Vortex defects in 2D SUSY Gauge theories

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Based on:

Sungjay Lee (KIAS), Takuya Okuda (U Tokyo) and KH, to appear

Introduction

"Vortex defect" — codimension-2 defect in gauge theory

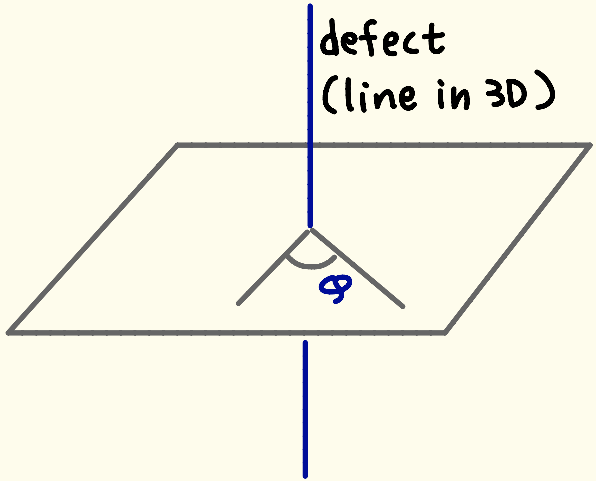
defined by a singular boundary condition

"Vorticity"
(Lie algebra element)

angle coordinate

$$A \simeq \eta \cdot d\varphi$$

Gauge field



Goal: study point-like vortex defects in

2D $\mathcal{N}=(2,2)$ SUSY gauge theories

The partition functions & some BPS observables for the theories on (squashed) spheres were evaluated using SUSY localization.

Benini-Cremonesi '12,
Doroud-Le Floch-Gomis-Lee '12, …

Method:

① Define the 2D $\mathcal{N}=(2,2)$ SUSY theory

of vector & chiral multiplet on S^2

※ Focus on BPS observables preserving a SUSY Q

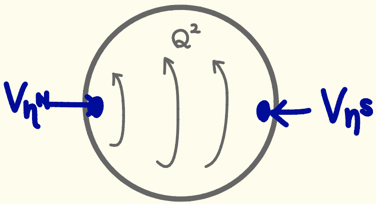
($\Leftrightarrow$ a Killing spinor)

$$Q^2 = (\text{rotation fixing North \& South poles}) + \cdots$$

② Introduce vortex defects V_{η^N}, V_{η^S} at NP & SP

so that Q is preserved

$\Rightarrow$ compute correlators



Results:

① The vortex correlators turn out to be trivial in many "simple" theories, but not always.

We see this in the examples with $U(1)$ gauge group

— GLSM for $\mathbb{CP}^{N-1}$, Quintic CY

② Even when their correlators are trivial, vortex defects can be used to derive

— twisted chiral ring relation

— Picard-Fuchs differential equation

Multiplet ①

Vector multiplet for gauge group G

A_m ···· gauge field

σ, ρ ···· real scalars

D ········ auxiliary scalar

$\lambda = \begin{pmatrix} \lambda^+ \\ \lambda^- \end{pmatrix}$ ··· gaugino, R-charge $(+1)$

$\bar{\lambda} = \begin{pmatrix} \bar{\lambda}^+ \\ \bar{\lambda}^- \end{pmatrix}$ ··· gaugino, R-charge (-1)

SUSY localization (I)

The path integral over vector multiplet fields

on the sphere with metric $ds^2 = \ell^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

localizes onto saddle point configurations

$$\sigma = \frac{a}{\ell}, \quad D = -\frac{a}{\ell^2},$$

$$\rho = -\frac{s}{\ell}, \quad A = s\cdot(\cos\theta \mp 1)\, d\varphi \quad \cdots \text{ on N/S hemispheres}$$

※ $a, s \in$ (Lie algebra), s is GNO quantized.

Saddle points in the presence of defects V_{η^N}, V_{η^S}

$$\sigma = \frac{a}{\ell}, \quad D = -\frac{a}{\ell^2} + 2\pi i \eta^N \cdot \delta^2(NP) + 2\pi i \eta^S \cdot \delta^2(SP)$$

$$\rho = -\frac{s}{\ell}, \quad A = \begin{cases} s\cdot(\cos\theta - 1)\,d\varphi + \eta^N \cdot d\varphi & (\text{North hemisphere}) \\ s\cdot(\cos\theta + 1)\,d\varphi + \eta^S \cdot d\varphi & (\text{South hemisphere}) \end{cases}$$

For $U(1)$ theories, the path integral simplifies to

$$\sum_{s \in \frac{1}{2}(\underbrace{\eta^N - \eta^S}_{\text{note}} + \mathbb{Z})} \int_{\mathbb{R}} \frac{da}{2\pi} \exp(-2ira + 2is\theta) \cdot \{\text{matter contrib}\}$$

note

the shift.

