Wronskians, Duality and FZZT - Cardy Branes

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Basic References:

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2. 2D Gravity and Random Matrices/ Phys. Report.
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3. Rational Theories of 2d Gravity from the Two-matrix Model. **hep-th/9303093**
   by J. M. Daul, V. A. Kazakov, & I. K Kostov.
4. Exact v.s. Semi classical Target Space of the minimal String **hep-th/0408039**
   by J. Maldacena, N. Seiberg, G. Moore, and D. Shih

Outline of this Talk

- Rush intro to the matrix model (m.m.)
- orthogonal polynomials in the m.m.
- Double-scaling limit of the m.m & the Baker-Akhiezer system (B.A.)
- From resolvent to the generalized wronskians.
- Kac table associated with the F-C branes
- Duality symmetries of the Kac table
- Summary

Rush intro to the (one) Matrix Model

Define the partition function (p.f.)

$$Z_N(\beta, \vec{g}) := \int_{\mathcal{H}} d^{N^2}M \, e^{-\beta \operatorname{tr} V(M;\vec{g})}$$

$\mathcal{H}$: ensemble of matrices (e.g. Hermitian ensemble. $M^\dagger = M$)

β : overall normalization ($\Rightarrow$ inverse Planck's constant)

V : matrix potential ($\Rightarrow$ usually of polynomial form)

$\vec{g}$: coupling constants ($\Rightarrow$ coefficients of the polynomial)

A perturbative expansion of the p.f w.r.t. $\vec{g}$

$\Rightarrow$ A perturbative expansion of the p.f w.r.t. $\vec{g}$ $\Rightarrow$ sum over 2-dim random surfaces

$\Rightarrow$ fish net $\Rightarrow$ sum over 2-dim random surfaces

$\Rightarrow$ **An exactly solvable model for 2d gravity!**

Eigenvalue (fermionic) picture of the M.M.

"Polar" decomposition of the matrix variable M.

$MV = V\Lambda$ (diagonalization) $\Leftrightarrow$ $M = V\Lambda V^{\dagger}$

degrees of freedom $\quad N + 2C_2^N = N^2 = N + 2C_2^N$

$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$ eigenvalue matrix

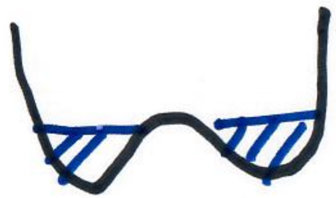
$$\Rightarrow \int_{V\in U(N)} d^{N^2}M = d^N\Lambda\,[\Delta(\vec{\lambda})]^2, \quad \Delta(\vec{\lambda}) = \prod_{i<j}(\lambda_i - \lambda_j)$$

Vandermonde deter..

$$\Rightarrow Z_N = \int d^N\Lambda\, e^{-\beta\sum_k V(\lambda_k) + \underbrace{2\sum_{i<j}\ln(\lambda_i-\lambda_j)}_{\text{repulsive interaction}}}$$

condensation of eigenvalues $\Rightarrow$ CUT(s)

e.g. $V \sim -M^2 + M^4 \Rightarrow$



Why DPE is of interest to the string theorists?

- Matrix model partition function for the hermitian matrix ensemble

$$Z = \exp^{-F} = \int dM \exp^{-trV(M)} = \int \Pi_{k=1}^{N} d\lambda_k \Delta^2(\lambda) \exp^{-\Sigma_k V(\lambda_k)}.$$

- Define orthogonal Polynomials $P_n(\lambda)$ w.r.t. weight $\exp^{-V(\lambda)}$

$$\int_{-\infty}^{\infty} d\lambda \exp^{-V(\lambda)} P_m(\lambda) P_n(\lambda) = h_n \delta_{mn},$$

where

$$P_n(\lambda) = \lambda^n + ..., \text{ and } \Delta(\lambda) = det\left(P_{j-1}(\lambda_i)\right).$$

- Partition function interms of normalization constant h_n

$$Z = \exp^{-F} = N!\left(\Pi_{k=0}^{N-1} h_k\right) = N! h_0^N \Pi_{k=0}^{N-1}\left(\frac{h_k}{h_{k-1}}\right)^{N-k}$$

From the Orthogonal Polynomials to the DPE

- Three terms recursion relation among the orthogonal polynomials

$$\lambda P_n = P_{n+1} + r_n P_{n-1}.$$

- Projection of $\lambda \cdot P_n$ onto P_{n-1}

$$h_n = \int \exp^{-V} P_n \left(\lambda P_{n-1}\right) = \int \exp^{-V} \left(\lambda P_n\right) P_{n-1} = r_n h_{n-1}.$$

- Projection of the derivative of P_n onto P_{n-1}

$$\begin{aligned} n h_{n-1} &= \int \exp^{-V} P_n' P_{n-1} = -\int P_n \frac{d}{d\lambda}\left(\exp^{-V} P_{n-1}\right) \\ &= \int \exp^{-V} P_n V' P_{n-1}. \end{aligned}$$

- The simplest case: quartic potential $\Rightarrow$ dP-I equation

$$V(\lambda) = \frac{N}{2g}(\lambda^2 + \lambda^4) \Rightarrow \frac{gn}{N} = r_n + 2r_n \cdot (r_{n+1} + r_n + r_{n-1}).$$

Resolvent operator in the M.M.

