Discrete Gauge Anomalies Revisited

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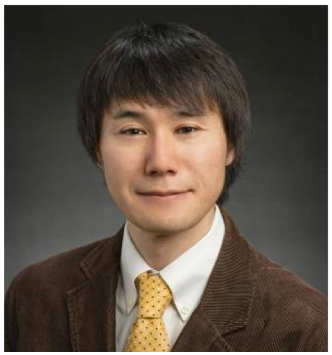

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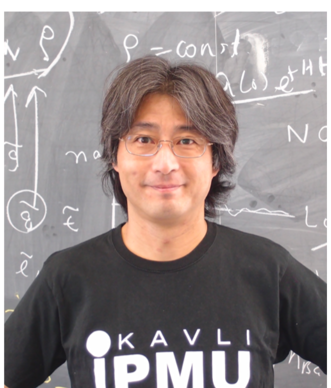

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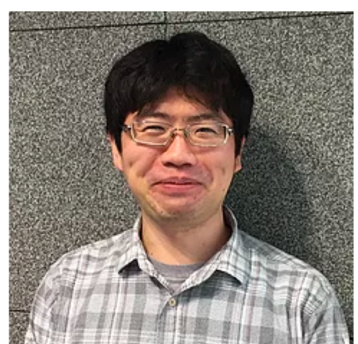

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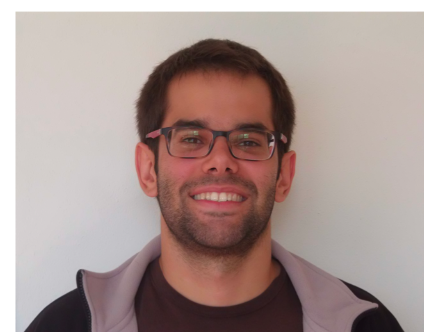

Outline

- Review of discrete gauge anomalies

- Anomalies revisited: Modern perspective and reformulation
  - Example: $\mathbb{Z}_n$ symmetry

- Role of symmetry extensions in discrete anomalies

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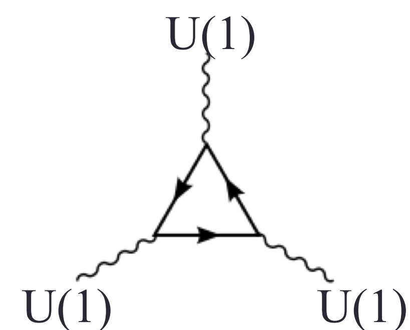
Anomalies in chiral gauge theories

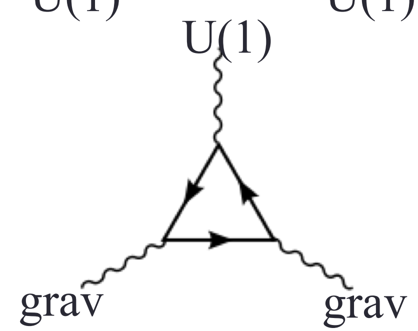
- Cancellation of gauge anomalies — in a chiral theory such as the *standard model* — is a fundamental constraint on a consistent quantum field theory.

- A U(1) chiral gauge theory is anomalous if the anomaly cancellation condition

$$\text{Purely gauge}: \sum_{\text{left}} q_L^3 - \sum_{\text{right}} q_R^3 = 0$$

$$\text{Mixed gauge and grav}: \sum_{\text{left}} q_L - \sum_{\text{right}} q_R = 0$$





is not satisfied. Here $\{q_L\}$ and $\{q_R\}$ are U(1) charges of Weyl fermions.

Q: While anomalies of cont. symm are well understood, how about the case of gauge anomalies associated with discrete symm?

- In this case, there are only global (non-perturbative) anomalies, and one can not use a “usual method” to calculate them
- In a paper by Krauss and Wilczek (1989), they also mentioned

PHYSICAL REVIEW LETTERS

Discrete Gauge Symmetry in Continuum Theories

Lawrence M. Krauss[a] and Frank Wilczek[b]

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mention two caveats. First, there are discrete symmetries—those associated with global anomalies—that cannot be consistently gauged. Identification of such anomalies is a difficult but well developed art,[1] into which we shall not enter here. Second, it is not quite true that the identifications we envisage in field space are

a difficult but well developed art!

Q: While anomalies of cont. symm are well understood, how about the case of gauge anomalies associated with discrete symm?

- For example, how do we couple Weyl fermions *consistently* to a (topological) $\mathbb{Z}_n$ gauge theory in 4d?

