Painlevé equations and four dimensional $\mathcal{N} = 2$ theories

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Part 1: existence of a precise correspondence

$$\boxed{\text{class } \mathcal{S} \text{ theories} \quad \Longleftrightarrow \quad \text{Painlevé equations}}$$

Class $\mathcal{S}$ theories: 4d $\mathcal{N} = 2$ theories constructed from 6d $\mathcal{N} = (2, 0)$

Painlevé: special 2° order non-linear Ordinary Differential Equations

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Painlevé: special 2° order non-linear Ordinary Differential Equations

Part 2: correspondence implies equivalence

$$\boxed{\text{solution Painlevé equations} \quad \Longleftrightarrow \quad \text{magnetic, dyonic } Z_{\text{Nek}}}$$

and can be used to compute Z_{Nek} at strong coupling

Class $\mathcal{S}$ theories and Hitchin systems

Class $\mathcal{S}$ theories [Gaiotto]: class of four dimensional $\mathcal{N}=2$ theories, constructed from 6d $\mathcal{N}=(2,0)$ theory via twisted compactification on $\mathcal{C}_{g,n}$

$$\text{6d } \mathcal{N}=(2,0) \text{ theory of type } A_1 \text{ on } \mathbb{R}^3 \times S^1_R \times C_{g,n}$$

$$\Big\Downarrow\, C_{g,n}$$

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Class $\mathcal{S}$ theories are naturally associated to Hitchin system [GMN]:

$$\text{4d } \mathcal{N}=2 \text{ theory of class } \mathcal{S}$$

$$\Big\Downarrow \; S^1_R$$

$$\text{3d } \mathcal{N}=4 \text{ theory on } \mathbb{R}^3;\ \sigma\text{-model with hyperkähler target } \mathcal{M}_H$$

The target $\mathcal{M}_H$ coincides with the moduli space of a Hitchin system associated to the punctured Riemann surface $C_{g,n}$

More in detail, $\mathcal{M}_H$ is the moduli space of solutions of the equations

$$\boxed{\begin{aligned} F_{z\bar{z}} + R^2[\varphi_z, \bar{\varphi}_{\bar{z}}] &= 0 \\ D_{\bar{z}}\varphi_z &= 0 \quad \mod G \\ D_z \bar{\varphi}_{\bar{z}} &= 0 \end{aligned}}$$

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- z: complex coordinate on $C_{g,n}$
- A_z: gauge connection on $C_{g,n}$ ($F_{z\bar{z}}$ field strength);
- φ_z: complex adjoint scalar ((1,0)-form after twist);
- two types of singular behaviour at the puncture $z = z_*$:

$$\varphi_z \sim \frac{1}{z - z_*}\,, \quad \text{regular singularity (simple pole)}$$

$$\varphi_z \sim \frac{1}{(z - z_*)^{1+r}}\,, \quad \text{irregular singularity (higher order pole, } r \geqslant 1)$$

Different $C_{g,n} \Longleftrightarrow$ different Hitchin system, different 4d theories;

focus on rank 1 theories with genus 0 Riemann surface:

- $SU(2)$ SQCD with flavor $N_F = 0, 1, 2, 3, 4$
- Argyres-Douglas theories H_0, H_1, H_2

Example: $SU(2)$ $N_F = 4$ SQCD from $C_{0,4}$ with four regular punctures

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Theories related by coalescence diagram (collision of punctures):

$$
\begin{tikzcd}
SU(2)_4 \arrow[r] & SU(2)_3 \arrow[r] \arrow[dr] & \begin{matrix} SU(2)_2 \\ SU(2)_2' \end{matrix} \arrow[r] \arrow[dr] & SU(2)_1 \arrow[r] \arrow[dr] & SU(2)_0 \\
 & & H_2 \arrow[r] & \begin{matrix} H_1 \\ H_1' \end{matrix} \arrow[r] & H_0
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$$D_{\bar{z}}\varphi_z = 0$$

- equivalent to torus fibration over Coulomb branch $\mathcal{B}_{4d}$;
- associated to complex algebraic integrable system [Donagi-Witten]: Hamiltonian $\Leftrightarrow$ u, spectral curve $\Leftrightarrow$ Seiberg-Witten curve

$$\det(y - \varphi_z) = 0 \quad \Longrightarrow \quad y^2 = \frac{1}{2}\mathrm{Tr}\varphi_z^2$$

