Buchdahl's Stability Bound in Eddington-inspired Born-Infeld Gravity

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NCTS Annual Theory Meeting 2018:
Particles, Cosmology and Strings
12/18/2018

Based on W. X. Feng, C. Q. Geng and L. W. Luo, arXiv:1810.06753 [gr-qc]

Buchdahl's stability bound in general relativity (GR)

- In 1959, Buchdahl showed that for a thermodynamically stable self-gravitating sphere in general relativity (GR), it must satisfy $r_s \geq (9/4)\mathcal{M}$

$$r_s : \text{radius of the sphere}$$

$$\mathcal{M} : \text{mass parameter of the sphere}$$

- Rather than $r_s \geq 2\mathcal{M}$, this inequality provides an stricter bound for a star to avoid collapsing into a black hole.

- This result is profound because it is independent of the equation of state (EoS) of the matter contained inside the sphere.

H. A. Buchdahl, Phys. Rev. 116, 1027 (1959)

Action with the Determinantal Form

- An alternative proposal for the gravitational action was proposed by Eddington (1924). He suggested at least in free, de Sitter space, the fundamental field should be the connection $$S_{\mathrm{Edd}}[\Gamma] = \frac{2\kappa}{8\pi} \int d^4x \sqrt{-\det(\boldsymbol{\mathcal{R}}(\Gamma))}$$

- Motivated by Born-Infeld electrodynamics, Deser and Gibbons (1998), discussed the possible forms of the gravitational analogue of Born-Infeld theory in the metric formalism, and the necessary conditions without running into the ghost problem.

A. S. Eddington, The Mathematical Theory of Relativity (1924)
M. Born and L. Infeld, Proc. Roy. Soc. Lond. A **144**, no. 852, 425 (1934)
S. Deser and G. W. Gibbons, Class. Quant. Grav. **15**, L35 (1998)

Eddington-inspired Born-Infeld (EiBI) Gravity

- Vollick (2004) first proposed the action (in Palatini formalism; the metric is minimally coupled to matter)

$$S_{\rm EiBI}[g,\Gamma] = \frac{2}{8\pi\kappa}\int d^4x \left[\sqrt{-\det(\boldsymbol{g}+\kappa\boldsymbol{\mathcal{R}}(\Gamma))} - \lambda\sqrt{-\det(\boldsymbol{g})}\right] + S_M[g,\Psi].$$

- Banados and Ferreira (2010) have shown that it resolves the singularity problem in GR to some extent. This model is equivalent to GR in vacuum.

D. N. Vollick, Phys. Rev. D 69, 064030 (2004)

M. Banados and P. G. Ferreira, Phys. Rev. Lett. 105, 011101 (2010)

EiBI field equations

- After varying the action w.r.t g and Γ independently, we obtain $$q_{\mu\nu} = g_{\mu\nu} + \kappa \mathcal{R}_{\mu\nu}$$ $$q^{\mu\nu} = \tau(g^{\mu\nu} - 8\pi\kappa T^{\mu\nu})$$

$T^{\mu\nu}$: **energy momentum tensor** $\tau = \sqrt{|g|/|q|}$

- The auxiliary metric $q_{\mu\nu}$ is used for raising/lowering index in the geometric sector; the metric tensor $g_{\mu\nu}$, on the other hand, is for the matter sector, hence it follows that

$$\delta^{\mu}{}_{\nu} - \kappa \mathcal{R}^{\mu}{}_{\nu} = q^{\mu\lambda} g_{\lambda\nu} = \tau(\delta^{\mu}{}_{\nu} - 8\pi\kappa T^{\mu}{}_{\nu})$$

