Instantons in 5 and 6 Dim Quantum Field Theories

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QFT on R^{1+3}

Quantum Field Theory

- Theory of Identical Particles

- James Clerk Maxwell: Atoms in Encyclopedia Britannica, 9th Ed. (year 1875) Atoms have been compared by Sir John Herschel to manufactured articles, on account of their uniformity.

- Planck (1900,Einstein (1905): photon
- Dirac (1927): quantization of electromagnetic field by using quantum harmonic oscillators and introducing photon as field quanta

Quantum Field Theories

- QFT as a modern calculus (Seiberg)
  - discovered for physics problem
  - enormously useful in diverse fields of science: particle physics, condensed matter physics, cosmology and string theory, mathematics
  - do not know the right formulation
  - sign of a deep idea

Feynman Path Integral

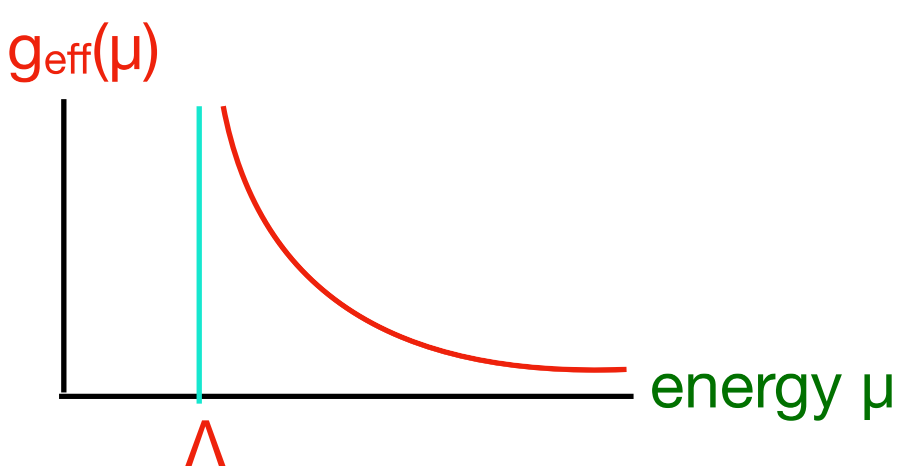
- 4d Yang-Mills theory of gauge group SU(N): A, $F=dA+A^2$

$$S_E = \int d^4x \left(\frac{1}{4g^2} \mathrm{tr} F_{\mu\nu} F_{\mu\nu} - \frac{i\theta}{32\pi^2} \mathrm{tr} F_{\mu\nu} {}^*F_{\mu\nu} \right)$$

- Euclidean time path integral

$$Z = \mathrm{Tr} e^{-HT} = \int [dA_\mu] \exp(-S_E)$$

- weakly coupled perturbative region:
  - ultraviolet physics: asymptotic freedom



- strongly coupled nonperturbative region
  - infrared physics: color confining, mass gap

instantons in R^4_E

- saddle point method: stationary point A^s such that $\delta S_E = 0$

$$Z = \sum_{A^s} e^{-S_E(A^s)} \int [d\delta A_\mu] \exp(-\frac{1}{2}\delta^2 S_E(A^*)\delta A^2)$$

- stationary points: self-dual instantons and anti-selfdual instantons such that *F= ±F
  - Counting instantons:
  - for U(N): the second Chern class

$$k = \frac{1}{32\pi^2} \int d^4x \, \mathrm{tr} F \wedge F, \quad k \in \mathbf{Z}$$

- instantons contribution to the vacuum energy of Yang-Mill theory with θ-therm

$$\mathcal{L}_{\text{eff}} = K e^{-\frac{8\pi^2}{g^2}} \cos(\theta)$$

- Highly nonperturbative, not yet color confining or mass gap…….

instantons in R^4_E

- N=2 supersymmetric Yang-Mills theory
- vector multiplet in the Coulomb branch: A^r, λ^r, a^r
- Coulomb branch: complex space of dimension r=rank of the gauge group
- The quantum correction to the moduli space metric: 1-loop exact + non-perturbative instanton corrections
- Free potential captures it well.

$$F(a),\ a_i^D = \partial F/\partial a^i,\ \tau_{ij} = \partial a_i^D/\partial a_j$$

$$\mathcal{L}_K = \frac{1}{2}\tau(a)_{ij}\partial_\mu a^i \partial^\mu a^j$$

- For SU(2) SYM,

$$F = \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2} + \sum_{k=1}^{\infty} F_k a^2 \left(\frac{\Lambda^4}{a^4}\right)^k$$

Quantum Field Theories

- How to extend QFTs
- Lower dimensions (0,1,2=1+2,3=1+2)
- Higher dimensions (5=1+4,6=1+5)

