The States in 2D CFT and Their Holographic Descriptions

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arXiv:1806.07595, arXiv:1808.02873(Phys. Rev. Lett. in press),
arXiv:1810.01258, work in process.

Key Words

Holography, Geometry and Entanglement

♦ Holography (AdS/CFT):

$$\text{Gravity on } AdS_{d+1} \Longleftrightarrow \text{CFT on the } R^d$$

Symmetry match and parameter match, specially for $\mathrm{AdS}_3/\mathrm{CFT}_2$, $c = \frac{3R}{2G}$

♦ Geometry:

$$\text{States in CFT} \Longleftrightarrow \text{Classical Geometries}$$

in the classical limit $c \to \infty$ or $G \to 0$.
But is this true for every states?

♦ Entanglement is an useful concept to understand this question.

• (Some) Entanglement measures are expected to be geometric quantities in context of AdS/CFT.

Geometric state

Definition and Simple Arguments

⧫ Definition:

• We define geometric state in the context of AdS/CFT, if a state(in CFT) has a classical geometric description, we call this state geometric state.

⧫ Arguments:

• Since the dual CFT of gravity is a quantum system, for some states, the quantum fluctuation may be very large. While the classical geometry **cannot** product the fluctuation.

• For a quantum system, we may define a state by superposition. But if two states have classical geometry descriptions respectively, what about the superposition of these two states?

• For some systems, one can find some states are more "classical" than others, such as the quantum harmonic oscillator. We know the coherent state can be seen as a classical state in this system, minimal uncertainty relation, follow the classical EoM.

Entanglement/Rényi entropy in CFT

Definitions and Calculations

⧫ Reduced Density Matrix $\rho_A := tr_B \rho$, ρ is the state of the system.
In general, ρ_A would be a mixed state even ρ is pure.

⧫ Rényi entropy is defined by

$$S_A^n := \frac{\log tr \rho_A^n}{1-n} \tag{1}$$

⧫ Entanglement entropy as the limit $S_A := \lim_{n\to 1} S_A^n$.

⧫ Replica methos to obtain $tr\rho_A^n$:

- Correlation functions or partition function on n-sheet Riemann surface Σ_n with singularity at the boundary of A.
- Correlators of n-copy CFT involving twist operator

$$\mathrm{tr}\rho_A \propto \langle \sigma(\ell)\tilde{\sigma}(0)\rangle_{\rho^n} \tag{2}$$

Entanglement/Rényi entropy in CFT

Twist operator OPE

In general,

$$\sigma(\ell)\tilde{\sigma}(0) = \frac{c_n}{(\ell)^{2h_\sigma}} \sum_K d_K \sum_{p=0}^{\infty} \frac{c_K^p}{p!} \ell^{h_K+p} \partial^p \Phi_K(0), \tag{3}$$

$c_K^p \equiv \frac{C^p_{h_K+p-1}}{C^p_{2h_K+p-1}}$ with C_x^y being the binomial coefficient, summation K is over all the orthogonalized (quasi-)primary operators. d_K are the constants that we need to calculate.

- At order ℓ^0 Φ_K is the identity operator. At order ℓ^2, it is T_j.
- At order ℓ^4, we have $T_j T_k$, $\mathcal{A}_j$ and $\partial^2 T_j$,with

$$\mathcal{A} := (TT) - \frac{3}{10}\partial^2 T \tag{4}$$

- Rényi entropy $\rightarrow$ the expectation value of these quasi-primary operators in state ρ

Holographic Entanglement and Rényi Entropy

Geometric Quantities

- Holographic Entanglement Entropy (RT formula)

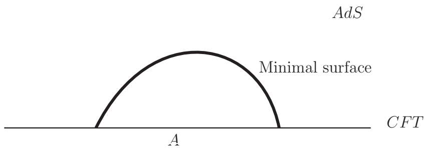
$$S_A = \frac{\text{Area(Minimal surface)}}{4G}, \tag{5}$$

- Holographic Rényi Entropy (Xi Dong's proposal)

$$\tilde{S}_n := n^2 \partial_n \left(\frac{n-1}{n} S_n \right) = \frac{\text{Area(Cosmic Brane}_n\text{)}}{4G}. \tag{6}$$

The tension of the brane is a function of the index n as

$$T_n = \frac{n-1}{4nG}. \tag{7}$$



Geometric State

Argument from holography

♦ The basic idea is to compare some quantities that we both can be calculated if the dual exists. We choose the geometric quantities, such as EE, and Rényi entropy.