GR-like field equations

- This field equations can be recast in GR-like field equations as

$$\mathcal{G}^{\mu}{}_{\nu}[q] = \mathcal{R}^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}\mathcal{R} = 8\pi\left(\tau T^{\mu}{}_{\nu} + \mathcal{P}\delta^{\mu}{}_{\nu}\right) \equiv 8\pi\mathcal{T}^{\mu}{}_{\nu}$$

$$\mathcal{P} = \frac{\tau - 1}{8\pi\kappa} - \frac{\tau T}{2} \text{ : isotropic pressure addition}$$

$$\mathcal{T}^{\mu}{}_{\nu} \text{ : auxiliary energy momentum tensor}$$

- Furthermore, taking the determinant of $q^{\mu\lambda}g_{\lambda\nu} = \tau(\delta^{\mu}{}_{\nu} - 8\pi\kappa T^{\mu}{}_{\nu})$, we can express τ solely in terms of $T^{\mu}{}_{\nu}$ as

$$\tau = \left[\det(\delta^{\mu}{}_{\nu} - 8\pi\kappa T^{\mu}{}_{\nu})\right]^{-\frac{1}{2}}$$

T. Delsate and J. Steinhoff, Phys. Rev. Lett. 109, 021101 (2012)

Perfect fluid assumption

- If we model the self-gravitating sphere by a perfect fluid,

$$T^{\mu}{}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + p\delta^{\mu}{}_{\nu}$$

ρ : **energy density**

p : **pressure**

u^{μ} : **four-velocity of the fluid element with** $u^{\mu}u_{\mu} = -1$

- We then obtain

$$\tau = [(1 + 8\pi\kappa\rho)(1 - 8\pi\kappa p)^3]^{-1/2} \equiv 1/ab^3$$

where $a \equiv \sqrt{1 + 8\pi\kappa\rho}$ and $b \equiv \sqrt{1 - 8\pi\kappa p}$ required to be positive

Auxiliary quantities

- We can define the “auxiliary” density $\tilde{\rho}$ and pressure $\tilde{p}$, respectively, in terms of a and b as follows

$$\mathcal{T}^0{}_0 = \frac{-a^2 + 3b^2 - 2ab^3}{16\pi\kappa ab^3} \equiv -\tilde{\rho}$$

$$\mathcal{T}^i{}_j = \frac{a^2 + b^2 - 2ab^3}{16\pi\kappa ab^3} \equiv \tilde{p}$$

- The “auxiliary” quantities are still potentially singular.

Spherically symmetric and static spacetime

- For considering a spherically symmetric and static spacetime, we make the Ansatz of the physical metric and auxiliary metric, given by

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + H^2(r)d\Omega^2 \\ &= -F^2(r)dt^2 + G^2(r)dr^2 + H^2(r)d\Omega^2 \\ q_{\mu\nu}dx^\mu dx^\nu &= -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \\ &= -A^2(r)dt^2 + B^2(r)dr^2 + r^2 d\Omega^2 \end{aligned}$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

- For perfect fluid, the relations between two sets of the metrics are given by

$$\begin{aligned} F^2 &= A^2 ab^{-3} \quad (\text{or} \quad \Phi = \alpha + \frac{1}{2}\ln a - \frac{3}{2}\ln b), \\ G^2 &= B^2/ab \quad (\text{or} \quad \Lambda = \beta - \frac{1}{2}\ln a - \frac{1}{2}\ln b), \end{aligned}$$

$$H^2 = r^2/ab$$

- Note that these two metrics are identical in the absence of matter, in which case $a = b = 1$.

Ansatz of two mass functions

- With the auxiliary metric at hand, if we make another Ansatz

$e^{2\beta} = B^2 \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1}$ $m(r)$: **auxiliary mass function within radial distance** r

- The tt-component of the "auxiliary" Einstein equation gives

$m'(r) = 4\pi r^2 \tilde{\rho}$ $'$: **derivative with respect to coordinate** r

- If we make another Ansatz $e^{2\Lambda} = G^2 \equiv \left(1 - \frac{2M(r)}{r}\right)^{-1}$

together with $G^2 = B^2/ab$, where $M(r)$ stands for the "physical" mass (the mass function appearing in the physical metric), we have