4 dim

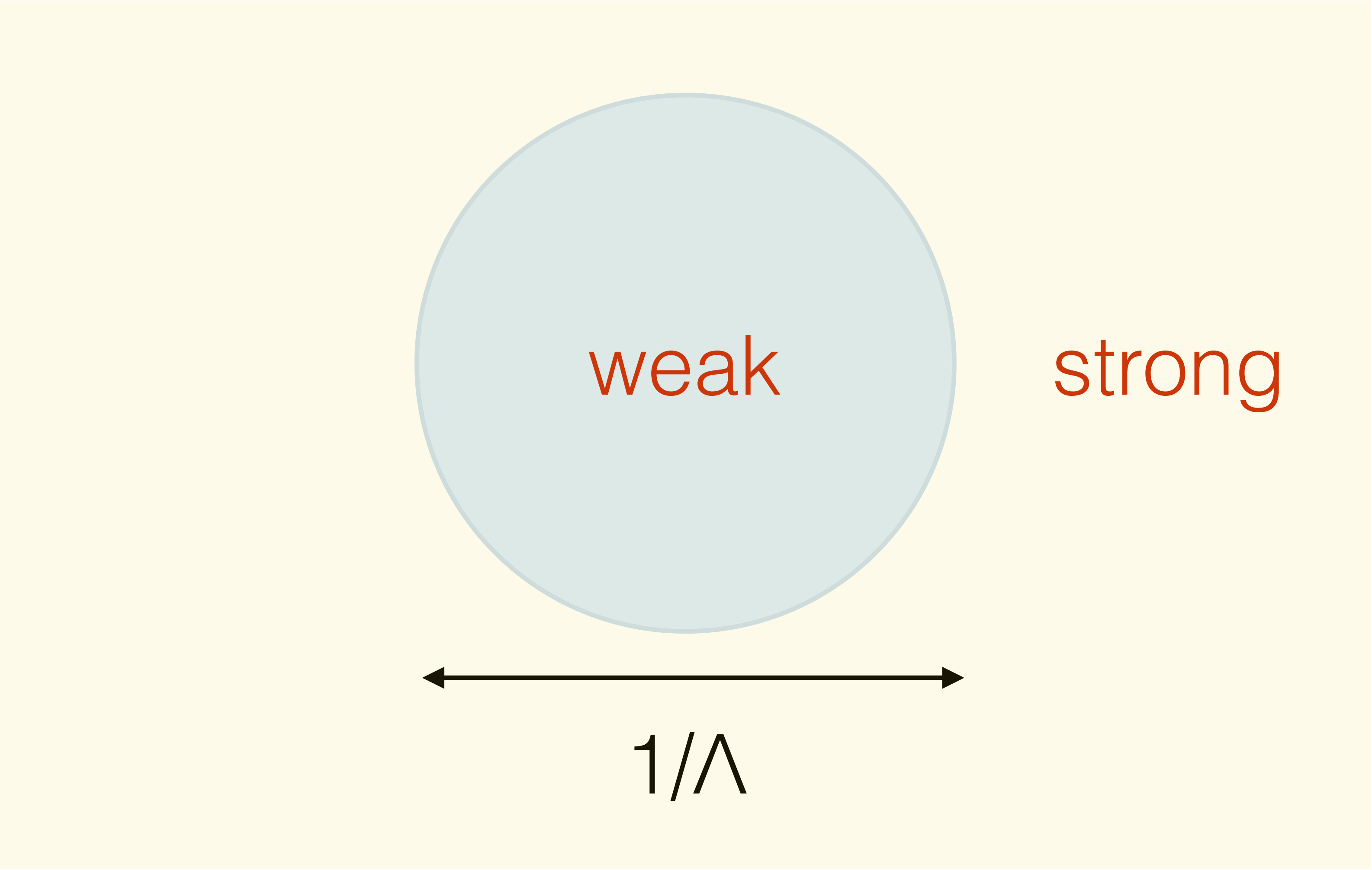
- Yang-Mills theory: $[g^2] = 0$
  - running coupling constant: $1/ g^2(p) \propto \ln (p/\Lambda)$
  - QCD: asymptotically free (UV free), confinement
  - QED: IR free, Landau pole
- 2,3 dim: well-defined, many relation to condensed matter physics

5 dim & 6 dim

- Yang-Mills theory: $[1/g^2] = M$ (5d), M^2 (6d)
  - coupling expansion = $1/M$ or $1/M^2$ expansion
  - non-renormalizable
  - IR free, UV strong, Landau pole?
- How to complete the physics of UV region?
  - 7,8,9,10: 10-dim superstring theory completion
  - 11-dim M theory completion
- 5,6 dim: 5,6 superconformal field theories and little string theories
  - instantons and instanton(self-dual) strings play a crucial role.

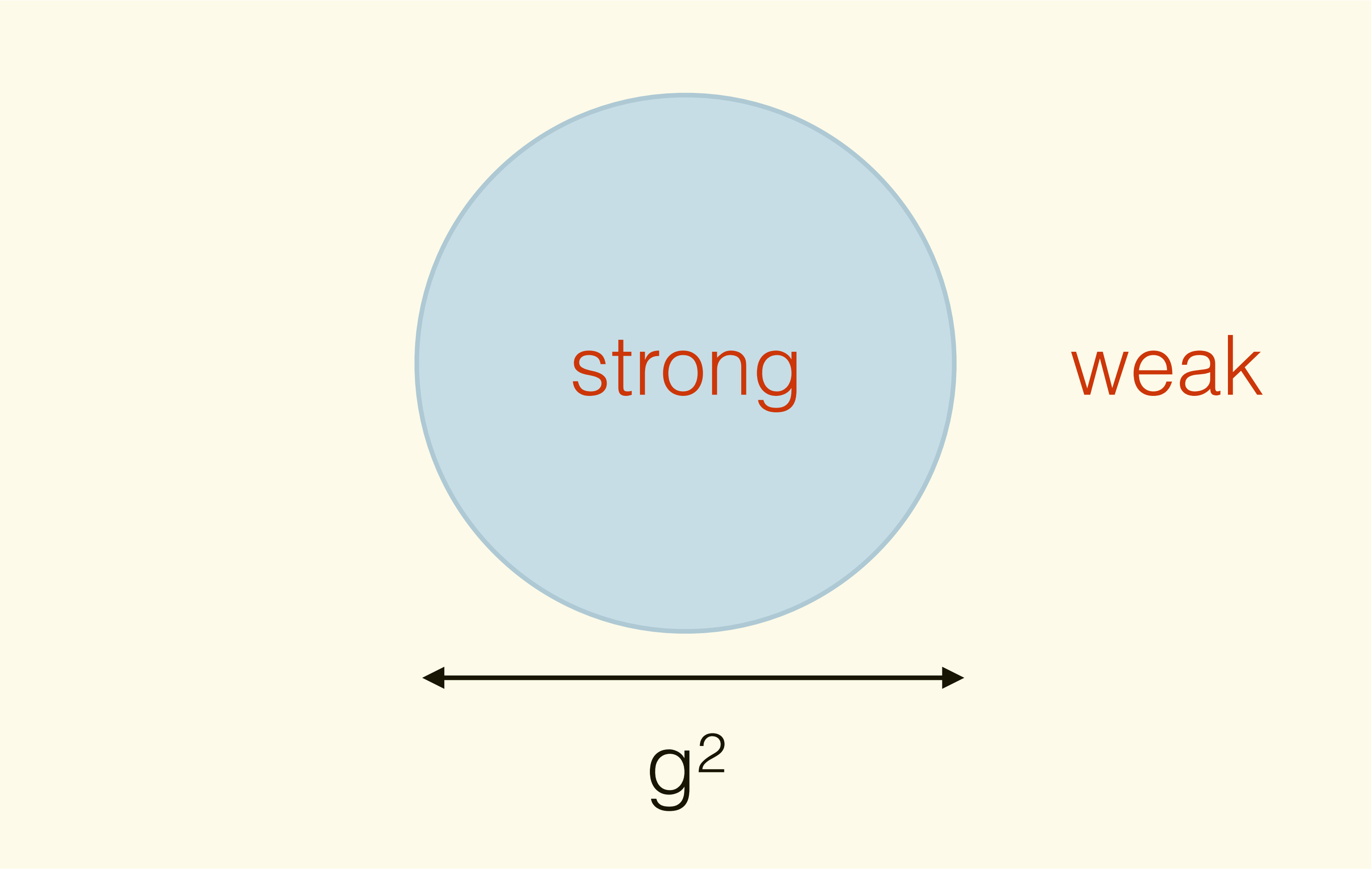
d=4 dim

- Yang-Mills theory: $[1/g^2](p) = c \ln (p/\Lambda)$, $c>0$
- weak interaction in short distance (asymptotically free)
- strong interaction in long distance (confinement)



