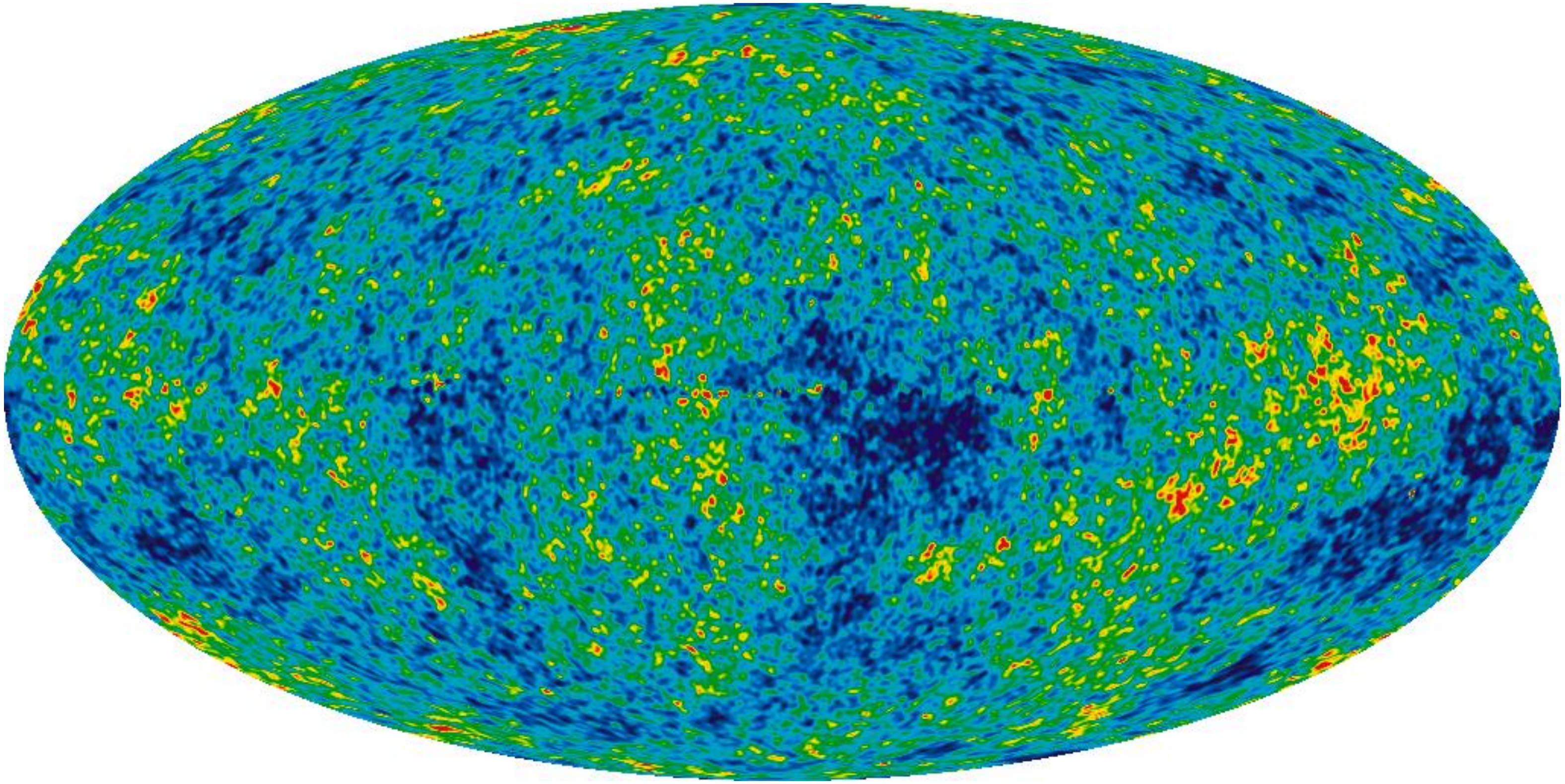
What is the B mode in CMB polarization

Contents

WMAP



https://map.gsfc.nasa.gov/media/121238/index.html

Temperature range ± 200 μK

What do we see?

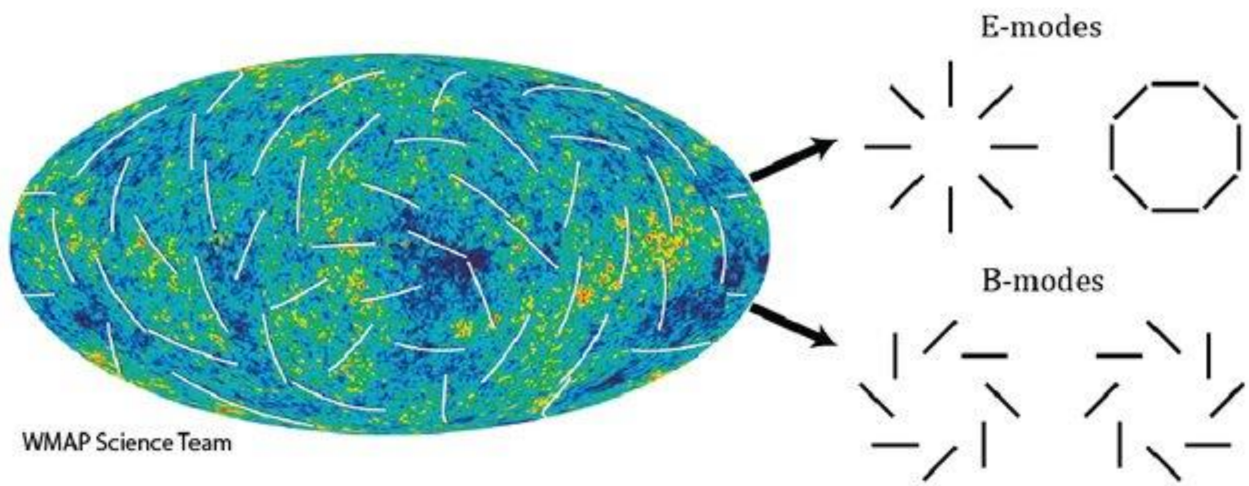
Tiny signals on the "frosted glass"

- 10^{-4} in intensity
- 10^{-7} or 10^{-8} in polarization

What do we see?