$$M = \frac{(1-ab)}{2} r + abm = m + \frac{1}{2}(1-ab)(r-2m)$$

Effective density corresponding to mass function of physical metric

- The "effective" density corresponding to $M(r)$ can be defined by $\rho_{\rm eff} \equiv M'(r)/4\pi r^2$, after some computations, we obtain

$$\rho_{\rm eff} = ab\tilde{\rho} + \frac{(1-ab)}{8\pi r^2} + \frac{\kappa}{2r}\left(1 - \frac{2m}{r}\right)\left(\frac{a^2c_s^2 - b^2}{ab}\right)\frac{d\rho}{dr}.$$

c_s : **sound speed**

- We may refer this effective density as physical density corresponding to GR, since it contributes to the physical mass function.

Monotonicity of energy density

- Is there a similar bound in EiBI?

- A crucial assumption in proving Buchdahl's stability bound in GR is the "monotonically decreasing" property of the physical density ρ . However, in EiBI the question turns out to be which density we should demand its monotonicity in order to minimize our assumptions.

- In other words, we have three types of density ρ , $\tilde{\rho}$ and $\rho_{\rm eff}$ in this model, is the monotonically decreasing ρ enough to imply the same monotonic behavior of $\tilde{\rho}(\rho, p)$ and $\rho_{\rm eff}(\rho, p)$?

Monotonicity of auxiliary density

- To figure out that, let us take the derivative of $\tilde{\rho}$ with respect to r, we have

$$\frac{d\tilde{\rho}}{dr} = \left[\frac{3a^2(a^2-b^2)c_s^2 + (3b^2+a^2)b^2}{4a^3b^5}\right]\frac{d\rho}{dr}.$$

- The constraint $24\pi\kappa(\rho+p)a^2c_s^2 + (3b^2+a^2)b^2 > 0$ is required to guarantee the monotonic decreasing of $\tilde{\rho}$ once ρ is so.

- For $\kappa > 0$, there is no question from this condition if the null energy conditions hold $\rho + p \geq 0$, however, the potential pathologies of $\kappa < 0$ may still exist.

Monotonicity of effective density

- For effective density $\rho_{\rm eff}$ corresponding to $M(r)$, the expression is much more complicated.

W. X. Feng, C. Q. Geng and L. W. Luo, arXiv:1810.06753 [gr-qc]

- Here, we just assume $\rho_{\rm eff}$ is a monotonically decreasing function and to see what it leads to.

- If $\rho_{\rm eff}$ is a monotonically decreasing function, then

$$M(r) \geq \frac{r^3}{r_s^3}\mathcal{M}$$

r_s : **the radius of the star at which** $p(r_s) = 0$

$$\mathcal{M} \equiv M(r_s)$$

Dual relation between two mass functions

- (1) If $\rho_{\rm eff}$ is a monotonic decreasing function, then

$$m(r) + \frac{1}{2}(1-ab)[r-2m(r)] \geq \frac{r^3}{r_s^3}\mathcal{M}$$

- (2) If $\tilde{\rho}$ is a monotonic decreasing function, then

$$M(r) + \frac{1}{2ab}(ab-1)[r-2M(r)] \geq \frac{r^3}{r_s^3}\mathcal{M}$$

where we have used the fact that $m(r_s) = M(r_s) \equiv \mathcal{M}$ at the surface ($a = b = 1$) of the star.

Which one is the stronger assumption?

- From these two inequalities, we see the dual relations between two mass functions. For not forming a black hole, we must have $r-2m(r)>0$ (also, $r-2M(r)>0$) throughout the interior of the star.

- The sign of the extra terms (which are absent in GR limit $a=b=1$) in the inequalities depends on $ab>1$ or $ab<1$.

- If $ab>1$, (1) is stronger than (2), i.e. the decreasing monotonicity of $\rho_{\rm eff}$ can imply $M(r) \geq \frac{r^3}{r_s^3}\mathcal{M}$ but (2) cannot.

- In contrast, if $ab<1$, (2) is stronger than (1) since (2) gives a more stringent inequality than $M(r) \geq \frac{r^3}{r_s^3}\mathcal{M}$.