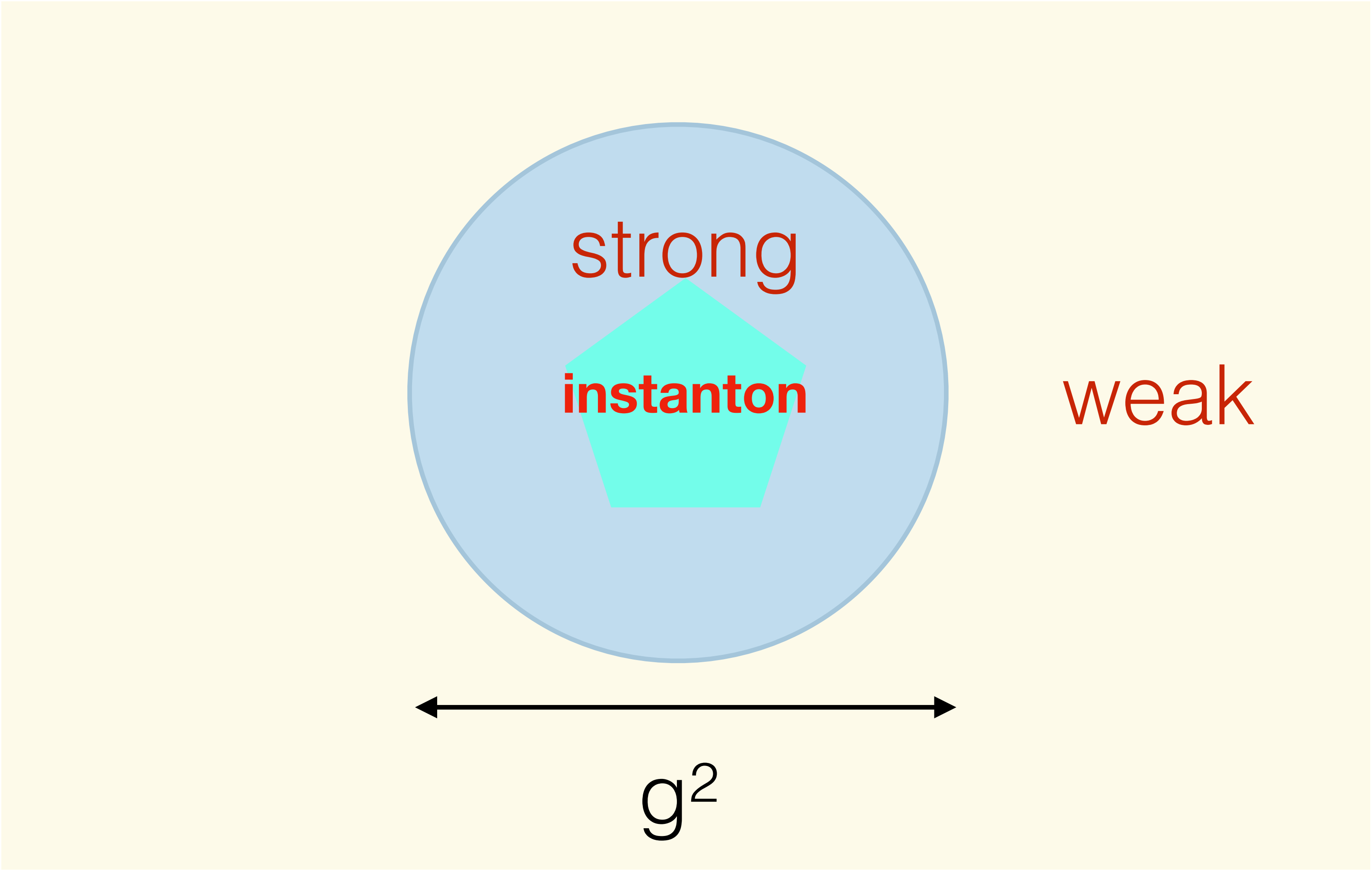
d=5,6 dim

- Yang-Mills theory: $[1/g^2] = M$ (5d), M^2 (6d)
  - strong interaction in short distance
  - weak interaction in long distance



d=5,6 dim

- Instanton solitons in 5d and Instanton (self-dual) strings in 6d provide the short distance completion of these theories.



5 and 6 Dim QFTs

5 & 6 dim superconformal group

W. Nahm 1978

- 5d: N=1 SCFTs with F(4) supergroup
- 6d: (1,0) SCFTs with OSp(2,6|1) supergroup
- 6d: (2,0) SCFTs with OSp(2,6|2) supergroup

conformal symmetry

- $p_\mu = \partial_\mu$ (space-time translation)
- $m_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$ (space-time translation)
- $d = x^\mu \partial_\mu$ (dilatation) $x^\mu \to \lambda x^\mu$
- $k_\mu = 2x_\mu(x^\nu \partial_\nu) - x^2 \partial_\mu$ (special conformal transformation)

$$x^\mu \to x'^\mu(x) = \frac{x^\mu - a^\mu x^2}{1 - 2(a \cdot x) + a^2 x^2}$$

No dimensionful parameter

5d $\mathcal{N}=1$ SCFTs

5 dim N=1

$$S_M = \frac{1}{g^2}\int d^5x\Big(\frac{1}{4}\mathrm{tr}F_{AB}F^{AB} + \frac{1}{2}D_A\Phi D^A\Phi ...\Big), \quad [g^2] = Length$$

- vector multiplet: A_μ, Φ, λ_A (symplectic Majorana)
- hyper multiplet: q_A, ψ (symplectic Majorana)

