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Plane Partition Realization of (Web of) W-algebra Minimal Models

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VOA Phase I: 1980s

- Virasoro algebra and the generalization (vertex operator algebra =VOA) is a fundamental symmetry in string theory and in exactly solvable statistical models.

  $$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

  Due to its importance, many variations of the symmetry was studied in the 80s. Famous examples are $N = 2$ superconformal algebra which describes Calabi-Yau compactifications and W-algebras which describes higher spin symmetry.
- The subject flourished in 80-90s when the perturbative study of string theory was essential, but became less popular since the mid-90s when the revolution of non-perturbative studies (such as Seiberg-Witten duality in $N = 2$ SYM) began.

VOA Phase II: 2010s

- The situation began to change when Nekrasov and his collaborators evaluated the instanton partition function of 4D $N = 2$ super Yang-Mills in the omega background. Localization technique implied that it is a sum of the contribution from the fixed point labeled by Young tables.

$$Z(q) = \sum_{\lambda} q^{|\lambda|} Z_{\lambda}(m, a)$$

- Another dramatic change occurs when Alday, Gaiotto and Tachikawa (AGT) conjectured that these Nekrasov factors may be identified with the conformal block of 2D conformal field theory. It became possible to use VOAs to describe the nonperturbative physics.
- In order to understand the dual role of VOA, one needs to notice that the VOA has apparently different descriptions.

Dual faces of VOA

- In Phase I, VOA was described by chiral fields (say, stress-energy tensor $T(z) = \sum_n L_n z^{-n-2}$) where z is the coordinate of Riemann surface. They are quantum fields on the world-sheet, and are appropriate to represent world sheet dynamics.
- On the other hand, their action on the Nekrasov factor Z_λ is complicated. Better set of operators is affine Yangian which is generated by Drinfeld currents $e(\zeta), f(\zeta), \psi(\zeta)$ where ζ is a spectral parameter. Their action of Nekrasov functions is simpler, for instance, e (f) is the operator which add (remove) a box to λ. On the other hand the algebra looks very different from VOA.
- 4D/2D duality implies that these two descriptions are equivalent.

W_∞ and affine Yangian of $\widehat{gl}_1$

- The correspondence between the first and the second picture is best described by enlarging VOA to W_∞ and affine Yangian of $gl(1)$ which are more universal. These two algebras are equivalent after the translation of parameters.
- W_∞ is a VOA which is described by infinite currents $W^{(n)}(z)$. If we write $W^{(n)}(z) = \sum_m W_{n,m} z^{-n-m}$, the generators have two indices $n, m \in \mathbb{Z}$. The power of z corresponds to the second index m.
- On the other hand, the Drinfeld currents $e(\zeta)$ (resp. $f(\zeta)$) is (roughly) identified as $\sum_{n\geq 0} W_{n,\pm 1}\zeta^{-n}$, expansion with respect to the first index which represents the spin of the operator.
- In this sense, the duality in VOA is realized by a rotation of indices

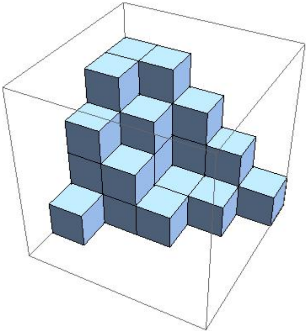
$$S : (n, m) \to (m, -n)$$

- Sometimes VOA Phase I (resp. II) is referred to as horizontal (resp. vertical) representation.
- The other VOA is obtained by a truncation of universal symmetry.

Gaiotto-Rapcak VOA and WoW

- Recently, Gaiotto and Rapcak introduced a family of VOAs (GR VOA) defined geometrically by intersecting branes.
- It may be either described by DS reduction of WZW model (horizontal picture) or a truncation of affine Yangian (vertical picture).
- General representation of affine Yangian is labeled by plane partition (3D Young diagram) which can have an infinite legs represented by three asymptotic Young diagrams.
- By connecting two plane partitions through the infinite legs, one may define a new algebra (tensor product of two Yangians with extra generators which modify the shared Young diagram). Each diagram represents a new VOA. It is called as Web-of W (WoW).
- Thus GR VOA and WoW give infinite families of algebras which have dual descriptions.