➢In some cases, we might be able to write down such a theory as

$$\int \sum_i \overline{\psi}_i (i\partial\!\!\!/ + q_i A\!\!\!/)\psi_i + \underline{\frac{in}{2\pi}\int B\wedge dA + \frac{ipn}{4\pi}\int B\wedge B}$$

[Kapustin-Seiberg JHEP (2014)]

Previous works

- There have been several attempts to tackle this problem, such as the works by Ibáñez-Ross, Banks-Dine, Csáki-Murayama, Araki *et al.*, etc.

- Let's review some of these works

Ibáñez-Ross

Their argument [Ibáñez-Ross PLB (1991)]

Z_n anomaly cancel. cond. = U(1) anomaly cancel. cond. + charge constraints on massive states through SSB of U(1)

The result (a *necessary* cond.):

$$\sum_i q_i^3 = pn + r\frac{n^3}{8}, \quad p, r \in \mathbb{Z};\ p \in 3\mathbb{Z} \text{ if } n \in 3\mathbb{Z},$$

$$\sum_i q_i = p'n + r'\frac{n}{2}, \quad p', r' \in \mathbb{Z}.$$

Contribution from Dirac and Majorana masses, respectively

Banks-Dine

Comments on Ibanez-Ross [Banks-Dine PRD (1992)]

- Only the linear constraint should be satisfied

➢ Can be understood in terms of **instantons** (at low energy)

- The nonlinear (cubic) constraint might be too restrictive and might not be required for consistency of the low energy theory

➢ *Not* solely from the low energy considerations and would depend on assumptions about UV embedding theories

Csáki-Murayama

Argument by *'t Hooft anomaly matching*. Two types of discrete anomalies are involved [Csáki-Murayama NPB (1998)]

- For Type I anomalies, the matching conditions have to be always satisfied *regardless of* the details of the massive bound state spectrum.

- The Type II anomalies have to be also matched *except* if there are **fractionally charged** massive bound states in the theory.

Motivation

- The Type I anomaly (linear constraint) is actually the **mixed anomaly** btw $\mathbb{Z}_n$ and gravity (i.e. Spin(4) spacetime symm of fermions)

- The **full anomalies** of Spin(4)$\times\mathbb{Z}_n$ should correspond to both Type I & II anomalies. But could we compute it *without* referring to any UV embedding theory with cont. symm?

I.e., could we determine discrete anomalies from ***first principles***?

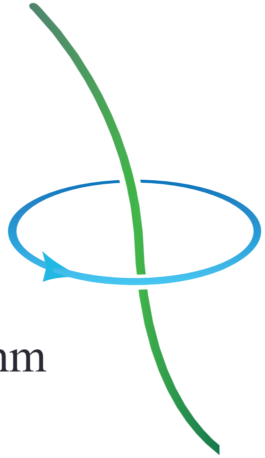
Outline

- Review of discrete gauge anomalies
- Anomalies revisited: Modern perspective and reformulation
  - Example: $\mathbb{Z}_n$ symmetry
- Role of symmetry extensions in discrete anomalies
- Conclusion

This work

- We study discrete gauge anomalies in (3+1)d chiral fermion theories from a more modern perspective, based on
1. the concept of symmetry-protected topological (SPT) phases
2. a *refined* definition of global anomalies by Witten (2016)

- In particular, we give a purely low energy description of discrete gauge anomalies — as gauge symmetries in many situations are ***emergent*** [Witten NatPhys (2018)]



A cosmic string
associated to a $\mathbb{Z}_n$ symm

This work

- We focus on the simplest case that the discrete internal symmetries are **cyclic** groups. I.e, the *full* symmetry group of fermions is $\underbrace{\mathrm{Spin}(4)}_{\text{spacetime}}\times\underbrace{\mathbb{Z}_n}_{\text{internal}}$

- Some of the discussions in this work can also be found in recent papers I. García-Etxebarria & M. Montero, arXiv:1808.00009 and S. Monnier & G. Moore arXiv:1808.01334