- Tiny signals on the "frosted glass"
- 10^{-4} in intensity
- 10^{-7} or 10^{-8} in polarization

Polarization



??

https://www.researchgate.net/publication/263811882_The_Primordial_Inflation_Polarization_Explorer_PIPER/figures?lo=1

“The CMB anisotropy polarization map may be decomposed into curl-free even-parity *E*-modes and divergence-free odd-parity *B*-modes. Primordial *B*-modes are only created by tensor perturbations (inflationary gravitational waves).“

Polarization

Polarization map may be decomposed into

- curl-free even-parity E modes
- divergence-free odd-parity B modes

Primordial B modes are only created by tensor perturbations (inflationary gravitational waves)

Looking back

- Now = 1.4×10^{10} y

  $\sim 4.4 \times 10^{17}$ s

4.5 orders

- CMB = 3.8×10^{5} y

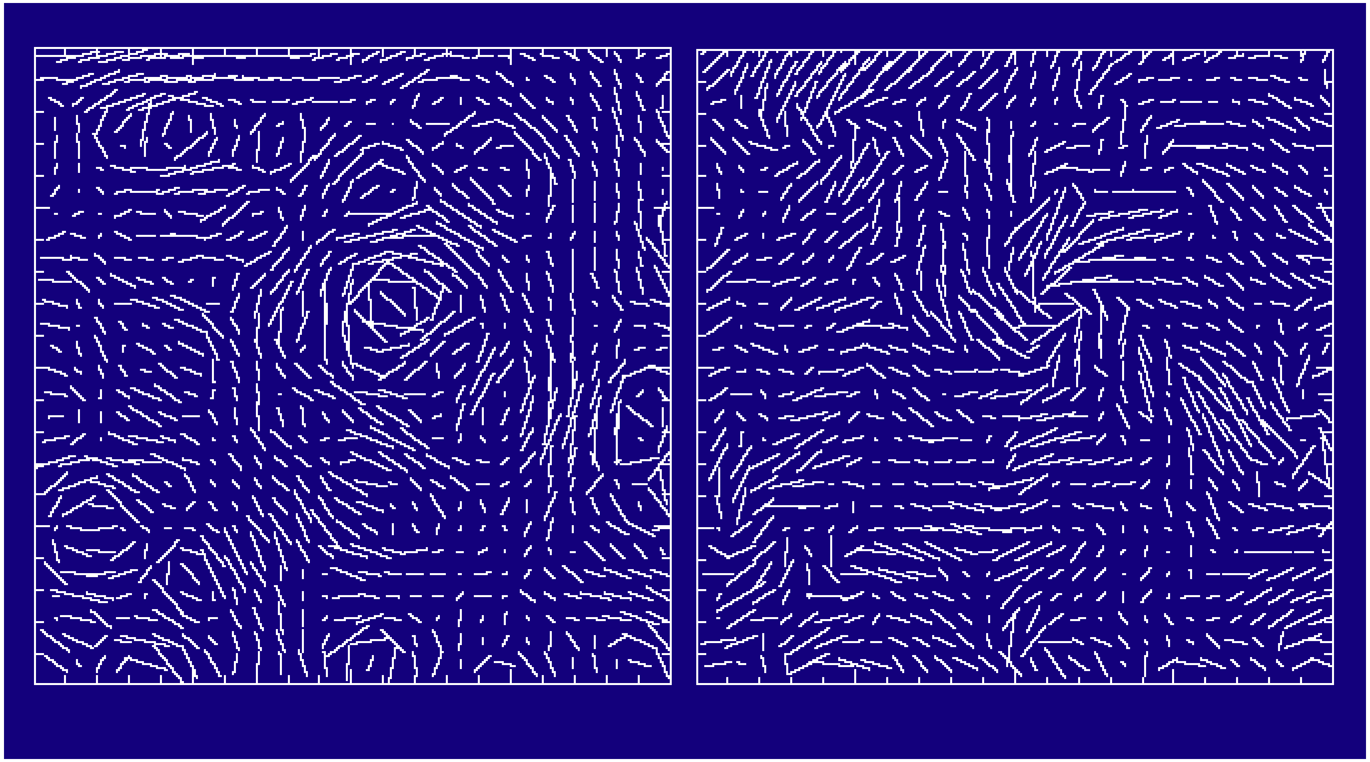
  $\sim 1.2 \times 10^{13}$ s

45 orders !!

- B-mode/Inflation

  $\sim 10^{-32}$ s

E mode and B mode



http://background.uchicago.edu/~whu/intermediate/Polarization/polar5.html

Heuristic statement

E mode

- Curl free
- It is a divergence
- $\sim 10^{-7}$

B mode

- Divergence free
- It is a curl
- $\sim 10^{-8}$??

B mode

“a twisting pattern, or spin, known as a curl or B mode”

Breaking New, *Nature* 507, 7492 (2014)

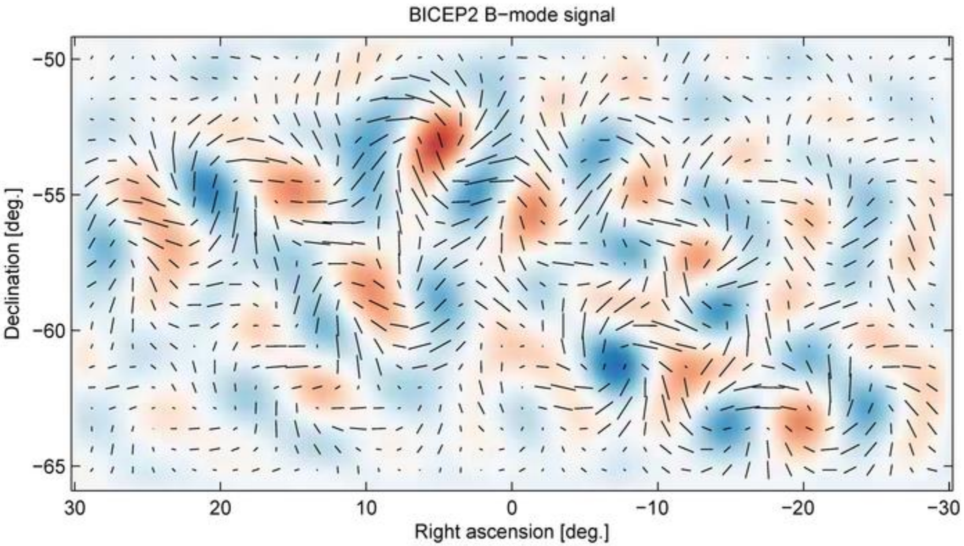
“the curl component (B mode)”