Plane partition – from Wikipedia



When it is sliced in one direction, it is decomposed into a set of partitions $\lambda_1 \succeq \cdots \succeq \lambda_n$. Plane partition may represent random surface or "melting crystal".

Minimal models for GR VOA or WoW

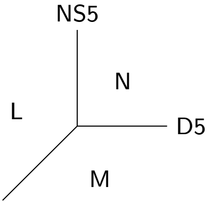
In this talk, I will explain the description of minimal model for GR VOA or WoW in the vertical frame.

- Minimal models are important since
    - Modular property by finite number of fields
    - They describe solvable statistical models such as Ising.
    - building block of Calabi-Yau compactification (Gepner)
- In the horizontal picture, we obtain the minimal models of VOA (W-algebra, for instance) by restricting the parameters to obtain the screening charges provide infinite null states.

In the vertical frame, GR VOA or WoW is given by a truncation of affine Yangian. We demonstrate that their minimal models by double truncation which is significantly simpler than horizontal picture. We can provide a graphical representation of the Hilbert space.

Gaiotto-Rapcak VOA

Gaiotto and Rapcak introduced a VOA Y_{LMN} through an intersecting 5-branes. Between two 5-branes, D3-branes with given number are placed. VOA is defined on the vertex of three 5-branes.



- On D3-branes in each domain, we have $U(L)$ ($U(M)$, $U(N)$) $\mathcal{N} = 4$ (twisted) SYM with coupling constant ψ
- On the boundary of each domain, we have 3D Chern-Simons with gauge groups. We obtain $U(M|L)$ and $U(N|L)$ CSs with $\psi = k + h$.
- Finally on the vertex, we have the chiral algebra $Y_{LMN}[\psi]$.
- The algebra is isomorphic when we shuffle three numbers L, M, N but we need perform $SL(2, \mathbb{Z}_2)$ transformation for ψ.

GR VOA as coset

- The CFT appearing at the vertex becomes a coset CFT (for $N > M$),
$$\frac{\mathcal{DS}_{N-M}[U(N|L;\Psi)]}{U(M|L;\Psi-1)}.$$
$\mathcal{DS}_{N-M}$: Drinfeld-Sokolov reduction with principal $su(2)$ embedding in $(N-M)\times(N-M)$ part in $U(N|L)$. The coset for $N < M$ case or $N = M$ case is similar.
- For $M = L = 0$ case, we obtain $\mathcal{DS}_N[U(N;\Psi)]$ which is identical to W_N algebra (with decoupled $U(1)$ boson). This is analogous to N-reduction of KP hierarchy.
- Thus, GR VOA is a generalization of W_N algebra.

Affine Yangian $Y(\widehat{gl}_1)$ and relation with $W_{1+\infty}$

(Part of) defining relations of affine Yangian (AY) of $\widehat{gl}_1$:

$$\begin{aligned} e(u)e(v) &\sim \varphi(u-v)e(v)e(u), \qquad f(u)f(v) \sim \varphi(v-u)f(v)f(u), \\ \psi(u)e(v) &\sim \varphi(u-v)e(v)\psi(u), \qquad \psi(u)f(v) \sim \varphi(v-u)f(v)\psi(u), \end{aligned}$$

where

$$\varphi(u) = \frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}.$$

h_i (with $\sum_{i=1}^{3} h_i = 0$) and ψ_0 are the parameters of the algebra. They have some redundancies. Irreducible set of parameters $\lambda_i = -\psi_0\sigma_3/h_i$ ($\sigma_3 := h_1 h_2 h_3, \sum_i \lambda_i^{-1} = 0$). Relation to the parameters of $W_{1+\infty}$:

$$c = 1 + \prod_{i=1}^{3}(\lambda_i - 1), \quad x^2 = 144(c+1)\prod_{i=1}^{3}(\lambda_i - 2)(\lambda_i - 3)^{-1},$$

c: central charge of Virasoro algebra

$x^2 = \frac{C_{44}^0 (C_{33}^4)^2}{(C_{33}^0)^2}$:OPE coefficients among primary fields.