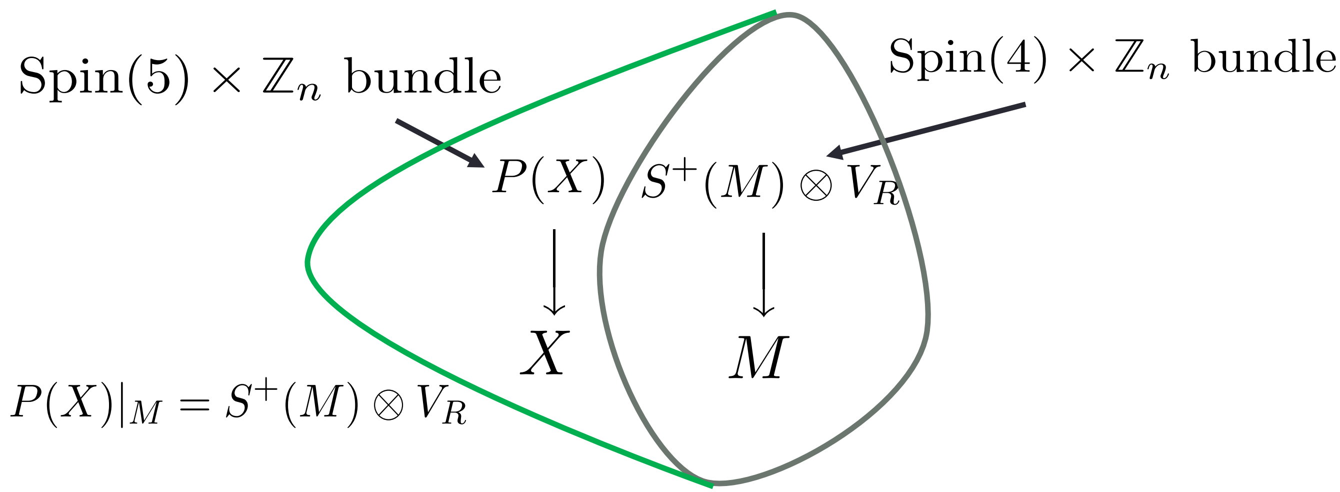
- Consider a set of left-handed Weyl ferm $\Psi = \{\psi_i\}$ with

$$\mathbb{Z}_n \text{ symm: } \psi_i \to e^{2\pi i s_i/n}\psi_i, \quad s_i \in \mathbb{Z}_n$$

The anomalies of Spin(4)×$\mathbb{Z}_n$ are computed as follows:

1. We formulate the above theory on a manifold endowed with both a **spin structure** and a $\mathbb{Z}_n$ **structure**

2. Then, we compute the global anomalies of the resulting theory, based on Dai-Freed Theorem for fermion partition functions
   [Dai-Freed JMP (1994), Witten RMP (2016)]



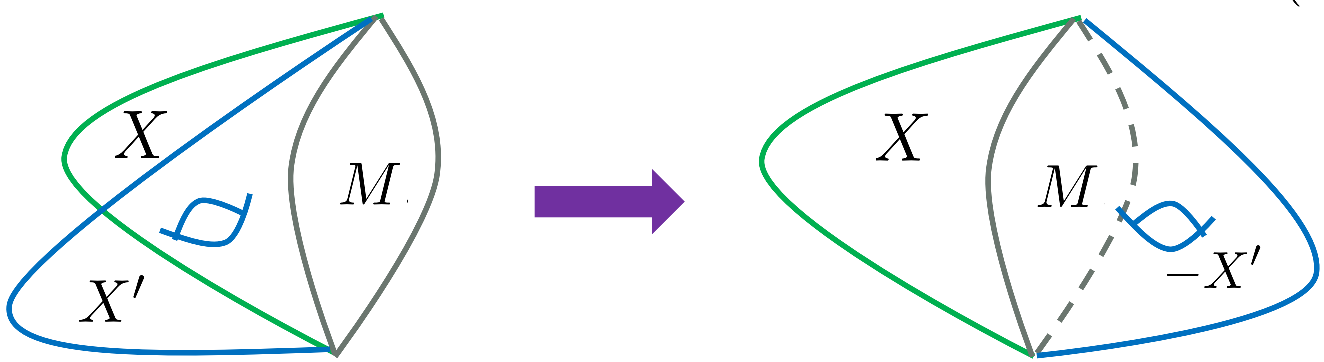
- Let M be a 4-manifold endowed with a spin$\times\mathbb{Z}_n$ structure.
- Let X be a 5-manifold w/ boundary $\partial X = M$ s.t. the spin$\times\mathbb{Z}_n$ structure on M extends over X.
- Then the Dai-Freed theorem gives a definition of the part. func. of fermions in the rep. R of $\mathbb{Z}_n$ on X:

$$\boxed{Z_\Psi(X) = |Z_\Psi(M)|\exp(-2\pi i \eta_R(X))}$$

"eta-invariant" of the Dirac op on X

- In order to have a purely 4d theory, the part. func. must not depend on how the theory extends in one dimension higher

$$X^* = X \cup (-X')$$



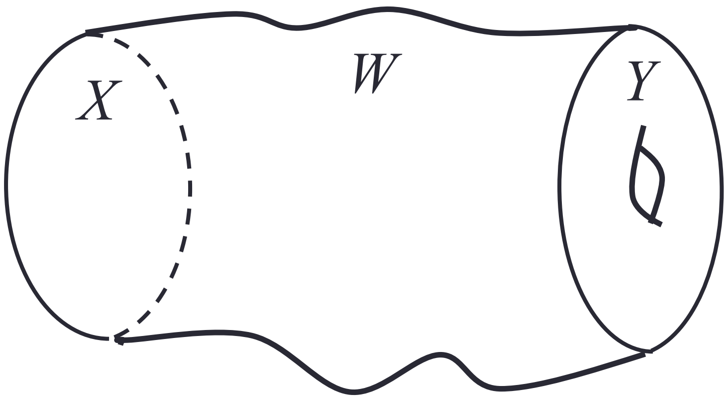
$$\frac{Z_\Psi}{Z'_\Psi} = \frac{\exp(-2\pi i \eta_R(X))}{\exp(-2\pi i \eta_R(X'))} = \exp(-2\pi i \eta_R(X^*))$$

- Anomaly-free condition: $\boxed{\exp(-2\pi i \eta_R(X^*)) = 1}$

for any closed X^* endowed with a spin$\times\mathbb{Z}_n$ structure

,

- Actually, $\exp(-2\pi i \eta_R(X))$ is an invertible TQFT part func (a cobordism invariant) on a (class of) 5d closed manifold with an associated structure [Witten 15, 16]



X is bordant to Y if $\partial W = X \sqcup Y$

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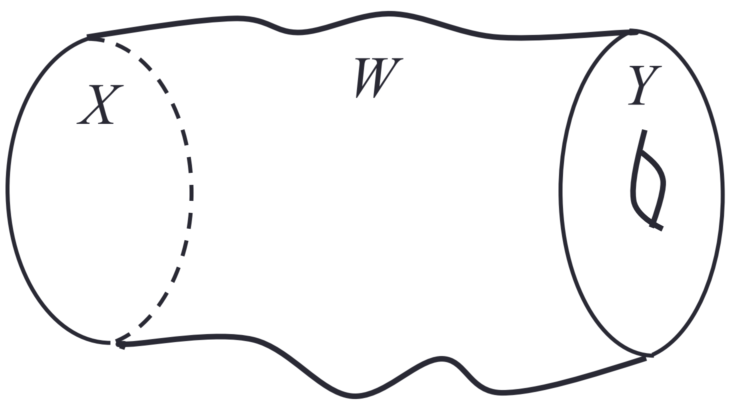
$$\exp(-2\pi i\eta_R(X)) = \exp(-2\pi i\eta_R(Y))$$



X (w/ a spin$\times\mathbb{Z}_n$ str) is bordant to Y (w/ a spin$\times\mathbb{Z}_n$ str) if $\partial W = X \sqcup Y$

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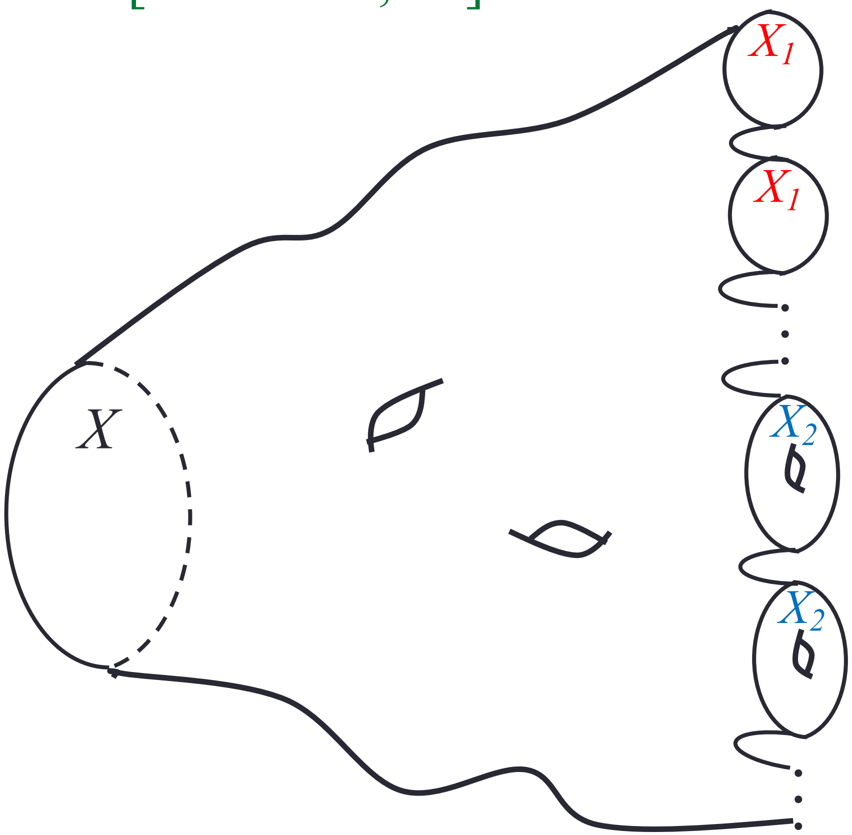