Kamionkowski & Kovetx
Ann Rev Astron & Astrophy, 54, 227 (2016)

B mode

- The “smoking gun”
- Primordial B mode can only be produced by tensor perturbations
- Would be evidence of inflation

B mode



https://www.cfa.harvard.edu/~cbischoff/cmb/

- Seen by BICEP2
- 2014
- High profile announcement

https://www.nytimes.com/2014/06/20/science/space/scientists-debate-gravity-wave-detection-claim.html

Announcement & Retraction

- BICEP2 (2014), Detection of B-Mode Polarization at Degree Angular Scales by BICEP2, Phys. Rev. Lett. 112, 24, 241101
- BICEP2/Keck & Planck (2015)

  Joint Analysis of BICEP2/Keck Array and Planck Data, Phys. Rev. Lett. 114, 10, 101301
- Foreground dust!

Questions

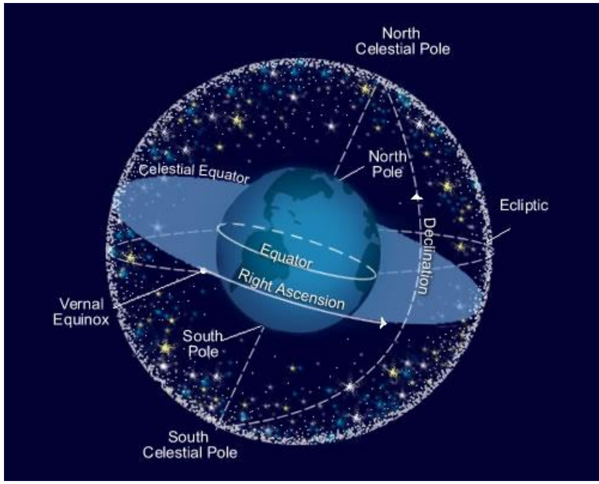
All these important physics questions rely on exquisite measurement of polarization on a surface and extracting the B mode

But

- How is polarization field decomposed?
- Is the E/B mode “like a grad/curl”?

What surface?

- Celestial sphere
- A plane
  - Math (toy) model
  - Small patch of sky
  - Any tangent plane of celestial sphere



http://planetary-science.org/astronomy/the-celestial-globe/

Decompose?

Literature seems hard, not accessible …, but

- No need for inflation, cosmology, GR, …
- Simple concept of plane surface (not sphere)
- Forget pixels

- It is a problem in classical electrodynamics in vacuum!

Like grad/ curl?

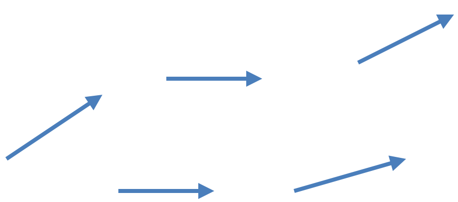
- No!
- Polarization is not a vector field, but a tensor field
- What does grad/ curl mean?

- Not like E and B field
- (E, B) modes $\longrightarrow$ (A, B) modes

Vector / tensor

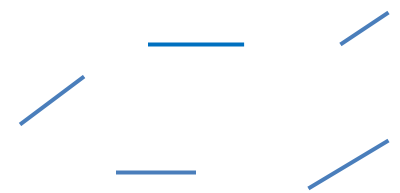
Vector field

- Arrows
- Returns after 360^{o}



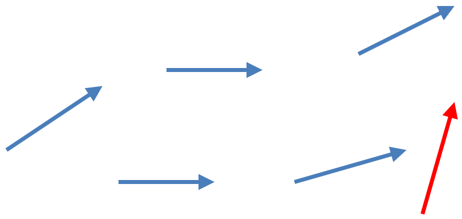
Polarization field

- Unsigned lines
- Returns after 180^{o}



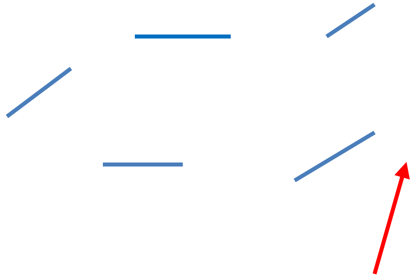
Curl?