Plane partition realization of AY

Plane partition realization: $|\Lambda\rangle$ the orthogonal basis labelled by a plane partition Λ,

$$\begin{aligned}
\psi(u)\,|\Lambda\rangle &= \psi_\Lambda(u)\,|\Lambda\rangle\,, \\
e(u)\,|\Lambda\rangle &= \sum_{\square\in\Lambda^+}\frac{1}{u-q-h_\square}\sqrt{-\frac{1}{\sigma_3}\mathrm{res}_{u\to q+h_\square}\psi_\Lambda(u)}\,|\Lambda+\square\rangle\,, \\
f(u)\,|\Lambda\rangle &= \sum_{\square\in\Lambda^-}\frac{1}{u-q-h_\square}\sqrt{-\frac{1}{\sigma_3}\mathrm{res}_{u\to q+h_\square}\psi_{\Lambda-\square}(u)}\,|\Lambda-\square\rangle\,, \\
\psi_\Lambda(u) &= \psi_0(u-q)\prod_{\square\in\Lambda}\varphi(u-q-h_\square)
\end{aligned}$$

with $h_\square = h_1 x + h_2 y + h_3 z$ when the box is located at (x, y, z).
The recursion relation among the orthogonal basis $|\Lambda\rangle$ is essentially the same as Nekrasov factor (while they are reduced to Young diagrams).

Affine Yangian and GR VOA

- Vertical GR VOA Y_{LMN} appears as a truncation of the affine Yangian through constraints on its parameters. (Prochazka and Rapcak)
$$\frac{L}{\lambda_1} + \frac{M}{\lambda_2} + \frac{N}{\lambda_3} = 1$$
- With this constraint, the plane partition contains a null state (pit) at $(L+1, M+1, N+1)$. Note: there is a shift symmetry $(L, M, N) \to (L+k, M+k, N+k)$ and one may set one of L, M, N to be zero.
- For instance, when $(L, M, N) = (0, 0, N)$, the height of the plane partition is limited by N. Such diagram can be sliced horizontally to describe N Young diagrams $\lambda_1 \preceq \lambda_2 \preceq \cdots \preceq \lambda_N$, which can be identified with the Hilbert space of W_N algebra (plus $U(1)$ boson). This agrees with the horizontal picture.

Plane partition with infinite legs

Plane partition may have nonvanishing asymptotic Young diagrams in three directions (x^1, x^2, x^3), say λ, μ, ρ. The partition function for original plane partition is MacMahon function,

$$\prod_{n=1}^{\infty}(1-q^n)^{-n} = M(q)$$

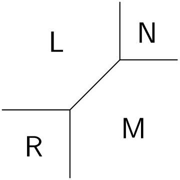
With the infinite legs attached, the partition function is modified to,

$$C_{\mu\nu\rho}(q)M(q)$$

where $C_{\mu\nu\rho}(q)$ is the (unrefined) topological vertex. The conformal weight and $U(1)$ charge becomes also nonvanishing. They are computed from the coefficients of $\psi(u)$. This function remains finite since the contributions of infinite boxes cancel with each other and only finite factor remains. This is a way to generate nontrivial representations.

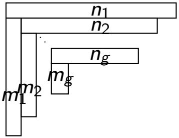
Connecting two Y-algebras: Web of W (WoW)

One may connect two affine Yangian by sharing one asymptotic Young diagram with another one.