Any 5d X (w/ a spin$\times\mathbb{Z}_n$ str) is bordant to $X_1 \sqcup X_1 \sqcup \cdots X_2 \sqcup X_2 \sqcup \cdots$ (w/ their spin$\times\mathbb{Z}_n$ strs)

generators

- Actually, $\exp(-2\pi i \eta_R(X))$ is an invertible TQFT part func (a cobordism invariant) on a (class of) 5d closed manifold with an associated structure [Witten 15, 16]

- By evaluating $\exp(-2\pi i \eta_R(X))$ in generators X_1 and X_2, we found the anomaly of Spin(4)×$\mathbb{Z}_n$ can be represented by

$$\alpha_R := (\eta_R(X_1) \mod \mathbb{Z},\ \eta_R(X_2) \mod \mathbb{Z})$$

CTH, 1808.02881

$$= \left(\frac{1}{6n}\left(n^2+3n+2\right)\sum_i s_i^3 \mod \mathbb{Z},\quad \underbrace{\frac{2}{n}\sum_i s_i \mod \mathbb{Z}}_{\text{mixed anomaly btw } \mathbb{Z}_n \text{ and Spin(4)}} \right)$$

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$$= \left(\frac{1}{6n}\left(n^2+3n+2\right)\sum_i s_i^3 \mod \mathbb{Z}, \quad \frac{2}{n}\sum_i s_i \mod \mathbb{Z} \right)$$

- Therefore, the anomaly cancellation condition should be

$$\left(n^2+3n+2\right)\sum_i s_i^3 = 0 \mod 6n,$$

$$2\sum_i s_i = 0 \mod n.$$

- In the presence of both chiralities, the cancel cond becomes

$$\boxed{\left(n^2+3n+2\right)\Delta s_3 = 0 \mod 6n, \quad 2\Delta s_1 = 0 \mod n}$$

where $\Delta s_3 := \sum_L s_L^3 - \sum_R s_R^3$ and $\Delta s_1 := \sum_L s_L - \sum_R s_R$

- Let's see some examples:

$$\underline{n=2}: \quad 12\Delta s_3 = 0 \mod 12, \quad 2\Delta s_1 = 0 \mod 2$$

=> any rep of $\mathbb{Z}_2$ is anomaly free!

$$\underline{n=3}: \quad 20\Delta s_3 = 0 \mod 18, \quad 2\Delta s_1 = 0 \mod 3$$

=> a (nontrivial) $\mathbb{Z}_3$ anomaly-free rep: 9 left-handed ferm with $s_{L,i}=1$

$$\underline{n=4}: \quad 30\Delta s_3 = 0 \mod 24, \quad 2\Delta s_1 = 0 \mod 4$$

=> a (nontrivial) $\mathbb{Z}_4$ anomaly-free rep: 4 left-handed ferm with $s_{L,i}=1$

- The anomaly-free conditions we derived basing on the Dai-Freed theorem have similar forms as the the Ibáñez-Ross cond
  - Only **linear terms** and **cubic terms** of the $\mathbb{Z}_n$ charges are involved

- However, our result should be a *necessary and sufficient* cond for consistently gauging a $\mathbb{Z}_n$ symm of a chiral ferm theory, while the Ibáñez-Ross cond is in principle a *necessary* cond