Vector field



$$\sum \vec{V}\cdot\Delta\vec{r}$$

Polarization field



$$\sum \bar{P}\cdot\Delta\vec{r}$$?? No such thing

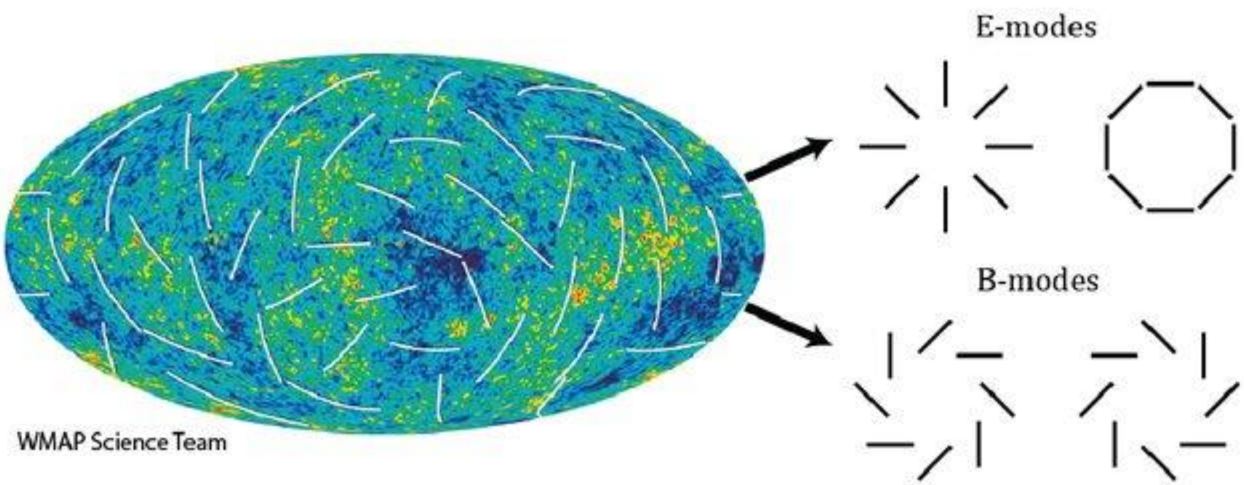
Factor of 2

Vector field

- V_i spin 1
- Returns after 360°
- Makes 0/90° with $\mathbf{k}$

Polarization field

- Av $E_i{}^*E_j$ spin 2
- Returns after 180°
- Makes ±45° with $\mathbf{k}$



Contents

Vector field on a plane

$\mathbf{V}(\mathbf{r})$ on a "patch" Ω on the plane

$$\mathbf{r} = (x, y)^{\mathrm{T}} = (x_1, x_2)^{\mathrm{T}}$$

$$\mathbf{d} = (d_1, d_2)^{\mathrm{T}} = (\partial_x, \partial_y)^{\mathrm{T}} = \nabla_{\perp}$$

$$\tilde{\mathbf{d}} = (\tilde{d}_1, \tilde{d}_2)^{\mathrm{T}} = (\partial_y, -\partial_x)^{\mathrm{T}}$$

$$\tilde{\mathbf{d}} = -\hat{\mathbf{n}} \times \mathbf{d} = [\mathbf{E}]\,\mathbf{d}$$

$$[\mathbf{E}] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Vector field on a plane

$$\mathbf{d} = (d_1, d_2)^{\mathrm{T}} = (\partial_x, \partial_y)^{\mathrm{T}}$$

$$\tilde{\mathbf{d}} = (\tilde{d}_1, \tilde{d}_2)^{\mathrm{T}} = (\partial_y, -\partial_x)^{\mathrm{T}}$$

$$\mathbf{d}^{\mathrm{T}}\,\mathbf{d} = \tilde{\mathbf{d}}^{\mathrm{T}}\,\tilde{\mathbf{d}} = \nabla_{\perp}^{2}$$

$$\tilde{\mathbf{d}}^{\mathrm{T}}\,\mathbf{d} = \mathbf{d}^{\mathrm{T}}\,\tilde{\mathbf{d}} = 0$$

Vector field on a plane

Given $\mathbf{V}(\mathbf{r})$, seek representation

$$\mathbf{V} = \mathbf{d}\Phi^a - \tilde{\mathbf{d}}\Phi^b = \nabla_\perp \Phi^a + \nabla_\perp \times (\hat{\mathbf{n}}\Phi^b)$$

$$\mathbf{d}^\mathrm{T}: \qquad \nabla_\perp^2 \Phi^a = \mathbf{d}^\mathrm{T}\mathbf{V} \equiv \varphi^a$$

$$\tilde{\mathbf{d}}^\mathrm{T}: \qquad \nabla_\perp^2 \Phi^b = -\tilde{\mathbf{d}}^\mathrm{T}\mathbf{V} \equiv \varphi^b$$

Can laplacian be inverted?

Yes for global data

Some ambiguity for a patch

Vector field on a plane

$$\mathbf{V} = \mathbf{d}\Phi^a - \tilde{\mathbf{d}}\Phi^b$$

$$V_1 = \partial_x \Phi^a - \partial_y \Phi^b$$

$$V_2 = \partial_y \Phi^a + \partial_x \Phi^b$$

$$V_1 \pm iV_2 = \partial_\pm (\Phi^a \pm i\Phi^b)$$

$$\mathbf{V} \pm i\tilde{\mathbf{V}} = (\mathbf{d} \pm i\tilde{\mathbf{d}})(\Phi^a \pm i\Phi^b)$$

Vector field on a plane

$$V_1 \pm iV_2 = \partial_\pm (\Phi^a \pm i\Phi^b)$$

$$\partial_\mp (V_1 \pm iV_2) = \nabla_\perp^2 (\Phi^a \pm i\Phi^b) = \varphi^a \pm i\varphi^b$$

$$\mathbf{V} \pm i\tilde{\mathbf{V}} = (\mathbf{d} \pm i\tilde{\mathbf{d}})(\Phi^a \pm i\Phi^b)$$

$$(\mathbf{d} \mp i\tilde{\mathbf{d}})(\mathbf{V} \pm i\tilde{\mathbf{V}}) = \nabla_\perp^2 (\Phi^a \pm i\Phi^b) = \varphi^a \pm i\varphi^b$$

Decomposition

$$\Phi^{a,b}(\mathbf{r})$$

$??\ \big\downarrow\ \mathbf{d}, \tilde{\mathbf{d}}$

$\nabla_\perp^2$

$$\mathbf{V}(\mathbf{r})$$

$\big\downarrow\ \mathbf{d}, \tilde{\mathbf{d}}$

$$\varphi^{a,b}(\mathbf{r})$$

Using two scalar fields

- Independent of axes
- Leads to expansion in terms of harmonic functions
- This method is unique to 2D

Contents

3. Polarization field

3.1 Review of polarization

3.2 Decomposition of polarization field

Review of polarization

Propagate along *z*, measure on *xy* plane

$$E_j = a_j\, e^{i\delta_j} \quad , j = 1, 2$$

$$S_{ij} = \text{Av}\, E_i^* E_j$$

$$|\Psi\rangle = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$$[\mathbf{S}] = \text{Av}\, |\Psi\rangle\langle\Psi| = \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}$$