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$\zeta \in \mathbb{C}^\times$: reduction $\mathcal{M}_H \longrightarrow \mathcal{M}_{flat}$, flat connections $\nabla = \frac{R}{\zeta}\varphi + D + R\zeta\bar{\varphi}$

- important limit: $R \to 0$, $\zeta \to 0$ with $\zeta/R = \hbar$ fixed (oper limit)

$$\boxed{\nabla \underset{oper}{\longrightarrow} \hbar\partial_z - \varphi_z}$$

- opers of rank 1 theories are naturally associated to Painlevé equations

Painlevé equations

Painlevé ($\sim$ 1900): classify all *non-linear* 2° order, degree 1 ODE

$$\ddot{q} = F(q, \dot{q}; t)$$

such that

- F rational in $q, \dot{q}$, meromorphic in t;
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Final result: six non-trivial equations

	PVI	PV	PIV	PIII	PII	PI
# parameters	4	3	2	2	1	0

Examples: $\ddot{q} = 6q^2 + t$ (PI) , $\ddot{q} = 2q^3 + tq + \alpha$ (PII)

Later developments (Lax pair formulation, analysis space initial conditions) led to a refinement of the classification; equations related by *confluence*:

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Diagram analogue to the gauge theory one:

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First hint towards the existence of a Painlevé/gauge theory correspondence

I) Painlevé arise as equations of motion classical Hamiltonian systems

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}_a(q,p;t)}{\partial p} \quad , \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}_a(q,p;t)}{\partial q}$$

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We can study time evolution of $\mathcal{H}_a(t) \Longrightarrow$ new 2° order, degree 2 ODE:

- σ-Painlevé equations: ODEs satisfied by

$$\boxed{\sigma_a(t) \ \propto \ \mathcal{H}_a(q(t), p(t); t)}$$

- τ-Painlevé equations: ODEs satisfied by $\tau_a(t)$

$$\boxed{\sigma_a(t) \ \propto \ \frac{d}{dt} \ln \tau_a(t)}$$

The $\tau_a(t)$ function is the one which usually enters in physical problems

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Consider a 2×2 system of linear ODE on a punctured Riemann sphere $C_{0,n}$

$$\boxed{\frac{d}{dz}\Psi(z) = A(z)\Psi(z)}$$

where $z \in C_{0,n}$ and the matrices $A(z) \in sl(2,\mathbb{C})$, $\Psi(z) \in GL(2,\mathbb{C})$

$$A(z) = \sum_{\nu=1}^{n} \frac{A^{(\nu)}(z)}{(z - z_\nu)^{r_\nu + 1}} \quad \text{with} \quad A^{(\nu)}(z) = \sum_{i=0}^{r_\nu} A_i^{(\nu)} (z - z_\nu)^{r_\nu - i}$$

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As for φ_z in the Hitchin system, a singularity $z = z_\nu$ can be

- regular $(r_\nu = 0) \implies \Psi$ has branch point, monodromy matrices M_ν
- irregular $(r_\nu \geqslant 1) \implies \Psi$ has essential singularity, Stokes matrices $S_k^{(\nu)}$

How do Painlevé equations arise in this setting?

- Choice of $A(z) \Longrightarrow$ fixed M_ν / $S_k^{(\nu)}$ matrices
- Choice of M_ν / $S_k^{(\nu)} \Longrightarrow$ many-parameters family of $A(z;\vec{t})$

Isomonodromic deformations of $A(z;\vec{t})$: deformations $\vec{t}$ preserving $M_\nu / S_k^{(\nu)}$

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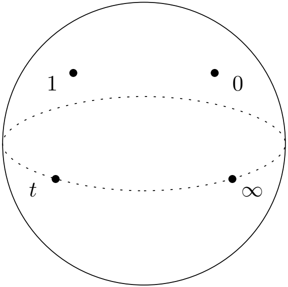
Example: $A(z;t)$ with 4 regular singularities at $0, 1, t, \infty$