♦ For AdS_3 the general solution of the vacuum Einstein equation is Bañado metric,

$$ds^2 = \frac{dy^2}{y^2} + \frac{L_\rho}{2}dz^2 + \frac{\bar{L}_\rho}{2}d\bar{z}^2 + \left(\frac{1}{y^2} + \frac{y^2}{4}L_\rho\bar{L}_\rho\right)dzd\bar{z}, \tag{8}$$

with

$$\langle T(z)\rangle_\rho = -\frac{c}{12}L_\rho(z), \quad \langle \bar{T}(\bar{z})\rangle_\rho = -\frac{c}{12}\bar{L}_\rho(\bar{z}). \tag{9}$$

♦ Arguments from holography

For a 2D CFT state of order c stress tensor expectation value to be holographic dual to a Bañados geometry, the entanglement/Rényi entropy obtained from CFT calculations should be at most order c in the large c limit. Otherwise, we call the CFT state non-geometric.

Geometric States

The necessary conditions

♦ With some calculations, we have (up to ℓ^6)

$$\langle T(w)\rangle_\rho = c\alpha(w) + \beta(w) + \frac{\gamma(w)}{c} + O\left(\frac{1}{c^2}\right),$$
$$\langle \mathcal{A}(w)\rangle_\rho = c^2\alpha(w)^2 + c\delta(w) + \epsilon(w) + O\left(\frac{1}{c}\right),$$
$$\langle \mathcal{B}(w)\rangle_\rho = c^2\left[\alpha'(w)^2 - \frac{4}{5}\alpha(w)\alpha''(w)\right] + c\zeta(w) + O(c^0),$$
$$\langle \mathcal{D}(w)\rangle_\rho = c^3\alpha(w)^3 + 3c^2\alpha(w)[\delta(w) - \alpha(w)\beta(w)] + c\eta(w) + O(c^0),$$

♦ Just by using EE, (up to ℓ^8)

$$\langle T(w)\rangle_\rho = c\alpha(w) + O(c^0),$$
$$\langle \mathcal{A}(w)\rangle_\rho = c^2\alpha(w)^2 + O(c). \tag{10}$$

Geometric State

Example: Thermal state, primary state

- Thermal states satisfy all the conditions we have. Specially, the canonical ensemble state can be checked non-perturbatively.
- The primary state $\phi(0)|0\rangle$, with $h_\phi = \epsilon c$.

$$\langle \phi^n(\infty)|\sigma(z)\tilde{\sigma}(1)|\phi^n(0)\rangle \simeq \exp[-\frac{nc}{6} f_0(h_\phi/c, 1-z)] \quad (11)$$

The gravity dual is expected to be AdS with with a defect (with a static massive particle in pure AdS).

- The "coherent" state on the primary state, $e^{z_0 L_{-1}}|\phi\rangle$. The gravity dual can be seen as a moving particle in AdS.

Non-geometric State

Example: The superposition state, (Some) Descendant state

- The superposition state $(h_{\phi_1} \neq h_{\phi_2})$

$$|\Phi\rangle := \cos(\theta)|\phi_1\rangle + \mathrm{e}^{\mathrm{i}\psi}\sin(\theta)|\phi_2\rangle. \tag{12}$$

$$\langle\Phi|\mathcal{A}|\Phi\rangle - \langle\Phi|T|\Phi\rangle^2 = \frac{16\pi^4(\epsilon_{\phi_1} - \epsilon_{\phi_2})^2}{L^4}\sin^2(\theta)\cos^2(\theta)c^2 + O(c) \neq 0. \tag{13}$$

- (Some) Descendant states,

$$\begin{aligned} &|\phi^{(m)}\rangle \quad \text{with} \quad h_\phi + m \sim O(c), \\ &|\tilde{\phi}\rangle \quad \text{with} \quad h_\phi \sim O(c), \\ &|\tilde{\phi}^{(m)}\rangle \quad \text{with} \quad h_\phi + m \sim O(c), \\ &|T^{(m)}\rangle \quad \text{with} \quad m \sim O(c), \\ &|\mathcal{A}^{(m)}\rangle \quad \text{with} \quad m \sim O(c), \end{aligned} \tag{14}$$

with $\tilde{\phi}$ is a quasi-primary operator with the definition $\tilde{\phi} \equiv (T\phi) - \frac{3}{2(h_\phi+1)}\phi''$

Mixed states

Microcanonical ensemble state

The microcanonical ensemble state is defined by

$$\rho_E := \frac{1}{\Omega(E)} \sum \delta(E_i - E)|E_i\rangle\langle E_i|, \quad \text{with} \quad \Omega(E) = \sum_i \delta(E - E_i). \qquad (15)$$