Stokes' parameters

Review of polarization

$$[\mathbf{S}] = \mathrm{Av}\left|\Psi\right\rangle\left\langle\Psi\right| = \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}$$

$I \sim$ total intensity

$Q, U \sim$ linear polarization

$V \sim$ circular polarization

Review of polarization

Ignore I

Ignore V because

(a) need CP violation,

(b) instruments cannot measure

$$[\mathbf{S}] \to [\mathbf{P}] = \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix} = \left(\mathbf{P}, \tilde{\mathbf{P}}\right)$$

$[\mathbf{P}] =$ 2×2 sym traceless tensor

$\mathbf{P} =$ doublet $\neq$ vector

Review of polarization

$$[\mathbf{P}] = \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix} = \left(\mathbf{P}, \tilde{\mathbf{P}}\right)$$

$[\mathbf{P}] =$ 2×2 sym traceless tensor

Under rotations by angle α

$$[\mathbf{P}] \rightarrow [\mathbf{P}'] = [\mathbf{R}]^{\mathrm{T}} [\mathbf{P}][\mathbf{R}] \qquad [\mathbf{R}] = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$Q \pm iU \rightarrow Q' \pm iU' = e^{\pm 2 i\alpha} (Q \pm iU)$$

Spin 2 , returns to itself under 180^{o} rotation

Review of polarization

$$Q \sim E_x^2 - E_y^2 = E^2(\cos^2\chi - \sin^2\chi) \sim \cos 2\chi$$

$$U \sim 2E_x E_y = 2E^2 \cos\chi \sin\chi \sim \sin 2\chi$$

$$\tan 2\chi = U / Q$$

2χ is determined mod 2π

3. Polarization field

3.1 Review of polarization

3.2 Decomposition of polarization field

Decompose polarization

$$V_1 \pm iV_2 = \partial_{\pm}(\Phi^a \pm i\Phi^b) \qquad \text{spin 1}$$

$$Q \pm iU = \partial_{\pm}^2(\Phi^a \pm i\Phi^b) \qquad \text{spin 2}$$

- Conceptually, this is all we need
- Scalars are independent of axes on tangent planes
- But this representation is not convenient for mapping to other (tangent) planes

Decompose polarization

$$V_1 \pm iV_2 = \partial_\pm(\Phi^a \pm i\Phi^b) \qquad \text{spin 1}$$

$$Q \pm iU = \partial_\pm^2(\Phi^a \pm i\Phi^b) \qquad \text{spin 2}$$

$$\partial_\pm^2 = D_1 \pm iD_2$$

$$D_1 = \partial_x^2 - \partial_y^2 \qquad D_2 = 2\partial_x\partial_y$$

$$Q = D_1\Phi^a - D_2\Phi^b$$

$$U = D_2\Phi^a + D_1\Phi^b$$

Decompose polarization

$$Q = D_1\Phi^a - D_2\Phi^b$$

$$U = D_2\Phi^a + D_1\Phi^b$$

$$\mathbf{D} = (D_1, D_2)^{\mathrm{T}} \qquad \tilde{\mathbf{D}} = (D_2, -D_1)^{\mathrm{T}} = [\mathbf{E}]\mathbf{D}$$

$$\mathbf{P} = \mathbf{D}\Phi^a - \tilde{\mathbf{D}}\Phi^b$$

Compare

$$\mathbf{V} = \mathbf{d}\Phi^a - \tilde{\mathbf{d}}\Phi^b$$

Decompose polarization

$$\mathbf{D} = (D_1, D_2)^{\mathrm{T}} \qquad \tilde{\mathbf{D}} = (D_2, -D_1)^{\mathrm{T}}$$

$$\mathbf{D}^{\mathrm{T}}\mathbf{D} = \tilde{\mathbf{D}}^{\mathrm{T}}\tilde{\mathbf{D}} = \nabla_{\perp}^{4}$$

$$\tilde{\mathbf{D}}^{\mathrm{T}}\mathbf{D} = \mathbf{D}^{\mathrm{T}}\tilde{\mathbf{D}} = 0$$

Compare

$$\mathbf{d}^{\mathrm{T}}\mathbf{d} = \tilde{\mathbf{d}}^{\mathrm{T}}\tilde{\mathbf{d}} = \nabla_{\perp}^{2}$$

$$\tilde{\mathbf{d}}^{\mathrm{T}}\mathbf{d} = \mathbf{d}^{\mathrm{T}}\tilde{\mathbf{d}} = 0$$

Decompose polarization

Given $\mathbf{P}(\mathbf{r})$, seek representation

$$\mathbf{P} = \mathbf{D}\Phi^a - \tilde{\mathbf{D}}\Phi^b$$

$$\mathbf{D}^{\mathrm{T}}: \qquad \nabla_\perp^4 \Phi^a = \mathbf{D}^{\mathrm{T}}\mathbf{P} \equiv \varphi^a$$

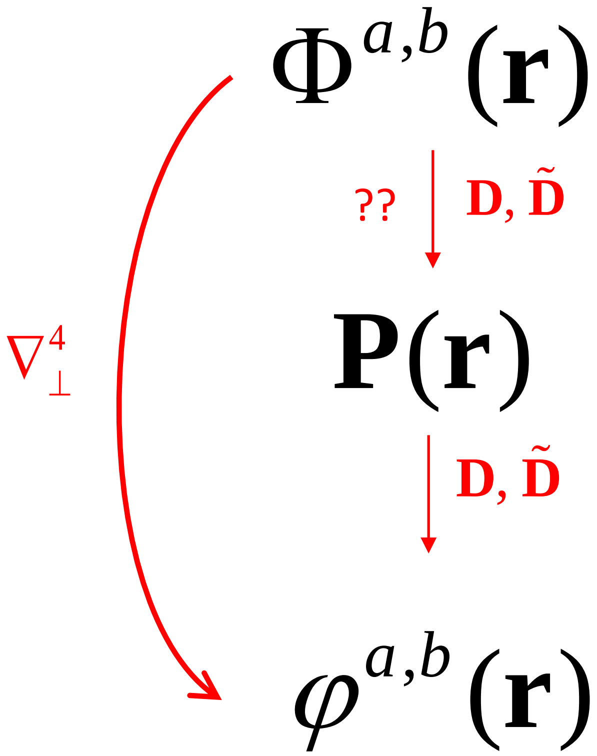
$$\tilde{\mathbf{D}}^{\mathrm{T}}: \qquad \nabla_\perp^4 \Phi^b = -\mathbf{D}^{\mathrm{T}}\mathbf{P} \equiv \varphi^b$$

Can bilaplacian be inverted?

