter we may need super gravity contribution to realize exact lattice SUSY ?