※ FI-θ coupling: $\mathcal{L} = -r \cdot D + \theta \cdot F_{12} + \cdots$

Multiplet ②

chiral multiplets

c.c.

ϕ .... complex scalar, R-charge $2q$

$\psi \equiv \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$ .... Dirac fermion, $2q-1$

F .... complex aux. field $2q-2$

$\bar{\phi}$

$\bar{\psi} \equiv \begin{pmatrix} \bar{\psi}^+ \\ \bar{\psi}^- \end{pmatrix}$

$\bar{F}$

furnishes a complex rep. of the gauge group.

Examples of $U(1)$ gauge theories

① N chirals $(\phi_1, \cdots, \phi_N)$ all with charge $+1$,

FI coupling $r\ (>0)$

$\Longrightarrow$ vacua : $\{ |\phi_1|^2 + \cdots + |\phi_N|^2 = r \} / U(1)$

$\cdots$ Non-linear sigma model (NLSM) on $\mathbb{CP}^{N-1}$ of size r.

Examples of $U(1)$ gauge theories

② chirals $(\phi_1, \cdots, \phi_5 ; P)$ with charges $(1, \cdots, 1 ; -5)$

FI coupling $r\ (>0)$

Superpotential $W \equiv P \cdot \underbrace{F_5(\phi_1, \cdots, \phi_5)}$ quintic polynomial

$\Longrightarrow$ vacua = Quintic hypersurface $F_5(\phi) = 0$

in $\mathbb{CP}^4$ of size r.

.... NLSM on a Calabi-Yau 3-fold.

SUSY localization (II)

Path integral over chirals localizes to $\phi = \psi = F = 0$.

Gaussian approx. around there gives an exact result.

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Take an $U(1)$ theory, and choose a saddle point (a, s).

Path integral over a single chiral of charge $+1$ gives

the "1-loop determinant"

$$Z_{1loop} = \frac{\Gamma(s+q-ia)}{\Gamma(s+1-q+ia)}$$

R-charge

Exact partition function

(Benini-Cremonesi '12, Doroud-Le Floch-Gomis-Lee '12)

example: $\mathbb{CP}^{N-1}$ model

$$Z_{S^2} = \sum_{s\in\frac{1}{2}\mathbb{Z}} \int_{\mathbb{R}} \frac{da}{2\pi} \cdot e^{-it(a+is)-i\bar{t}(a-is)} \qquad (t \equiv r+i\theta)$$

$$\cdot \prod_{j=1}^{N} \frac{\Gamma(s-ia+q_j)}{\Gamma(1+s+ia-q_j)}$$

※ $q_j = \frac{1}{2}(\text{R-charge})_j - i\cdot(\text{mass})_j$

Correlators $\langle V_{\eta^N} V_{\eta^S} \rangle$

- Sketch of the computation of $Z_{1loop}(a, s, \eta^N, \eta^S)$

① components in a chiral multiplet form 2 pairs.

$$\phi \overset{Q}{\longleftrightarrow} \Psi_1 , \quad \Psi_2 \overset{Q}{\longleftrightarrow} F$$

② one can find a 1st order differential op. J such that

- $\phi, \Psi_1 \in \mathcal{H} \underset{J^\dagger}{\overset{J}{\rightleftarrows}} \mathcal{H}' \ni \Psi_2, F$

- $[Q, J] = [Q, J^\dagger] = 0$

Correlators $\langle V_{\eta^N} V_{\eta^S} \rangle$

③ The evaluation of $Z_{1\text{loop}} = \dfrac{\det Q^2|_{\mathcal{H}'}}{\det Q^2|_{\mathcal{H}}} = \dfrac{\det Q^2|_{\mathrm{Ker}\, J^\dagger}}{\det Q^2|_{\mathrm{Ker}\, J}}$

needs eigenfunctions of $Q^2 = \frac{\partial}{\partial\varphi} + \cdots$ and $(J^\dagger J$ or $J J^\dagger)$.