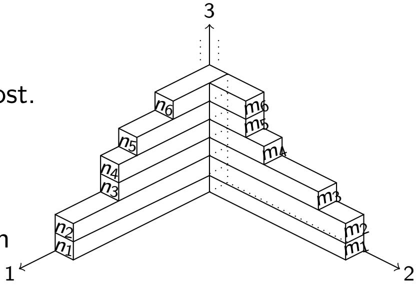
We have two AYs acting on two plane partitions. Since it adds (or remove) only one box to each plane partition, they cannot change the shape of shared leg. We need to introduce extra generators which changes the shape of the asymptotic Young diagram. Thus, WoW consists of tensor product of two W_∞ with extra generators (typically fermionic operator).

Free field example



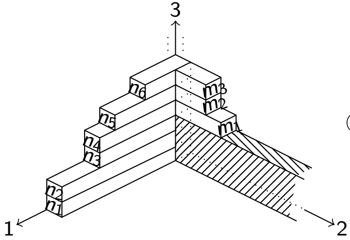
This is the simplest example of Y-algebra Y_{001} which represents free fermion.
The basis of the state may be written as,
$b_{-n_1} \cdots b_{-n_g} c_{-m_1} \cdots c_{-m_g} |0\rangle$
with $n_1 > \cdots > n_g \geq 1$
and $m_1 > \cdots > m_g \geq 1$
which gives aYoung diagram.

Another example is Y_{110}
which is described by bosonic ghost.
$\beta_{-n_1} \cdots \beta_{-n_g} \gamma_{-m_1} \cdots \gamma_{-m_g} |0\rangle$
with $n_1 \geq \cdots \geq n_g \geq 1$
and $m_1 \geq \cdots \geq m_g \geq 1$
It is described by a plane partition with a pit $(2, 2, 1)$



$bc\beta\gamma$ system: the simplest WoW

One may connect these two to generate the simplest example of WoW $Y_{110} - Y_{001}$. Originally the AF is described by bilinear combination of bc and $\beta\gamma$ separately. In the combined system new generators $b\gamma$ and βc appear which modifies tensor product of AY to WoW. In this case, the number of bc and $\beta\gamma$ may not balance separately. It generates an infinite leg with the same shape appearing in both sides. Fermionic generators operate on them to modify the shape.



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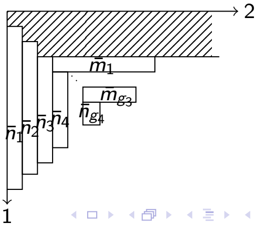
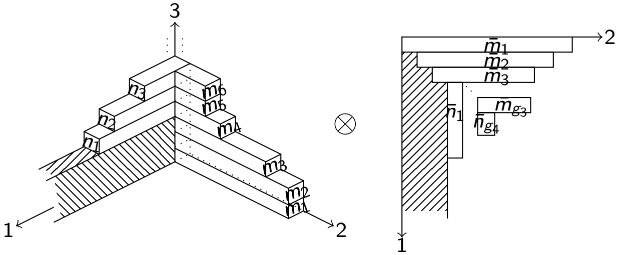


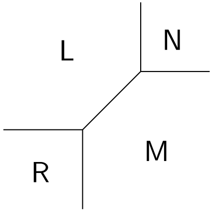
$bc\beta\gamma$ system: negative $U(1)$ charge

The number $\#(c) - \#(b)$ may be either positive or negative. Depending on the sign, the direction of the leg changes. For the positive (resp. negative) case, it goes to x^2 (resp. x^1) direction. In diagrammatic representation, it implies that the direction of the connected legs depends on the sign. Since this is inconvenient, we fix the direction of the leg and allows the Young diagrams to have negative rows as an analytic continuation.



WoW: generalization

By changing L, M, N, R, we obtain different types of VOA.