Equation of state (EoS) matters!

- When proving the Buchdahl's stability bound, we will use the stronger assumption depending on which region ($ab > 1$ or $ab < 1$) we are considering.
- These conditions depend on the EoS inside the star and will affect the Buchdahl's stability bound in contrast to GR (which is independent of the EoS):

$$\frac{\rho}{1+8\pi\kappa\rho} > p \text{ or } \frac{\rho}{1+8\pi\kappa\rho} < p$$

Buchdahl's stability bound in EiBI

Theorem. *If (i) both $\tilde{\rho}$ and $\rho_{\rm eff}$ are finite and monotonic decreasing functions and (ii) A^2 and B^2 are positive definite, the Buchdahl's stability bound in EiBI gravity for $ab > 1$ is given by*

$$r_s\left(1 - \frac{1}{2}g - \frac{1}{2}g^2\right) \geq \frac{9}{4}\mathcal{M},$$

where

$$g \equiv \frac{\mathcal{M}}{r_s^3}\int_0^{r_s} \frac{\sqrt{ab} - 1}{\sqrt{1 - \frac{2r^2}{r_s^3}\mathcal{M}}} r dr.$$

Remarks

- If $ab > 1$ throughout the interior of the sphere, g is positive definite. This means the lower bound of the stable radius in EiBI is larger than GR.

- By expansion in terms of the order of κ, we have
  $$r_s/2\mathcal{M} \geq 9/8 + (3\pi/8)\kappa(\bar{\rho} - \bar{p}) + \mathcal{O}(\kappa^2)$$

E.g. for neutron star, the typical density $\bar{\rho} \sim 10^{18}\ \mathrm{kg/m^3}$ with $\mathcal{M} \sim 3\ \mathrm{km}$

yielding $\kappa \lesssim 10^9 \mathrm{m}^2$

- For $ab < 1$, the proof goes similarly but with different mass inequality in use (the one which is stronger for $ab < 1$). The result is the same as GR, $r_s \geq (9/4)\mathcal{M}$.

An Intuitive Explanation: Repulsive effect of EiBI

- The intuitive explanation is the EoS “switch-on” of the “repulsive effect” in EiBI gravity.

- It becomes clear by expanding the auxiliary quantities

$$\tilde{\rho} = \rho - \pi\kappa(5\rho^2 - 6\rho p - 3p^2) + \mathcal{O}(\kappa^2),$$
$$\tilde{p} = p + \pi\kappa(\rho^2 + 2\rho p + 9p^2) + \mathcal{O}(\kappa^2)\,.$$

- We observe that the repulsive effect ($\tilde{\rho} < \rho$ and $\tilde{p} > p$) in EiBI is significant only when $\rho > [(3 + 2\sqrt{3})/5]p \simeq 1.29p$

- Otherwise, we will have $\tilde{\rho} > \rho$ and $\tilde{p} > p$ the repulsive effect reduces.

An Intuitive Explanation: Repulsive effect of EiBI

- When including all order of κ, the former case corresponds to $ab > 1$ and the latter corresponds to $ab < 1$

- The borderline near $ab \simeq 1$ marks $\tilde{\rho} \simeq \rho$, where the repulsive effect switches on/off.

- Moreover, as we shall see, the critical value $ab = 1$ determining the borderline of the repulsive effect corresponds to an exotic EoS.