④ General eigenfunctions take separated form,

$$f(\theta,\varphi) \sim e^{im\varphi} \cdot (\sin\theta)^{\pm(m-\underset{\sim}{\eta^N})} \quad \text{near NP } (\theta \sim 0)$$

$$f(\theta,\varphi) \sim e^{im\varphi} \cdot (\sin\theta)^{\pm(m-\underset{\sim}{\eta^S})} \quad \text{near SP } (\theta \sim \pi)$$

fractional

※ $m \in \mathbb{Z}$

⑤ When $\eta^N, \eta^S \notin \mathbb{Z}$, there are

2 possible boundary conditions at NP, SP.

- Ordinary b.c. $\boxed{\phi = \Psi_1 = 0, \quad J^\dagger \Psi_2 = J^\dagger F = 0}$ at poles.

  ※ Ψ_2, F may diverge mildly as $\sim (\sin\theta)^\gamma \quad (\gamma > -1)$

- Flipped b.c. $\boxed{J\phi = J\Psi_1 = 0, \quad \Psi_2 = F = 0}$ at poles.

recall

$$\phi, \Psi_1 \in \mathcal{H} \underset{J^\dagger}{\overset{J}{\rightleftarrows}} \mathcal{H}' \ni \Psi_2, F$$

1-loop determinants

① ordinary b.c. $\Rightarrow$ $Z_{1loop} = \dfrac{\Gamma(\lceil \eta^N \rceil - \eta^N + s - ia + q)}{\Gamma(-\lceil \eta^S \rceil + \eta^S + 1 + s + ia - q)}$ ceiling fn.

② flipped b.c. $\Longrightarrow$ $Z_{1loop} = \dfrac{\Gamma(\lfloor \eta^N \rfloor - \eta^N + s - ia + q)}{\Gamma(-\lfloor \eta^S \rfloor + \eta^S + 1 + s + ia - q)}$ floor fn.

Note that Z_{1loop} is periodic in η^N, η^S in both cases.

(Large gauge transformation can shift η^N, η^S by integers)

The correlator therefore satisfies, for $k, h \in \mathbb{Z}$,

$$\langle V_{\eta^N + k} V_{\eta^S + h} \rangle = e^{-kt - h\bar{t}} \langle V_{\eta^N} V_{\eta^S} \rangle$$

(Non-) triviality

$\mathbb{CP}^{N-1}$ case :

$$\langle V_{\eta^N} V_{\eta^S} \rangle = \sum_{s \in \frac{1}{2}(\mathbb{Z}+\eta^N-\eta^S)} \int \frac{da}{2\pi}\, e^{-it(a+is-i\eta^N)-i\bar{t}(a-is-i\eta^S)}$$
$$\cdot \prod_{j=1}^{N} \frac{\Gamma(\lceil \eta^N \rceil - \eta^N + s + q_j - ia)}{\Gamma(-\lceil \eta^S \rceil + \eta^S + 1 + s - q_j + ia)}$$

※ chose ordinary b.c.

By a shift of integration contour of a which does not cross the poles of the integrand one can actually show the triviality,

$$\langle V_{\eta^N} V_{\eta^S} \rangle = e^{-t\lceil \eta^N \rceil - \bar{t}\lceil \eta^S \rceil} \cdot \langle 1 \rangle$$

However, a different way of contour-shift leads to the identification

$$\boxed{\text{Vortex defect } V_{\eta^N} \longleftrightarrow \text{polynomial of } \underbrace{\Sigma \equiv -\ell(\rho+i\sigma)}_{\text{twisted chiral op.}}}$$

$$\eta^N \in (-1, 0] \cdots\cdots\cdots V_{\eta^N} = 1 \qquad = 1 \quad (0)$$

$$\eta^N \in (0, 1] \cdots\cdots\cdots V_{\eta^N} = \prod_{j=1}^{N} (\Sigma + q_j) \qquad = e^{-t} \quad (1)$$

$$\eta^N \in (1, 2] \cdots\cdots\cdots V_{\eta^N} = \prod_{j=1}^{N} (\Sigma + q_j)(\Sigma + q_j + 1) \quad = e^{-2t} \quad (2)$$

$$(1) \qquad e^{-t} = \prod_{j=1}^{N} (\Sigma + q_j)$$

$$(2) \qquad e^{-2t} = \prod_{j=1}^{N} (\Sigma + q_j)(\Sigma + q_j + 1)$$

(1) is the twisted chiral ring relation.

$\because$ For generic large $\langle \Sigma \rangle$ all the chirals are massive.