5 dim N=1

- Seiberg (9608111):
  - 5d N=1 SYM theories with SU(2)=Sp(1) gauge group and N_f fundamental hyper multiplets have a UV fixed point with enhanced global symmetry $E_{n+1} \supset$ $SO(2N_f) \times U(1)_I$
  - type I', HE,HO T-duality
- instantons: the number of 1/2 BPS instantons is $U(1)_I$ charge

5 dim N=1

- Morrison, Seiberg (9609070):
  - For $N_f=0$, discrete $\theta=0,\pi$ with E_1, $\bar{E}_1$ enhanced symmetry.
  - non-Lagrangian theory of rank 1 E_0 without any global symmetry.
- E_0, $\bar{E}_1=U(1)$, $E_1=SU(2)$, $E_2=SU(2)xU(1)$, $E_3=SU(3)xSU(2)$, $E_4=SU(5)$, $E_5=SO(10)$, E_6, E_7, E_8
- $\tilde{E}_8$: SU(2) + 8 fundamental hypermultiplets: completed in 6d (1,0) SCFT (E-string theory)
  - M5 branes exploring the Horava-Witten E_8 Wall in Heterotic E_8xE_8 theory

Dyonic Instantons

Lambert-Tong(9907), Kim-KL(0111)

- Dyonic Instantons
  - $$E = \frac{1}{2g^2}\int d^4x\ \mathrm{tr}\Big(\frac{1}{2}F_{\mu\nu}^2 + F_{\mu 0}^2 + (D_0\Phi)^2 + (D_\mu\Phi)^2 + \ldots\Big)$$
    $$= \frac{1}{2g^2}\int d^4x\ \mathrm{tr}\Big(\frac{1}{4}(F_{\mu\nu} \mp {}^*F_{\mu\nu})^2 + (F_\mu \mp D_\mu\Phi)^2 + (D_0\Phi)^2\Big) \pm M_{\mathrm{central}}$$
- Instantons: $F={}^*F$ (self-dual), and mass = $8\pi^2/g^2\ k$
  - zero modes for SU(2) case: 8k= 4 (c.m. position)+ 4(overall scale + SU(2) global symmetry)
- Dyonic one: $E_\mu=D_\mu\Phi$, $D_0\Phi=0$
  - Gauss law: $D_\mu E_\mu=D_\mu D_\mu\Phi=0$
  - Boundary condition $\langle\Phi\rangle= \mu\ \sigma_3/2$

Dyonic Instantons

Lambert-Tong(9907), Kim-KL(0111)

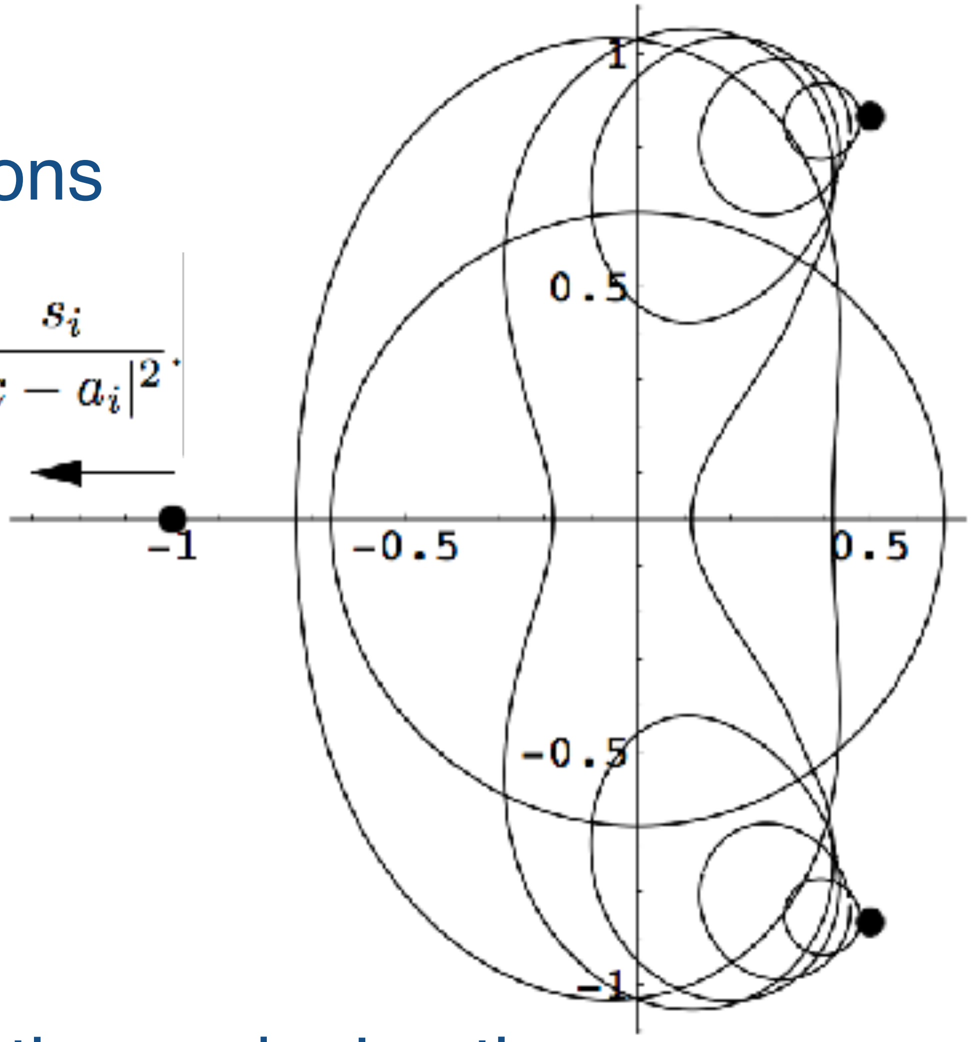
- The zeros of Φ is gauge invariant and depend on the instanton moduli space (supertube, Townsend 0103.): composition of D4,D0,F1,D2 loop.
- Nontrivial angular momentum (generalization of monopole-electric charge)
- For a given instanton number k, and electric charge Q with the given boundary condition, there is a hugh degeneracy with many different angular momentum (J+,J-) of SO(4)=SU(2xSU(2) spatial rotation.