Dai-Freed + Cobordism theory

$$(n^2+3n+2)\sum_i s_i^3 = 0 \mod 6n,$$
$$2\sum_i s_i = 0 \mod n.$$

$\Leftarrow$

Ibáñez-Ross (Csáki-Murayama)

$$\sum_i q_i^3 = pn + r\frac{n^3}{8},$$
$$\sum_i q_i = p'n + r'\frac{n}{2}.$$

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Dai-Freed + Cobordism theory

$$(n^2+3n+2)\sum_i s_i^3 = 0 \mod 6n,$$
$$2\sum_i s_i = 0 \mod n.$$

$$\overset{?}{\Longleftrightarrow}$$

Ibáñez-Ross (Csáki-Murayama)

$$\sum_i q_i^3 = pn + r\frac{n^3}{8},$$
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Role of symm extensions in discrete anomalies

- We know that the Ibanez-Ross cond. are subject to the issue of symm extensions [Banks-Dine 92], which is also crucial in our situation – discrete anomalies can in general change or even disappear when symm are extended.

- This is the essential difference btw the anomaly cancel. cond. of a cont. symm and the one of a discrete symm:

Indep of the normalization of U(1) charges

Sensitive to symm extensions, e.g. a lift from $\mathbb{Z}_n$ to $\mathbb{Z}_{ln}$

- For example, let's consider the following symm extension

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin} \times \mathbb{Z}_8 \to \mathrm{Spin} \times \mathbb{Z}_4 \to 1$$

- Take a set of $\mathbb{Z}_4$ charges R={1, 1, 2}, which is anomalous. When $\mathbb{Z}_4$ is extended to $\mathbb{Z}_8$ w/ R'={2, 2, 4}, the anomaly is gone ("**trivialized**")

$$\alpha_R = \left(\frac{1}{6 \cdot 4} \left(4^2 + 3 \cdot 4 + 2\right) \left(1^3 + 1^3 + 2^3\right) = \frac{1}{2} \mod \mathbb{Z}, \quad \frac{2}{4}(1+1+2) = 0 \mod \mathbb{Z} \right)$$

$$\downarrow$$

$$\alpha_{R'} = \left(\frac{1}{6 \cdot 8} \left(8^2 + 3 \cdot 8 + 2\right) \left(2^3 + 2^3 + 4^3\right) = 0 \mod \mathbb{Z}, \quad \frac{2}{8}(2+2+4) = 0 \mod \mathbb{Z} \right)$$

- For example, let's consider the following symm extension

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin} \times \mathbb{Z}_8 \to \mathrm{Spin} \times \mathbb{Z}_4 \to 1$$

- This means three left-handed ferm w/ $\mathbb{Z}_4$ charges $\{1, 1, 2\}$ cannot consistently couple to a $\mathbb{Z}_4$ gauge field, but can couple to a $\mathbb{Z}_8$ gauge field (with rescaled $\mathbb{Z}_8$ charges $\{2, 2, 4\}$)
- On the other hand, the linear anomaly $\frac{2}{n}\sum_i s_i \mod \mathbb{Z}$ (present for a single ferm w/ a unit $\mathbb{Z}_4$ charge) can ***never*** be trivialized upon any symm extension

➢ A $\mathbb{Z}_n$ symm w/ non-vanishing linear anomaly cannot be gauged!

In summary:

➢ For any consistent chiral gauge theory w/ a **definite full symm group**:

$$\overbrace{\underbrace{\text{massless}}_{\text{The discrete charges of the massless Weyl ferm must strictly satisfy the whole anomaly cancel cond}} + \text{massive (topological)}}^{\text{full symm group (e.g. Spin(4)}\times\mathbb{Z}_8\text{) is known}}$$

In summary:

- If only the (effective) symm on the massless ferm is known:

only eff symm (e.g. Spin(4)×$\mathbb{Z}_4$) is detected

massless + massive (topological)

The anomaly constrains for the massless ferm should only be respected ***up to*** symmetry extensions

In some situations—with the knowledge of "anomalies"—we can predict the existence of massive particles carrying "fractional" charges

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Conclusion

- We *revisit* discrete gauge anomalies in chiral fermion theories in 3 + 1 dimensions — from a more modern perspective based on the concept of SPT phases.

- Focusing on the simplest case that the internal symm are **cyclic** groups, a *reformulation* of the "discrete anomaly cancellation" conditions, first proposed by Ibáñez and Ross in 1991, is given.

- The role of symmetry extensions in discrete anomalies is clarified in a formal fashion, respecting the viewpoint in the previous work by Banks and Dine.

Thank You!