Integrating them out yields the twisted superpotential,

$$\widetilde{W}(\Sigma) = -\underbrace{t \cdot \Sigma}_{\text{gives FI-}\theta\text{ term}} - \sum_{j=1}^{N} (\Sigma + q_j) \left\{ \log(\Sigma + q_j) - 1 \right\}$$

$$\partial \widetilde{W} / \partial \Sigma = 0 \longrightarrow (1)$$

$$(1) \qquad e^{-t} = \prod_{j=1}^{N} (\Sigma + q_j)$$

$$(2) \qquad e^{-2t} = \prod_{j=1}^{N} (\Sigma + q_j)(\Sigma + q_j \underline{+1}) \ \star$$

(1) & (2) contradict due to $\star$ (an effect of Ω-deformation).

They both make sense if we replace

$$\Sigma \longrightarrow \frac{\partial}{\partial t} = -z\frac{\partial}{\partial z} \qquad (z \equiv e^{-t})$$

$$(1) \longrightarrow \quad z \cdot Z_{S^2} = \prod_{j=1}^{N} \left(-z\frac{\partial}{\partial z} + q_j\right) \cdot Z_{S^2}$$

$$(2) \longrightarrow \quad z^2 Z_{S^2} = \prod_{j=1}^{N} \left(-z\frac{\partial}{\partial z} + q_j\right)\left(-z\frac{\partial}{\partial z} + q_j + 1\right) \cdot Z_{S^2}$$

Remark :

N independent solutions to $z \cdot Z_{S^2} = \prod_{j=1}^{N} \left(-z\frac{\partial}{\partial z} + q_j\right) \cdot Z_{S^2}$

agree precisely with "vortex partition functions".

The contour of a-integration in

$Z_{S^2} = \sum_{s} \int \frac{da}{2\pi} (\cdots\cdots)$ can be closed.

Then Z_{S^2} becomes a bilinear of vortex partition fn.

Quintic

Let us define V_{η^N} by

ordinary b.c. for $\phi_{1,\cdots,5}$ (R-charge $2q$)

flipped b.c. for P (R-charge $2-10q$)

Then the contour-shift analysis gives a non-trivial relation between V_{η^N} and $\Theta \equiv \ell(\rho + i\sigma) - q$.

The relations

$$\eta^N \in (-1, -4/5] \qquad V_{\eta^N} = (1+5H)(2+5H)(3+5H)(4+5H)$$

$$(-4/5, -3/5] \qquad V_{\eta^N} = (1+5H)(2+5H)(3+5H)$$

$$(-3/5, -2/5] \qquad V_{\eta^N} = (1+5H)(2+5H)$$

$$(-2/5, -1/5] \qquad V_{\eta^N} = (1+5H)$$

$$(-1/5, 0] \qquad V_{\eta^N} = 1$$

$$(0, 1/5] \qquad V_{\eta^N} = -H^4/5$$

Remark :

$V_{\eta^{N+1}} = e^{-t} V_{\eta^N}$ and $\Theta \equiv -\frac{\partial}{\partial t} - q = z\frac{\partial}{\partial z} - q \quad \left(z \equiv -5^5 e^{-t} \right)$

lead to Picard-Fuchs equation

$$\left\{ \left(z\frac{\partial}{\partial z}\right)^4 - z\left(z\frac{\partial}{\partial z}+\frac{4}{5}\right)\left(z\frac{\partial}{\partial z}+\frac{3}{5}\right)\left(z\frac{\partial}{\partial z}+\frac{2}{5}\right)\left(z\frac{\partial}{\partial z}+\frac{1}{5}\right)\right\}\left(z^{-q} \cdot Z_{S^2}\right) = 0 \qquad \bigstar$$

Z_{S^2} for Calabi-Yau GLSM is known to coincide with e^{-K},

(Jockers-Kumar-Lapan-Morrison-Romo '12)

where $K(t,\bar{t}) =$ Kähler potential for the "conformal manifold"

$=$ bilinear of "periods" (solutions to $\bigstar$)

Conclusions

vortex defects in 2D $\mathcal{N}=(2,2)$ SQEDs were studied.

① $\mathbb{CP}^1$ model ... V_η itself is trivial, but it can be used to explain chiral ring relation & differential equation in a new way.

② Quintic model .... V_η is nontrivial.

Other interesting issues

- behavior of V_η under mirror symmetry
- non-abelian gauge theories