Dyonic Instantons

- The zeros of Φ is gauge invariant and depend on the instanton moduli space parameters

- **Jackiw-Nohl-Rebbi (77)** for two instantons

$$A_\mu(x) = \frac{i}{2}\sigma^a \bar{\eta}^a_{\mu\nu}\partial_\nu \log H(x), \quad H(x) \equiv \sum_{i=0}^{\kappa} \frac{s_i}{|x-a_i|^2}.$$



- **Kim-Lee (0111)**

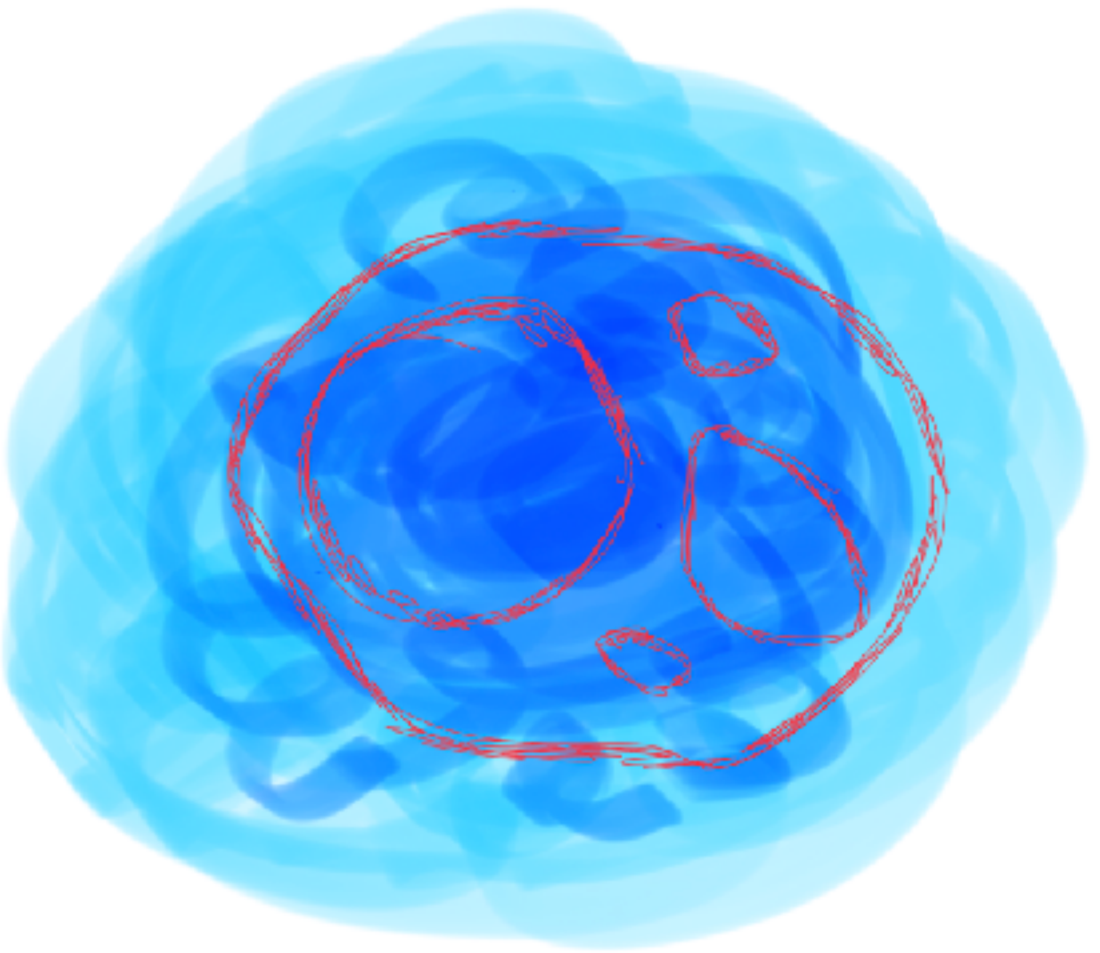
$$D_\mu D_\mu \Phi = 0, \quad \langle\Phi\rangle = \frac{v}{2}\sigma_3$$

  - Circle of D2 =the zeroth of Φ
  - For a given instanton configuration and <Φ>, the electric charge is fixed.
  - For a given number of instanton and electric charge there is a huge degeneracy with different angular momentum.

SU(2) Dyonic Instantons

Lambert-Tong(9907), Kim-KL(0111)

- k instanton with 8k zero modes
- 4 position, 3 overall SU(2) orientation, 1 overall size
- 8k-8: internal zero modes: relative siz and orientations
- For a given number of instanton and electric charge there is a huge degeneracy with different angular momentum.



5 dim N=1 SU(2) gauge

- Supercharge $Q_{\alpha A}, Q_{\dot{\alpha} A}$
- SO(4)=SU(2)$_-$xSU(2)$_+$:($\alpha, \dot{\alpha}$)
- SU(2)$_R$: A
- Supersymmetric Algebra

$$\{Q^A_M, Q^B_N\} = P_\mu (\Gamma^\mu C)_{MN} \epsilon^{AB} + i \frac{4\pi^2 k}{g^2_{YM}} C_{MN} \epsilon^{AB} + i \mathrm{tr}(v\Pi) C_{MN} \epsilon^{AB}$$

- BPS states in the Coulomb phase: instantons, electric charges and magnetic monopole strings

$$M_{central} = \frac{8\pi^2 k}{g^2} + vQ_{electric}$$

Quantum dyonic instantons

- Testing the enhancement of the global symmetry
- BPS states are invariant under the continuous deformation
- Counting the BPS states with a given supercharge

$$Q = \epsilon_{\dot{\alpha}A} Q_{\dot{\alpha}A}$$

- Generalized Witten Index=Index partition function

$$Z = \mathrm{Tr}(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-i\epsilon_+(J_+ + J_R) - i\epsilon_- J_- - i\alpha_i \Pi_i}$$

- $\varepsilon_1, \varepsilon_2$ Ω-deformation, localize the instanton on R^4
    - chemical potential for the angular momentum J_1, J_2
- α_i: localize the instanton size
    - chemical potential for the electric charge Π_i

5 dim N=1

- Fermion zero modes (gaugino λ and higgsinos ψ)

- Partition functions in the path integral is composed of perturbative and nonperturbative contributions