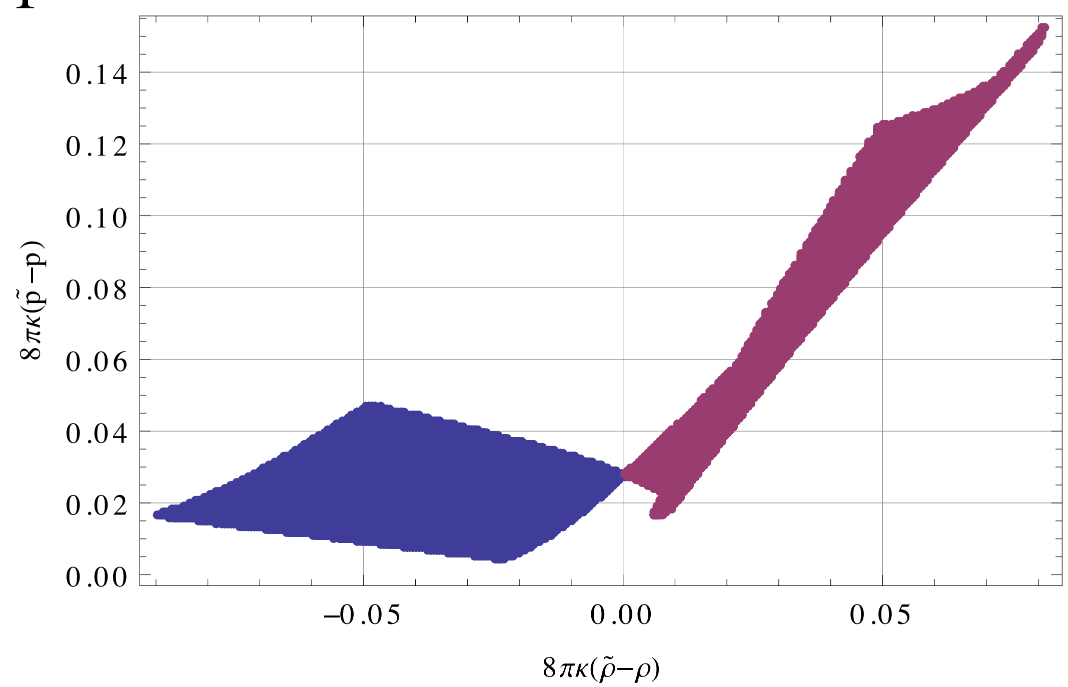

FIG. 1. Blue (Left) region: $ab \gtrsim 1$ (with $1.1 < a < 1.2$ and $0.942 < b < 1.0$) v.s. Purple (Right) region: $ab \lesssim 1$ (with $1.0 < a < 1.1$ and $0.85 < b < 0.942$). The borderline near $ab \simeq 1$ marks $\tilde{\rho} \simeq \rho$, where the repulsive effect is about to be significant or insignificant regions.

Singularity avoidance and pathologies in EiBI

- The "singularity avoidance" feature of this model (Delsate and Steinhoff) relies on the fact that as $b \to 0$,i.e. $8\pi\kappa p \to 1$, the "auxiliary" energy density and pressure diverges but with finite "physical" energy density and pressure.

T. Delsate and J. Steinhoff, Phys. Rev. Lett. 109, 021101 (2012)

- In this regard, κ can be taken as a "cutoff scale" near the Planck scale, where the non-perturbative quantum gravity effects become important.

- However, as we noted previously, if we assume $b \neq 0$, inside the star (since we expect things inside a compact star are still far from Planck scale even at the center of the star),

$$\rho_{\rm eff} = ab\tilde{\rho} + \frac{(1-ab)}{8\pi r^2} + \frac{\kappa}{2r}\left(1 - \frac{2m}{r}\right)\left(\frac{a^2 c_s^2 - b^2}{ab}\right)\frac{d\rho}{dr}.$$

is still potentially divergent due to $(1-ab)/8\pi r^2$ as $r \to 0$.

An exotic EoS sharing the similarity with the Hagedorn temperature

- The remedy to cure the pathology is to set $ab = 1$, or equivalently

$$p = \frac{\rho}{1 + 8\pi\kappa\rho} \quad \text{, near} \quad r = 0.$$

- Physically, this leads to an exotic EoS controlled by κ near the center ($r \lesssim \sqrt{\kappa}$) of a star "regardless" of the real matter contents.

- In this exotic EoS, the physical pressure is bounded by $1/8\pi\kappa$, however, there is no bound on the physical density.

- Gibbons (2002) has shown a similar EoS in a Born-Infeld string model.