Isomonodromic deformation: variations of t which do not change M_ν

1-parameter case: deformation of $A(z;t)$ by t is isomonodromic if

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Matrix $B(z;t)$ constrained: we have an overdetermined system

$$\begin{cases} \partial_z \Psi(z;t) = A(z;t)\Psi(z;t) \\ \partial_t \Psi(z;t) = B(z;t)\Psi(z;t) \end{cases} \qquad A(z;t), B(z;t) \text{ Lax pair}$$

$\implies$ need compatibility condition $\Psi_{zt} = \Psi_{tz}$ equivalent to

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This condition gives a set of equations which reduce to a Painlevé equation;
choice of singularities in $C_{0,n}$ $\Longleftrightarrow$ choice of Painlevé equation

Introduce an overall scale κ by rescaling parameters:

$$\boxed{(\kappa\partial_z - A(z;t))\,\Psi(z;t) = 0}$$

Equivalence Painlevé $\kappa\partial_z - A(z;t) \iff$ Hitchin oper $\hbar\partial_z - \varphi_z$

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Painlevé isomonodromic problem	$\mathcal{N}=2$ **theory Hitchin system**
Riemann surface $C_{0,n}$	compactification surface $C_{0,n}$
Painlevé connection $\kappa\partial_z - A$	oper $\hbar\partial_z - \varphi_z$
Painlevé time t	gauge coupling Λ
Painlevé σ-function (Hamiltonian)	Coulomb branch parameter u
Painlevé free parameters	masses $\mathcal{N}=2$ theory
curve $y^2 = \frac{1}{2}\mathrm{Tr}A^2$	Seiberg-Witten curve $y^2 = \frac{1}{2}\mathrm{Tr}\varphi_z^2$

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$\boxed{\text{Painlevé } \tau\text{-function}}$	$\boxed{\text{(dual) Nekrasov partition function}}$

Painlevé τ-functions and CFT

Consider PVI ($C_{0,4}$); the solution to the associated system of linear ODE

$$\frac{d\Psi}{dz} = A(z;t)\Psi$$

can be expressed in terms of a $c = 1$ free fermion CFT [Jimbo-Miwa-Sato]:

$$\boxed{\Psi_{\alpha\beta}(z_0; z) = (z - z_0)\frac{\langle \overline{\psi}_\beta(z_0)\psi_\alpha(z)\mathcal{O}_0\mathcal{O}_t\mathcal{O}_1\mathcal{O}_\infty\rangle}{\langle \mathcal{O}_0\mathcal{O}_t\mathcal{O}_1\mathcal{O}_\infty\rangle}}$$

with free fermions $\psi_\alpha, \overline{\psi}_\beta$ $(\alpha, \beta = 1, 2)$ and "twist" fields $\mathcal{O}_{z_\nu}$

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Idea: matrix $A(z;t)$ determines monodromies M_ν of Ψ

$$\Psi \to \Psi M_\nu \quad \text{when} \quad z - z_\nu \to e^{2\pi i}(z - z_\nu)$$

Having a correlator with such M_ν requires the Operator Product Expansion

$$\text{OPE}: \quad \overline{\psi}_\beta(z_0)\psi(z) \sim \frac{\delta_{\alpha\beta}}{z - z_0} \quad , \quad \psi_\alpha(z)\mathcal{O}_{z_\nu} \sim \mathcal{O}^{(0)}_{z_\nu,\alpha}(z - z_\nu)^{A_0^{(\nu)}}$$

Do fields $\mathcal{O}_{z_\nu}$ with such properties exist?