    - $Z = Z_{\text{perturbative}} Z_{\text{non-perturbative}}$

- perturbative: W-bosons and matter hyper

- nonperturbative: dyonic instantons = solitons with nontrivial angular momentum

Partition Function

- Ω-deformation, localized on R^4 : $\varepsilon_1=\varepsilon_++\varepsilon_-$, $\varepsilon_2=\varepsilon_+-\varepsilon_-$
- localized on the size of instantons: μ=α (Coulomb parameter)
- Z_k: Nekrasov partition function

$$Z_{non-pert} = 1 + \sum_{k=1}^{\infty} q^k Z_k, \quad q = e^{-\frac{8\pi^2}{g^2}}$$

- ADHM supersymmetric quantum mechanics for instantons
- localization: O(k) gauge theory, vector, hyper, fermi
(Kim,Kim,Lee 1206….)

$$I^{k=1}_{Sp(N)} = \frac{1}{32i^2}\left[\frac{\prod_{l=1}^{N_f} 2i\sin\frac{m_l}{2}\prod_{i=1}^{N} 2i\sin\frac{m\pm\alpha_i}{2}}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\sin\frac{m\pm i\gamma_1}{2}\prod_{i=1}^{N} 2i\sin\frac{i\gamma_1\pm\alpha_i}{2}} + \frac{\prod_{l=1}^{N_f} 2\cos\frac{m_l}{2}\prod_{i=1}^{N} 2\cos\frac{m\pm\alpha_i}{2}}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\sin\frac{m\pm i\gamma_1}{2}\prod_{i=1}^{N} 2\cos\frac{i\gamma_1\pm\alpha_i}{2}}\right] - I^{k=1}_{D0}$$

Partition Function (Gopakumar-Vafa Invariants)

- the free energy F: $Z = \exp(F) = \mathrm{PE}(F_s)$
- F_s = single particle free energy

$$Z_b = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$
$$= \exp(-\ln(1-x)) = \exp(\sum_{n=1}^{\infty} \frac{x^n}{n})$$
$$= \mathrm{PE}(x)$$

$$Z_f = 1 - x = \exp(-\ln(1-x))$$
$$= \exp(-\sum_{n=1}^{\infty} \frac{x^n}{n})$$
$$\equiv \mathrm{PE}(-x)$$

- $A = e^{-\mu} q^{2/(8-N_f)}$ **Mitev,Pomoni,Taki,Yagi,1411**

$$F_s = \sum_{n=0}^{\infty} \sum_{j_L, j_R = 0} N_{(j_L, j_R)} f(j_L, j_R) A^n$$

$$f(j_L, j_R; t, q) \equiv f(j_L, j_R) = \frac{(-1)^{2j_L + 2j_R + 1} \left((tq)^{-j_L} + \cdots + (tq)^{j_L} \right) \left((\frac{t}{q})^{-j_R} + \cdots + (\frac{t}{q})^{j_R} \right)}{(t^{1/2} - t^{-1/2})(q^{1/2} - q^{-1/2})},$$

$$t = e^{\epsilon_1}, q = e^{-\epsilon_2},$$

Partition Function

- some examples

$$f(0,0) = -\frac{\sqrt{qt}}{(1-t)(1-q)},$$
$$f(0,\frac{1}{2}) = \frac{t+q}{(1-t)(1-q)},$$
$$f(\frac{1}{2},\frac{1}{2}) = -\frac{(t+q)(1+qt)}{\sqrt{qt}(1-t)(1-q)},$$
$$f(\frac{1}{2},1) = -\frac{(1+qt)(q^2+qt+t^2)}{qt(1-t)(1-q)}.$$

- SU(2)+N_f=5 : E_6

$$F_{E_6} = -\frac{\sqrt{qt}}{(1-t)(1-q)}\chi_{\bar{\mathbf{27}}}\tilde{A} + \frac{t+q}{(1-t)(1-q)}\chi_{\mathbf{27}}\tilde{A}^2 + \mathcal{O}(\tilde{A}^3)$$

- $\overline{27}= 10_0 + 16_1+ 1_2$, $27=1_0+16_1+10_2$

- Ã measures the charge

- F_s has a simple pole in t and q.

6d (2,0) SCFTs

6d (2,0) SCFTs

Witten'95, Seiberg Witten '96

- superconformal symmetry: $OSp(2,6|2) \supset O(2,8) \times Sp(2)_R$
  - fields: B, Φ_I, Ψ
  - selfdual strength $H=dB={}^*H$, purely quantum $\hbar=1$
  - superconformal symmetry: $OSp(2,6|2) \supset O(2,8) \times Sp(2)_R$
- tensor branch: M2 branes connecting M5 branes = selfdual
- We do not know how to write down the theory for nonabelian case.
- N^3 degrees of freedom
- A, D, E type: (type IIB on ADE singularity)

Counting BPS states on $R^{1+4} x S^1$

- $x^5 \sim x^5 + 2\pi R$
- small R limit: 5d N=2 SYM with instanton being KK modes
  - instanton counting $Z = Z_{pert} Z_{inst}$
  - $Z_{inst} = 1 + q Z_1 + q^2 Z_2 + \ldots.$

Kim[2],Koh,Lee[2],1110

- large R limit: 6d self-dual elliptic genus in tensor branch
  - elliptic genus of self-dual strings
  - $Z = Z'_{pert} Z_{stri}$

$$Z_{\text{string}} = 1 + \sum_{k_1, k_2, .., k_r} w_1^{k_1} w_2^{k_2} .. w_r^{k_r} Z_{k_1 ., , k_r}$$

Haghighat,Iqbal,Kozcaz,
Lockhart,Vafa, 1305

Counting BPS states on $R^{1+4}xS^1$

Nekrasov'02,Nekrasov,Okounkov '03

- U(1) theory

$$Z_1 = \frac{\sin\frac{m\pm\epsilon_-}{2}}{\sin\frac{\epsilon_+\pm\epsilon_-}{2}} \qquad Z_{\rm inst} = 1 + qZ_1 + q^2 Z_2 + .. = \exp(\frac{q}{1-q}Z_1)$$

Kim[2],Koh,Lee[2],1110

  - sum over Young tableau Y

- U(N) theory

$$I_{k=1} = \left(\frac{\sin\frac{\gamma_1+\gamma_2}{2}\sin\frac{\gamma_1-\gamma_2}{2}}{\sin\frac{\gamma_1+\gamma_R}{2}\sin\frac{\gamma_1-\gamma_R}{2}}\right)\sum_{i=1}^{N}\prod_{j(\neq i)}\frac{\sinh\frac{\mu_{ij}+i\gamma_2-i\gamma_R}{2}\sinh\frac{\mu_{ij}-i\gamma_2-i\gamma_R}{2}}{\sinh\frac{\mu_{ij}}{2}\sinh\frac{\mu_{ij}-2i\gamma_R}{2}}$$