- $(L, M, N, R) = (0, 1, 1, 0)$ $bc\beta\gamma$ system.
- $(L, M, N, R) = (0, 1, 2, 0)$ $N = 2$ SCA
- $(L, M, N, R) = (0, 1, 3, 0)$ Polyakov-Bershadsky VOA

One may connect the third vertex (WoWoW) and so on, and it describes infinitely many unknown algebras. The treatment of negative Young diagrams may be more complicated for the general models.

Double truncation

In the following, we consider a special case where we put the second constraint on the parameters.

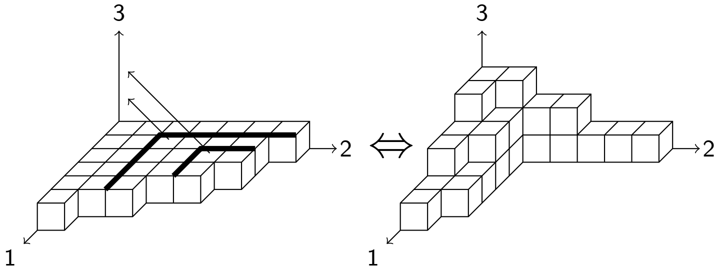
$$\frac{L_1}{\lambda_1}+\frac{M_1}{\lambda_2}+\frac{N_1}{\lambda_3}=1\,,\qquad \frac{L_2}{\lambda_1}+\frac{M_2}{\lambda_2}+\frac{N_2}{\lambda_3}=1\,.$$

In this case, we seem to have two pits at (L_i, M_i, N_i) $(i=1,2)$. This is, however, too naive. With such parametrization, we have periodicity in h function,

$$h_{x+L_1,y+M_1,z+N_1}=h_{x+L_2,y+M_2,z+N_2}$$

Since h governs the representation of the plane partition, above relation implies that the plane partition becomes degenerate, namely the boxes in the different locations should be identified. We will denote $Y_{L_1M_1N_1:L_2M_2N_2}$ to describe such algebra. With two constraints, there is no free parameter.

We illustrate the degeneration phenomena in the simplest case $Y_{120:001}$.

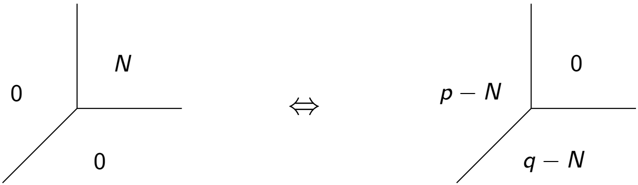


The algebra may be identified as Y_{001} described by one Young diagram or Y_{120} by plane partition with a pit at $(2,3,1)$. One may construct the latter diagram by cutting Young diagram into pieces. One needs to impose that both diagrams is consistent as a plane partition which imposes tighter constraints.

Outline

W_N-algebra

W_N algebra is realized as GR VOA Y_{00N}. In order to describe its minimal model, we restrict the parameters to have second pit : $Y_{00N:(p-N)(q-N)0}$ where $p, q > N$.



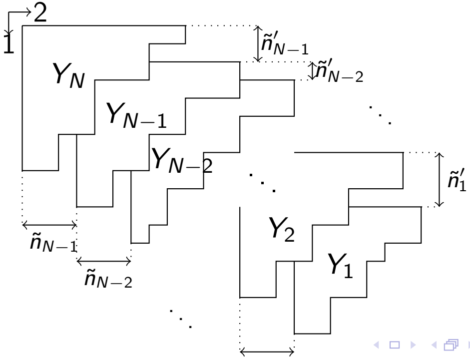
The central charge becomes

$$c = (N-1)\left(1 - Q^2 N(N+1)\right)+1\,, \quad Q = \sqrt{\beta}-1/\sqrt{\beta}, \quad \beta = p/q$$

The identification of boxes is $(x, y, z+N) \sim (x+p-N, y+q-N, z)$. It generates the correct characters of W_N minimal models.