$E = \frac{2\pi}{L}(\Delta - \frac{c}{12})$. For an operator χ, in high temperature $L \gg \beta$

$$\langle\chi\rangle_\beta = \mathcal{Z}^{-1}(\beta) \sum_i e^{-\beta E_i} \langle E_i|\chi|E_i\rangle = \mathcal{Z}^{-1}(\beta) \int_{E_0}^{+\infty} dE \langle\chi\rangle_E e^{-\beta E}, \qquad (16)$$

where $\langle\chi\rangle_E := \sum_i \langle E_i|\chi|E_i\rangle \delta(E_i - E)$. By an inverse Laplace transformation

$$\overline{\langle\chi\rangle}_E := \frac{\langle\chi\rangle_E}{\Omega(E)} \qquad (17)$$

For examples,

$$\overline{\langle T\rangle}_E = -\frac{\pi^2 c}{6\lambda^2}, \quad \overline{\langle \mathcal{A}\rangle}_E = \frac{\pi^4 c(5c+22) I_3}{180\lambda^4 I_1}, \qquad (18)$$

where $\lambda := \sqrt{\frac{\pi c L}{6E}}$, the notation I_ν for $I_\nu(\frac{\pi c L}{3\lambda})$ ($I_\nu(z)$ is the modified Bessel function of the first kind)

Thermal state

canonical ensemble v.s. microcanonical ensemble

♦ Consider energy density of microcanonical ensemble is $\frac{\pi c}{6\lambda^2}$ (with $\beta = \lambda$) and thermodynamic limit $L \to \infty$

- For small subsystem, $\ell \ll \beta \ll L$

$$S^n_{A,\beta} - S^n_{A,E} = O(\frac{\beta}{L}) \qquad (19)$$

$$S(\rho_{A,\beta}|\rho_{A,E}) = O(\frac{\beta}{L}). \qquad (20)$$

- For large subsystem, $\ell \sim L$ (Say, $\ell > L - \frac{\log(2)\beta}{2\pi}$)

$$S^n_{A,E} - S^n_{A,\beta} \simeq \frac{\pi c L}{3\beta} - \frac{\pi c(n+1)L}{6n\beta} + I_n(1 - e^{-2\pi(L-\ell)/\beta}), \qquad (21)$$

EE difference(n=1) is quantum c^0.

Guo-Lin-Zhang, appear soon

Ensemble Average

♦ For microcanonical average, the descendant states dominate,

$$\frac{\Omega_p(E)}{\Omega(E)} \sim \mathrm{e}^{-L\left[\sqrt{\frac{2\pi c\varepsilon}{3}}-\sqrt{\frac{2\pi(c-1)}{3}\varepsilon}\right]} \to 0 \text{ as } L \to \infty. \tag{22}$$

♦ Microcanonical ensemble state satisfies the geometric state constraints, the descendants don't.

- The descendant states we consider are special.
- Ensemble average erases the difference (quantum fluctuations).

♦ Quantum KdV EoM (currents) to Classical KdV EoM (currents)

$$J_2 = T, \quad J_4 = \mathcal{A} + \frac{3}{10} T'', \ldots \tag{23}$$

These currents form the mutually commuting KdV charges, $Q_{2k-1} := \int_0^L \frac{dw}{L} J_{2k}$. If choosing Q_2 as Hamiltonian, we have the quantum KdV EoM,

$$\dot{T} = -\frac{5c+22}{30} T''' - 3\mathcal{A}' \to \dot{U} = U''' + 6UU', \tag{24}$$

with $\alpha(w) = U(w)/6$.
♦ Fluctuation of observables such as $T(f) := \int T(x)f(x)$,

$$\frac{\sqrt{\langle T(f)^2\rangle_\rho - \langle T(f)\rangle_\rho^2}}{\langle T(f)\rangle_\rho} \sim \frac{1}{\sqrt{c}} \tag{25}$$

Summary

♦ Entanglement and spacetime

• Non-entanglement state (such as Boundary state) $\Longrightarrow$ trivial spacetime
• Entanglement of different spacetime region $\Longrightarrow$ spacetime connectivity

Absence of entanglement (Too classical) $\Longrightarrow$ No proper classical geometry

• Entanglement is large (in the sense of c) (Too quantum) $\Longrightarrow$ No proper classical geometry description

♦ How to describe the non-geometric state in the context of AdS/CFT?