Yes: twist fields can be realized in terms of fermion bilinears

$$\mathcal{O}_{z_\nu} = \exp\left(\int_{\mathcal{C}_\nu} \mathrm{Tr}[A_0^{(\nu)} J(y)]dy\right) \;, \qquad J_{\beta\alpha} = \overline{\psi}_\beta \psi_\alpha \;\; \widehat{sl}(2)_1 \text{ current}$$

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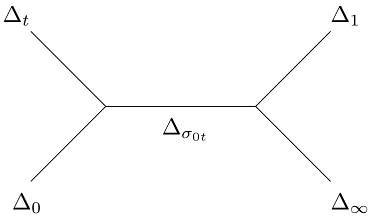
Bonus: from this construction the PVI τ-function can be realized as

$$\boxed{\tau_{\mathrm{VI}}(t) = \langle \mathcal{O}_0 \mathcal{O}_t \mathcal{O}_1 \mathcal{O}_\infty \rangle}$$

Need to consider the $c = 1$ four-point conformal block



Remark: subtlety with dimension $\Delta_{\sigma_{0t}}$

- $\psi_\alpha(z)$: monodromy $M_t M_0$ around fields in the OPE $\mathcal{O}_0\mathcal{O}_t$
- Let $e^{\pm 2\pi i \sigma_{0t}}$ be eigenvalues of $M_t M_0$: σ_{0t} defined up to $n \in \mathbb{Z}$
- Expect infinitely many primaries in OPE $\mathcal{O}_0\mathcal{O}_t$ with $\Delta_{\sigma_{0t}} = (\sigma_{0t} + n)^2$

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We can now use AGT (instanton) representation of the 4-point correlators

$\Longrightarrow$ express $\tau_{\rm VI}(t)$ in terms of $SU(2)$ $N_F = 4$ instantons

Explicitly, obtain $\tau_{\rm VI}(t)$ as a $t \sim 0$ series expansion [Gamayun-Iorgov-Lisovyy]

$$\boxed{\tau_{\rm VI}(t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n \eta_{0t}}\, Z_{\rm Nek}^{N_F=4}(\vec{\theta}, \sigma_{0t} + n; t)}$$

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Dictionary with $\mathcal{N}=2$ $SU(2)$ $N_F=4$ theory:

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- time variable $t \sim 0 \longleftrightarrow$ instanton parameter Λ/ϵ (weak coupling)

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Following coalescence diagram, obtain small t series for $\tau_{\rm V}$, $\tau_{{\rm III}_1}$, $\tau_{{\rm III}_2}$, $\tau_{{\rm III}_3}$:

$$\boxed{\tau_{\rm V}, \tau_{{\rm III}_1}, \tau_{{\rm III}_2}, \tau_{{\rm III}_3}\ (t\sim 0)} \quad \Longleftrightarrow \quad \boxed{SU(2)\ \ N_F = 3,2,1,0\ \ (\Lambda \sim 0)}$$

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$\Longrightarrow$ extract “Z_{Nek}” at strong coupling ($\Lambda \to \infty$) and $\epsilon_1 + \epsilon_2 = 0$:

$$\boxed{\tau_{\mathrm{IV}}, \tau_{\mathrm{II}}, \tau_{\mathrm{I}} \ (t \to \infty)} \quad \Longleftrightarrow \quad \boxed{\text{Argyres-Douglas } H_2, H_1, H_0 \ (\Lambda \to \infty)}$$

Correspondence Stokes sectors $\Longleftrightarrow$ strong coupling points

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We can analyse τ-functions at $t \to \infty$ (in a fixed Stokes sector)

- Use τ-functions $t \to \infty$ asymptotic behaviour [Jimbo] + make ansatz

$$\tau(t) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta} \text{“} Z_{\mathrm{Nek}}(\vec{\theta}, \sigma + n; t) \text{”}$$

- Plug ansatz into τ-Painlevé equation and determine $t \to \infty$ expansion

$\Longrightarrow$ extract “Z_{Nek}” at strong coupling $(\Lambda \to \infty)$ and $\epsilon_1 + \epsilon_2 = 0$:

$$\boxed{\tau_{\mathrm{IV}}, \tau_{\mathrm{II}}, \tau_{\mathrm{I}} \ (t \to \infty)} \quad \Longleftrightarrow \quad \boxed{\text{Argyres-Douglas } H_2, H_1, H_0 \ (\Lambda \to \infty)}$$

Correspondence Stokes sectors $\Longleftrightarrow$ strong coupling points

Proceed in the same way for $\tau_{\mathrm{V}}, \tau_{\mathrm{III}_1}, \tau_{\mathrm{III}_2}, \tau_{\mathrm{III}_3}$

$$\boxed{\tau_{\mathrm{V}}, \tau_{\mathrm{III}_1}, \tau_{\mathrm{III}_2}, \tau_{\mathrm{III}_3} \ (t \to \infty)} \quad \Longleftrightarrow \quad \boxed{SU(2) \ \ N_F = 3, 2, 1, 0 \ (\Lambda \to \infty)}$$