- Colored Young tableaus

$$I_{\{Y_1,Y_2,\cdots,Y_N\}} = \prod_{i,j=1}^{N}\prod_{s\in Y_i}\frac{\sinh\frac{E_{ij}-i(\gamma_2+\gamma_R)}{2}\sinh\frac{E_{ij}+i(\gamma_2-\gamma_R)}{2}}{\sinh\frac{E_{ij}}{2}\sinh\frac{E_{ij}-2i\gamma_R}{2}}$$

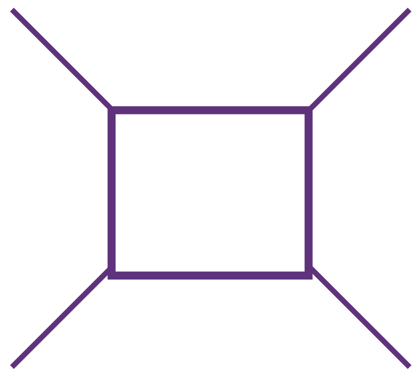
$$E_{ij} = \mu_i - \mu_j + i(\gamma_1-\gamma_R)h_i(s) + i(\gamma_1+\gamma_R)(v_j(s)+1)$$

6d (1,0)
SCFTs & LSTs

6d (1,0) SCFTs

Seiberg'96,Danielsson et.al.'97

- supercharge Q (1,0), ε-spinor (0,1)
  - (1,0) vector multiplet: gaugino (0,1)
  - (1,0) hypermultiplet: higgsino (1,0)
  - tensor multiplet: (1,0)



- gauge anomaly due to vector and matter 1-loop
- anomaly polynomial

$$\mathrm{Tr}_R F^4 = \alpha_R \mathrm{tr} F^4 + c_R (\mathrm{tr} F^2)^2$$

$$\alpha_R = 0 \ for \ SU(2), SU(3), G_2, SO(8), F_4, E_6, E_7, E_8$$

$$c_{\mathrm{tot}} = \left[c_{Ad} - \sum_{R \ \mathrm{matter}} c_R \right] \geq 0$$

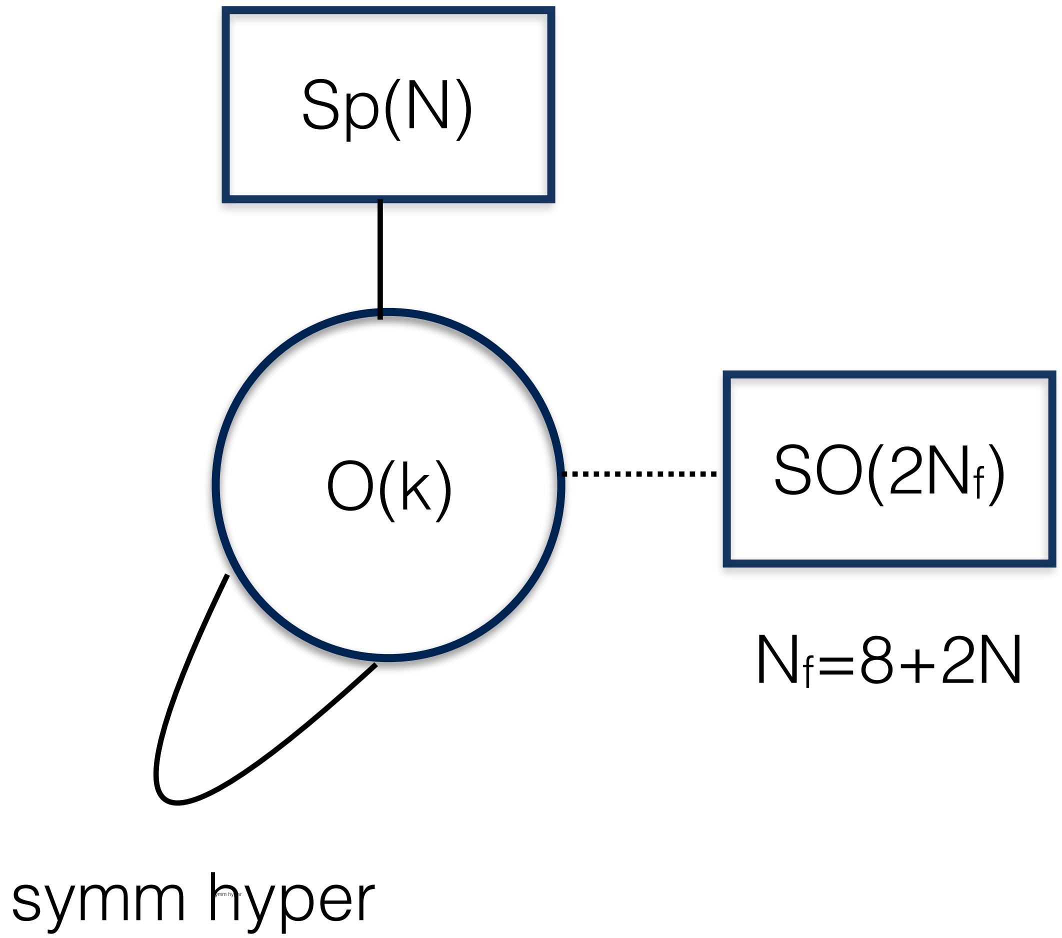
6d (1,0) rank 1 with E_8 global symmetry

- M5 branes on Horava-Witten M9 Wall
  - $O8^-$+8D8 branes + 1 NS5 brane + 2N D6 branes:
  - two 1/2 NS5 branes on O6- plane + (4+N)D6 branes
    - $Sp(N)_T$ - [SO(2N+16)]
    - k instanton strings=self-dual strings: 2d O(k) group
- 1/2 NS5 branes on $O8^-$+8D8+ 1 NS5 brane + N D6 branes
  - $SU(N)_T$-[SU(N+8)]+ antisymmetric hyper:
  - tensor multiplet: (1,0)
  - k instanton strings=self-dual strings: 2d U(k) gauge group