Yes for global data

Some ambiguity for a patch

Decomposition



Decompose polarization

Given $\mathbf{D}$ (doublet), introduce $[\mathbf{D}]$ $(2\times 2$ matrix$)$

$$[\mathbf{D}]=\left(\mathbf{D},\tilde{\mathbf{D}}\right)$$

In Cartesian representation

$$[\mathbf{D}]\pm i[\tilde{\mathbf{D}}]=(\mathbf{d}\pm i\tilde{\mathbf{d}})\otimes(\mathbf{d}\pm i\tilde{\mathbf{d}})$$

$$=(\mathbf{d}\otimes\mathbf{d}-\tilde{\mathbf{d}}\otimes\tilde{\mathbf{d}})\pm i(\mathbf{d}\otimes\tilde{\mathbf{d}}+\tilde{\mathbf{d}}\otimes\mathbf{d})$$

in terms of which

$$[\mathbf{P}]\pm i[\tilde{\mathbf{P}}]=([\mathbf{D}]\pm i[\tilde{\mathbf{D}}])(\Phi^{a}\pm i\Phi^{b})$$

Decompose polarization

$$[\mathbf{P}] = \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix} = \left(\mathbf{P}, \tilde{\mathbf{P}}\right)$$

Tensor representation says the same thing twice. Advantage?

- Indices are Cartesian
- Can regard all quantities as in 3D space
- Specified by normal vector
- Normal components zero

Contents

Global analysis (1)

$$\mathbf{V} \sim (\mathbf{d}, \tilde{\mathbf{d}})\, \Phi^{a,b}$$

local

$$(\mathbf{d}, \tilde{\mathbf{d}})\, \mathbf{V} \quad \sim \varphi^{a,b}$$

$$\nabla_{\perp}^{2} \Phi^{a,b} = \varphi^{a,b}$$

global

$$\varphi^{a,b}(\mathbf{r}) = \sum_{\alpha} c_{\alpha}^{a,b}\, F_{\alpha}(\mathbf{r}) \quad , \text{ e.g., plane waves } \mathbf{k}$$

$$\Phi^{a,b}(\mathbf{r}) = \sum_{\alpha} C_{\alpha}^{a,b}\, F_{\alpha}(\mathbf{r})$$

$$C_{\alpha}^{a,b} = \lambda_{\alpha}^{-1} c_{\alpha}^{a,b}$$

$$\lambda_{\alpha} = \text{ eigenvalue of } -\nabla_{\perp}^{2}, \text{ e.g., } \mathbf{k}^{2}$$

Global analysis (2)

$$\mathbf{V} = \mathbf{d}\Phi^{a} - \tilde{\mathbf{d}}\Phi^{b}$$

$$\Phi^{a,b}(\mathbf{r}) = \sum_{\alpha} C_{\alpha}^{a,b} F_{\alpha}(\mathbf{r})$$

$$\mathbf{V}(\mathbf{r}) = \sum_{\alpha} C_{\alpha}^{a} \boxed{\mathbf{d}F_{\alpha}(\mathbf{r})} - \sum_{\alpha} C_{\alpha}^{b} \boxed{\tilde{\mathbf{d}}F_{\alpha}(\mathbf{r})}$$

vector harmonics

Global analysis (3)

$$\mathbf{P} \sim (\mathbf{D}, \tilde{\mathbf{D}})\, \Phi^{a,b}$$

local

$$(\mathbf{D}, \tilde{\mathbf{D}})\, \mathbf{P} \quad \sim \varphi^{a,b}$$

$$\nabla_{\perp}^{4} \Phi^{a,b} = \varphi^{a,b}$$

global

$$\varphi^{a,b}(\mathbf{r}) = \sum_{\alpha} c_{\alpha}^{a,b}\, F_{\alpha}(\mathbf{r}) \quad , \text{ e.g., plane waves } \mathbf{k}$$

$$\Phi^{a,b}(\mathbf{r}) = \sum_{\alpha} C_{\alpha}^{a,b}\, F_{\alpha}(\mathbf{r})$$

$$C_{\alpha}^{a,b} = \lambda_{\alpha}^{-2} c_{\alpha}^{a,b}$$

$$\lambda_{\alpha} = \text{ eigenvalue of } -\nabla_{\perp}^{2}, \text{ e.g., } \mathbf{k}^{2}$$

Global analysis (4)

$$\mathbf{P} = \mathbf{D}\Phi^{a} - \tilde{\mathbf{D}}\Phi^{b}$$

$$\Phi^{a,b}(\mathbf{r}) = \sum_{\alpha} C_{\alpha}^{a,b} F_{\alpha}(\mathbf{r})$$

$$\mathbf{P}(\mathbf{r}) = \sum_{\alpha} C_{\alpha}^{a} \boxed{\mathbf{D}F_{\alpha}(\mathbf{r})} - \sum_{\alpha} C_{\alpha}^{b} \boxed{\tilde{\mathbf{D}}F_{\alpha}(\mathbf{r})}$$

tensor harmonics

- Ordering of operators?