Example: τ-function for PII / Argyres-Douglas H_1

Expansion 1: $\arg t = \pi, \pm\frac{\pi}{3}$

$$\tau_{II}(t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n\eta}\mathcal{G}(\sigma+n,s), \qquad 4t^3 = 9s^2$$

$$\mathcal{G}(\sigma,s) = e^{-\frac{3s^2}{32}+\sigma s} s^{-\frac{1}{12}-\frac{\sigma^2}{2}+\frac{\theta^2}{3}} 12^{-\frac{\sigma^2}{2}} G(1+\sigma)\left[1+\sum_{k=1}^{\infty}\frac{D_k(\sigma)}{s^k}\right]$$

$$D_1(\sigma) = \frac{\sigma(34\sigma^2-96\theta^2+31)}{72}\,, \quad D_2(\sigma) = \dots$$

Expansion 2: $\arg t = 0, \pm\frac{2\pi}{3}$

$$\tau_{II}(t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n\eta}\mathcal{G}(\sigma+n,s), \qquad 8t^3 = 9s^2$$

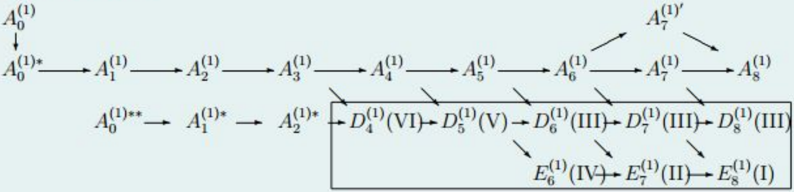
$$\mathcal{G}(\sigma,s) = e^{i\sigma s+\frac{i\pi\sigma^2}{2}} s^{-\sigma^2+\frac{\theta^2}{12}} 6^{-\sigma^2} G(1+\sigma+\frac{\theta}{2})G(1+\sigma-\frac{\theta}{2})\left[1+\sum_{k=1}^{\infty}\frac{D_k(\sigma)}{s^k}\right]$$

$$D_1(\sigma) = -\frac{i\sigma(68\sigma^2-9\theta^2+2)}{36}\,, \quad D_2(\sigma) = \dots$$

Perspectives

Differential Painlevé equations are part of a more general story [Sakai]

The list of D



0) $A_0^{(1)}$ is a **boss** of Painlevé equations (**elliptic Painlevé**)

1) The A-series give **q-difference Painlevé equations**

2) $A_0^{(1)**}$, $A_1^{(1)*}$, $A_2^{(1)*}$: **difference Painlevé** (**Boalch**) corresponding to

$$[(111111, 222, 33)], \quad [(1111), (1111), (22)], \quad [(111), (111), (111)].$$

3) Eight in the box are **Painlevé differential equations**.

0),1) rank 1 6d $\mathcal{N} = (1, 0)$ and 5d $\mathcal{N} = 1$ $SU(2)$ $N_F = 0, \ldots, 7$

2) rank 1 4d Minahan-Nemeschansky (from a talk by [Ohyama])

Conclusions

Hitchin systems 4d $\mathcal{N} = 2 \Longleftrightarrow$ Painlevé isomonodromic problems:

- oper connection $\hbar\partial_z - \varphi_z \Longleftrightarrow$ Painlevé connection $\kappa\partial_z - A$
- "dual" instanton partition function $\Longleftrightarrow$ Painlevé τ-function

Time evolution $\tau(t)$ determined by isomonodromic deformation condition; used to extract strongly-coupled "$Z_{\text{Nek}}^{c=1}$" of Argyres-Douglas and SQCD

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Future directions:

- Argyres-Douglas at superconformal point?
- flat sections: "dual" *ramified* partition function?
- $c \neq 1$: quantization of Painlevé Hamiltonian?
- q-difference Painlevé: relation to non-perturbative topological strings?

Thanks!