Primary fields by asymptotic Young diagrams

In order to describe the primary fields of W_N algebra, we need to have nontrivial asymptotic Young diagrams in x^1, x^2 directions specified ($\vec{n}$ and $\vec{n}'$) (x^3 direction is not allowed since we have a pit at $(0, 0, N)$). We decompose the plane partition in z direction which gives N Young diagrams.



In order that they should be consistent as a plane partition and the periodicity constraint

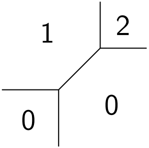
$$Y_{i,R} - Y_{i+1,R+n_i-1} \geq -(n'_i - 1)\,,$$

with the periodicity $Y_{N+1} \sim Y_1$. This constraint is referred to as N-Burge condition and it is known to characterize the Hilbert space of minimal models properly.
The conformal dimensions of the primary fields can be derived geometrically from the shape of the asymptotic Young diagrams as,

$$\Delta(\vec{n}, \vec{n}') = \frac{12(\sum_{i=1}^{N-1}(pn_i - qn'_i)\vec{\omega}_i)^2 - N(N^2-1)(p-q)^2}{24pq}$$

which agrees with the known formula in the textbook.

$N = 2$ SCFT is a WoW ($Y_{210} - Y_{001}$) described by following diagram



- Y_{001} is described by a decoupled free fermion. It provides, however, the degree of freedom of intermediate channel. Supercharges modify them.
- We impose the parameter of Y_{210} such that it has the second pit at $(1, 1, n+1)$. Thus the algebra may be written as $Y_{210:00n}$. The central charge of the whole algebra becomes,

$$c = \frac{3n}{n+2} + 1$$

which corresponds to the minimal models (plus decoupled $U(1)$ factor).

$N = 2$ SCA

- There is a free leg in $Y_{210;00n}$ (between 2 and 1). Putting a nonvanishing asymptotic Young diagram corretly describes every primary field. By tuning asymptotic Young diagrams, one may describe all the known primary fields

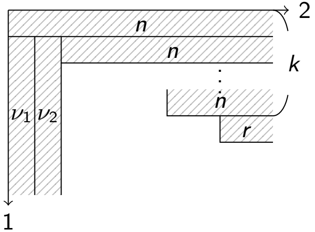
$$h_{l,m} = \frac{l(l+2) - m^2}{4(n+2)}, \quad J_{l,m} = \frac{m}{n+2}$$

with $0 \leq l \leq n, \quad -l \leq m \leq l, \quad l - m \equiv 0 \pmod{2}$.
- The number of states of the plane partition is identical to the known character of $N = 2$ SCA (Dobrev, Kiritsis, Matsuo 1986)

$$\chi(\tau, z) = \sum_{r \in \mathbb{Z}_n} c^{(n)}_{l,m+2r}(\tau)(\Theta_{2m+2r(n+2),2n(n+2)}(\tau, \frac{z}{n+2}) + \Theta_{2m+2(r+n)(n+2),2n(n+2)}(\tau, \frac{z}{n+2}))$$

Some technical issue

Here is a picture of the plane partition in $Y_{210;00n}$



- Two columns in x^1 direction describes a free leg. The heights ν_1, ν_2 are related to l, m which parametrizes the primary fields.
- The vacancy between two legs appears when we translate infinite legs. These vacancy should be filled to make a consistent plane partition.
- In $k < 0$ case, we need to have both legs in x^1 direction.

Conclusion and outlook

- Construction of minimal models in the conventional method is sophiscated– DS reduction, screening currents, counting of null states
- In the vertical frame, the reduction is straightforward and geometrical – adding the second pit in the plane partition.
- It works for known VOA, W_N algebra and $N = 2$ SCA. We hope that it applies universally for other WoW as well. An interesting exercise is to study the next simplest case $Y_{310:00n} - Y_{001}$ which produces Bershadsky-Polyakov algebra (Bershadsky 1991, Arakawa 2010).
- Use of plane partition may provide a new insight to understand the relation between the algebra and geometry in string theory.