$$\mathbf{P}(\mathbf{r}) = \sum_\alpha C_\alpha^a \mathbf{D} F_\alpha(\mathbf{r}) - \sum_\alpha C_\alpha^b \tilde{\mathbf{D}} F_\alpha(\mathbf{r})$$

$$\mathbf{D} \sim \begin{pmatrix} \partial_x^2 - \partial_y^2 \\ 2\partial_x \partial_y \end{pmatrix} \sim -\begin{pmatrix} k_x^2 - k_y^2 \\ 2k_x k_y \end{pmatrix} \sim -k^2 \begin{pmatrix} \cos^2\phi - \sin^2\phi \\ 2\cos\phi \sin\phi \end{pmatrix}$$

$$\begin{pmatrix} Q \\ U \end{pmatrix} = -\sum_{\mathbf{k}} k^2 \times$$

$$\left[C_{\mathbf{k}}^a \begin{pmatrix} \cos 2\phi \\ \sin 2\phi \end{pmatrix} + C_{\mathbf{k}}^b \begin{pmatrix} -\sin 2\phi \\ \cos 2\phi \end{pmatrix} \right] F_{\mathbf{k}}$$

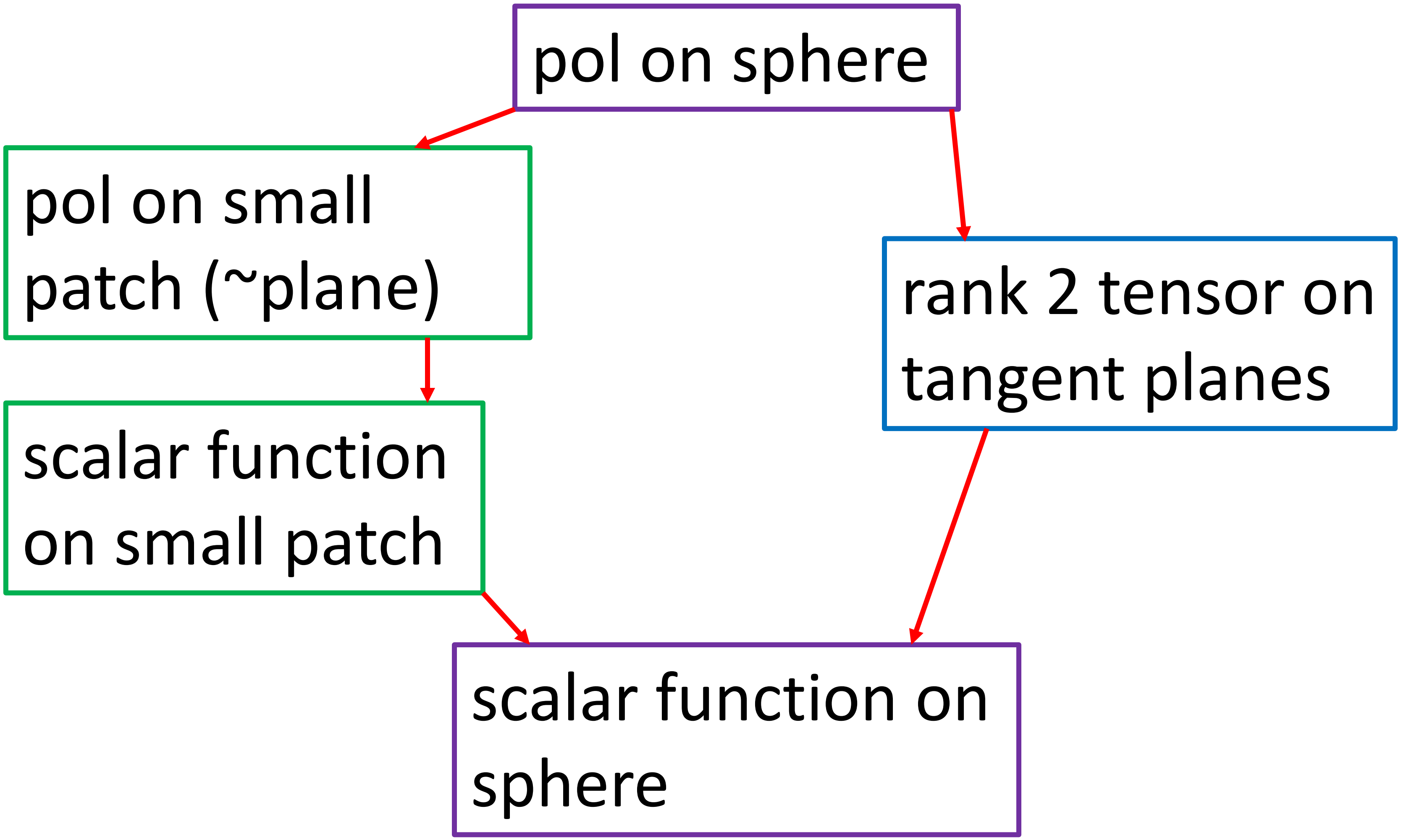
$\tan 2\chi = U / Q$ direction of polarization

A mode: $\tan 2\chi = \tan 2\phi$, $\chi - \phi = n\pi / 2$

B mode: $\tan 2\chi = -\cot 2\phi$, $\chi - \phi = \pi / 4 + n\pi / 2$

Contents

Two paths


pol on sphere
pol on small patch (~plane)
scalar function on small patch
rank 2 tensor on tangent planes
scalar function on sphere

Sphere

Recall

$$[\mathbf{P}] \pm i[\mathbf{P}] = ([\mathbf{D}] \pm i[\tilde{\mathbf{D}}])(\Phi^a \pm i\Phi^b)$$

$$[\mathbf{D}] \pm i[\tilde{\mathbf{D}}] = (\mathbf{d} \pm i\tilde{\mathbf{d}}) \otimes (\mathbf{d} \pm i\tilde{\mathbf{d}})$$

$$= (\mathbf{d} \otimes \mathbf{d} - \tilde{\mathbf{d}} \otimes \tilde{\mathbf{d}}) \pm i(\mathbf{d} \otimes \tilde{\mathbf{d}} + \tilde{\mathbf{d}} \otimes \mathbf{d})$$

All of these can be regarded as objects in 3D

But having normal component zero

But what are $\mathbf{d}$, $\tilde{\mathbf{d}}$?