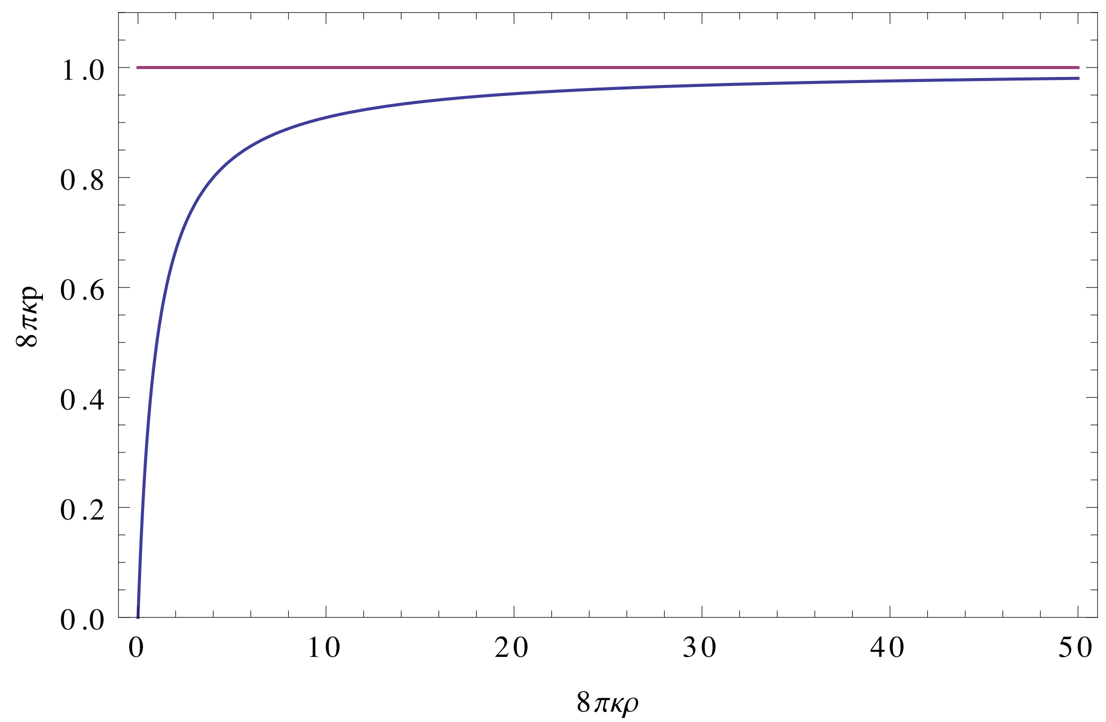

FIG. 2. Exotic EoS with an upper bound of pressure that is unreachable. This phenomenon is analogous to the effect of Hagedorn temperature.

G. W. Gibbons, Grav. Cosmol. **8**, 2 (2002)

An exotic EoS sharing the similarity with the Hagedorn temperature

- Remarkably, this situation is in close analogy to the Hagedorn temperature (Hagedorn(1965) Nuovo Cim.Suppl), in which the energy and entropy diverge but with a fixed and finite (Hagedorn) temperature.

R. Hagedorn, Nuovo Cim. Suppl. 3, 147 (1965)

- However, the discussions above are only at the classical level, the pressure near the "cutoff scale" may signal the breakdown of EiBI or a Hagedorn-like phase transition (Atick and Witten (1988) NPB).

J. J. Atick and E. Witten, Nucl. Phys. B 310, 291 (1988)

- Whether this divergence of ρ really occurs during the gravitational collapse requires further understanding of EiBI or even the quantized version of it.

Conclusions

- The proof relies on assuming the monotonically decreasing property of effective density ρ_{eff} , which corresponds to physical mass function in GR.
- This assumption may restrict the possible classes of EoSs due to the highly nonlinear matter coupling in EiBI gravity.
- The Buchdahl's stability bound in EiBI depends on the EoSs contained inside the self-gravitating spheres.
- A Hagedorn-like EoS at the center of the star during collapse may trigger some unknown Hagedorn-like phase transition.