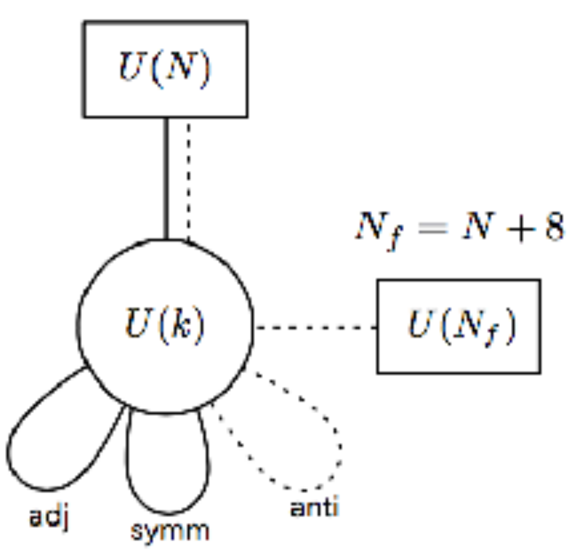
6d (1,0) rank 1 with E_8 global symmetry

- a part of E_6 conformal matter
  - a single M5 brane on E_6 singularity
  - fractionalization: $[E_6]$-T-$SU(3)_T$-T-$[E_6]$
  - $[E_6]$-T-[SU(3)]
- a part of E7 conformal matter : $[E_7]$-T-$SU(2)_T$-$SO(7)_T$-$SU(2)_T$-T-$[E_7]$
  - [E7]-T-[SU(2)]
- a part of E8 conformal matter k instanton strings=self-dual strings: 2d U(k) gauge group

6d (1,0) rank 1 with E_8 global symmetry



Kim[2],KL,Park,Vafa '14
Kim[2],KL'15



(a)

Field	Type	$U(k)$	$U(N)$	$U(N_f)$	$U(1)_A$
$(A_\mu, \lambda^{\dot{\alpha}A})$	vector	**adj**	–	–	0
$(a_{\alpha\dot{\beta}}, \chi^A_\alpha)$	hyper	**adj**	–	–	0
$(q_{\dot{\alpha}}, \psi^A)$	hyper	**k**	$\overline{\mathbf{N}}$	–	0
(Ξ_l)	Fermi	**k**	–	$\overline{\mathbf{N_f}}$	0
$(\varphi_A, \Phi^{\dot{\alpha}})$	twisted hyper	**sym**	–	–	+1
(Ψ_α)	Fermi	**anti**	–	–	+1
(ψ)	Fermi	**k**	**N**	–	+1

(b)

6d (1,0) rank 1 with E_8 global symmetry

- Partition function $Z = Z_{pert}\, Z_{string}$, $Z_{string} = 1 + w\, Z_1 + w^2\, Z_2 + \ldots$
  - single string partition function

Klemm,Mayr,Vafa'96

$$Z_1 = \sum_{i=1}^{4} \frac{Z_{1(i)}}{2} = -\frac{\Theta(q, m_l)}{\eta^6 \theta_1(\epsilon_1)\theta_1(\epsilon_2)}\ , \quad \Theta(\tau, m_l) = \frac{1}{2}\sum_{n=1}^{4}\prod_{l=1}^{8}\theta_n(\tau, m_l)$$

  - two strings

Haghighat,Lockhart,Vafa,1406

$$Z_2 = \frac{1}{576\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_1)}\Big[4A_1^2(\phi_{0,1}(\epsilon_1)^2 - E_4\phi_{-2,1}(\epsilon_1)^2) \\ + 3A_2(E_4^2\phi_{-2,1}(\epsilon_1)^2 - E_6\phi_{-2,1}(\epsilon_1)\phi_{0,1}(\epsilon_1)) + 5B_2(E_6\phi_{-2,1}(\epsilon_1)^2 - E_4\phi_{-2,1}(\epsilon_1)\phi_{0,1}(\epsilon_1))\Big] + (\epsilon_1 \leftrightarrow \epsilon_2) \qquad (3.26)$$

- multi-strings (eg. four strings)

$$Z_{4(1)} = -\oint \left[\eta^4 du_1 du_2 \cdot \frac{\theta_1(2\epsilon_+)^2\theta_1(2\epsilon_+ \pm u_1 \pm u_2)\theta_1(\pm u_1 \pm u_2)}{\eta^{10}}\right]_{\text{vec}} \\ \cdot \left[\frac{\eta^{20}}{\theta_1(\epsilon_{1,2})^2\theta_1(\epsilon_{1,2} \pm u_1 \pm u_2)\theta_1(\epsilon_{1,2} \pm 2u_1)\theta_1(\epsilon_{1,2} \pm 2u_2)}\right]_{\text{sym}} \cdot \left[\prod_{l=1}^{8}\frac{\theta_1(m_l \pm u_1)\theta_1(m_l \pm u_2)}{\eta^4}\right]_{\text{Fermi}} \qquad (3.43)$$

$$\sum_{i=1}^{4} \frac{\theta_1(2\epsilon_1+\epsilon_2)\theta_1(-\epsilon_1)\prod_l \theta_i(m_l \pm (\epsilon_1 - \frac{\epsilon_2}{2}))\theta_i(m_l \pm \frac{\epsilon_2}{2})}{2\eta^{24}\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_1-\epsilon_2)\theta_1(2\epsilon_2-\epsilon_1)\theta_1(3\epsilon_1-\epsilon_2)\theta_1(2\epsilon_2-2\epsilon_1)} + (\epsilon_1 \leftrightarrow \epsilon_2)$$

Issues

Find some general principle for Z_{string}

- Anomaly polynomial on the string dynamics — delZotto,Lockhart 1609
    - Holomorphic anomaly equation on partition function
    - employ the ring structure of Jacobi forms
- Find the bound on the angular momentum, use the modularity — Gu,Huang,Kashini-Poor,Klemm1701
- Find some smart way to write the instanton string dynamics — H.Kim,S.Kim,J.Park 1608
- many more….

Conclusion

5, 6 d Quantum Field Theories

- Ultra-violet physics of these theories is not well-defined unless the detail of the instanton dynamics is provided.

- One has to supply the detail of instanton dynamics or the self-dual strings (some highly nontrivial cases have been attacked very successfully but much remain to be done.)

- Not only interesting by their own, they imply that many new insights to the 4d quantum field theories   A lot of them have been discovered but still a lot more to be discovered.