Sphere

No need to say perp

Now write as

$$\tilde{\mathbf{d}} = -\hat{\mathbf{n}} \times \nabla \qquad \mathbf{d} = \mathbf{n} \times \hat{\mathbf{d}}$$

Can be ported to any tangent plane

On a sphere

$$\tilde{\mathbf{d}} = -i\mathbf{L} \qquad \mathbf{d} = i\tilde{\mathbf{L}}$$

Contents

Summary (1)

is the curl

$$V_1 \pm iV_2 = \partial_{\pm}(\Phi^a \pm i\Phi^b)$$

$$Q \pm iU = \partial_{\pm}^2(\Phi^a \pm i\Phi^b)$$

"like the curl"
But second-order derivative

Like the curl only in this formal sense

Summary (2)

schematic

$$\mathbf{V} \rightarrow \mathbf{dV} = \varphi^{a,b} \rightarrow \text{invert laplacian gives } \Phi^{a,b}$$

$$\mathbf{P} \rightarrow \mathbf{DV} = \varphi^{a,b} \rightarrow \text{invert bilaplacian gives } \Phi^{a,b}$$

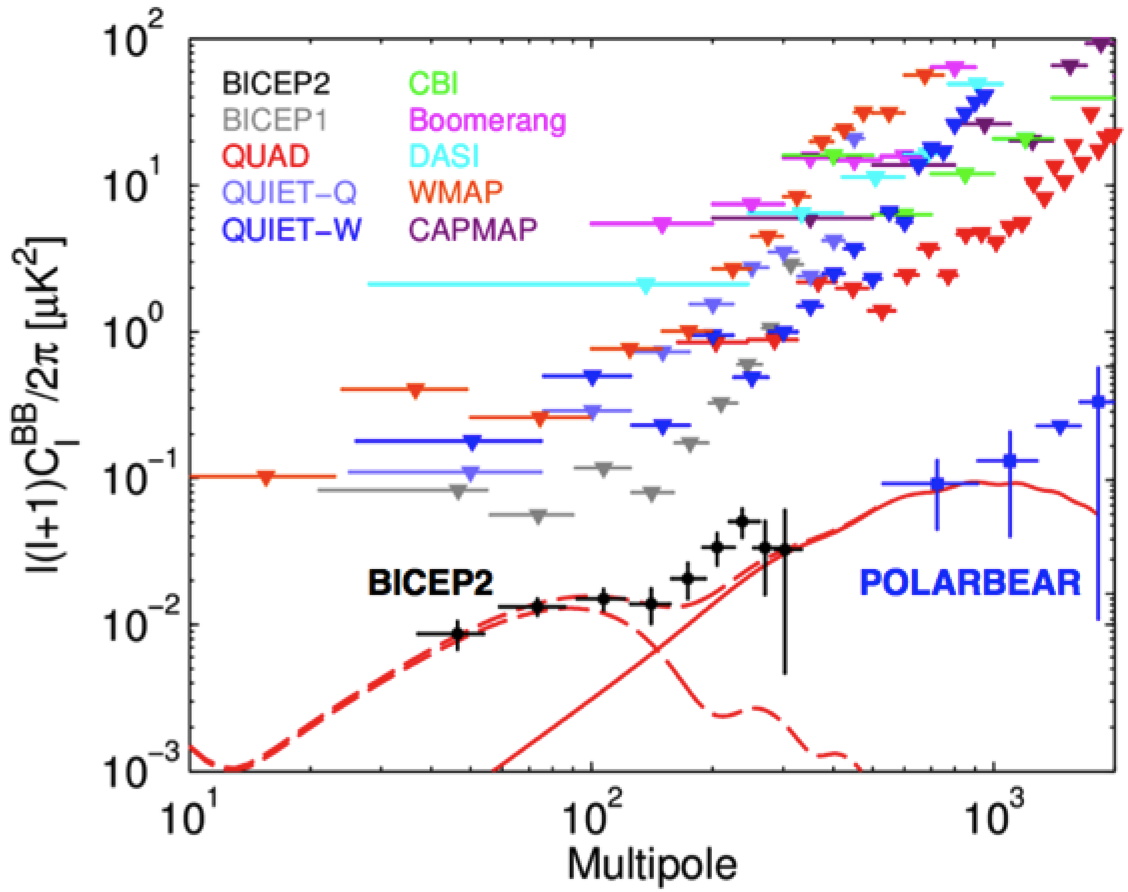
"Amount of B mode" given by

$$|\varphi^{b}| \text{ locally or } |\Phi^{b}| \text{ globally}$$

Or globally in terms of the mode

amplitudes of Φ^{b}

Observational limits (2014)

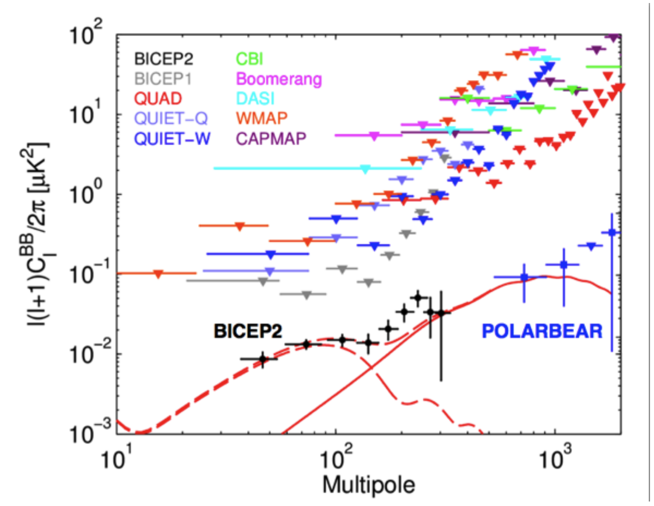


https://www.andrewjaffe.net/blog/2014/03/gravitational-w.html

Observational limits (2014)

Vertical axis is essentially the squared mode coefficients in

$$\Phi^b = \sum_{\ell m} C^b_{\ell m} \, Y_{\ell m}$$



https://www.andrewjaffe.net/blog/2014/03/gravitational-w.html

Conclusion

- Polarization
- Measurement very difficult
- Separation into two modes conceptually simple
- Removing other effects (dust, lensing, ...) tricky
- Potential implication is huge