Macroscopic loop operator:

$$\hat{\phi}(x) \sim \frac{1}{N}\int^{x} \mathrm{tr}\left(\frac{1}{\zeta - X}\right) d\zeta \sim \frac{1}{N}\,\mathrm{tr}\ln(x - X)$$

$$e^{\hat{\phi}(x)} \sim \det(x - X) \sim \prod_{k=1}^{N} (x - \lambda_k)$$

$$\partial_x \hat{\phi}(x) \sim \mathrm{tr}\frac{1}{x - X} = \sum_{k=0}^{\infty} x^{-(k+1)}\,\mathrm{tr}(X^k)$$

$$Q(x) \sim \langle \partial_x \hat{\phi}(x) \rangle \Rightarrow \underbrace{F(x, Q) = 0}_{\text{Spectral curve!}}$$

$\langle e^{\hat{\phi}} \rangle$ = FZZT ~~partition function~~.

Orthogonal Polynomial technique in the M.M.

Explicit computation of the mult-point correlator among FZZT branes (A. Morozov. hep-th/9303139)

$$\langle \det(x-M) \rangle = P_N(x)$$

$$\left\langle \prod_{i=1}^{n} \det(x_i - M) \right\rangle = \frac{\det\left(P_{N+i-1}(x_j)\right)}{\Delta(\vec{x})} \quad n \geq 2$$

$$= \frac{1}{\Delta(\vec{x})} \begin{vmatrix} P_{N+n-1}(x_1) & P_{N+n-1}(x_2) & \cdots \\ P_{N+n-2}(x_1) & P_{N+n-2}(x_2) & \cdots \\ \vdots & & \\ P_N(x_1) & P_N(x_2) & \cdots \end{vmatrix}$$

Double Scaling Limit (D.S.L.)

$$e^{-\partial_n} \to \frac{\partial}{\partial t}, \quad P_n(x) \to \psi(t;\zeta) \Rightarrow \textbf{wronskian!}$$

Double scaling limit of the M.M.

$N \to \infty \quad g \to g_c$

"lattice" spacing $a \sim N^{-*} \to 0$

$$\left\{ \begin{array}{l} t \sim a^{-\#} \left(\dfrac{g_c - g}{g_c} \right) \\ \zeta \sim \left(\dfrac{g_c - g \frac{n}{N}}{g_c} \right) a^{-\Delta} \end{array} \right.$$

moduli of the FZZT brane

deg of the orthogonal polynomial.

$\Rightarrow$ Baker-Akhiezer system & the Lax Pair

$$\zeta \psi(t;\zeta) = P(t;\partial)\, \psi(t;\zeta)$$

$$\frac{\partial}{\partial \zeta} \psi(t,\zeta) = Q(t:\partial)\, \psi(t:\zeta)$$

$$P(t:\partial) = 2^{p-1}\partial^{p} + \sum_{n=2}^{p} u_n(t)\, \partial^{p-n} \ldots$$

Spectral Curves of the FZZT-Cardy Branes

elemental FZZT brane: $\psi^{(j)}(\zeta) = e^{\hat{\varphi}^{(j)}(\zeta)}$

resolvent: $Q^{(j)}(\zeta) = \langle \partial_\zeta \hat{\varphi}^{(j)}(\zeta) \rangle$

Spectral curve $F(\zeta, Q) = \prod_{j=1}^{p} \left(Q - Q^{(j)}(\zeta) \right) = 0$

$$= \frac{1}{2^{p-1}} \left[T_p\left(\frac{Q}{\#} \right) - T_q\left(\frac{\zeta}{\sqrt{\mu}} \right) \right]$$

$T_p(x)$: p-th Chebyshev polynomial of 1st kind.

$T_n(\cos\theta) = \cos(n\theta)$.

Seiberg-Shih relation

$$|(r,s)\rangle_{\zeta} = \sum_{\substack{k=-(r-1) \\ \text{step}2}}^{(r-1)} \sum_{\substack{l=-(s-1) \\ \text{step}2}}^{(s-1)} |(1,1)\rangle_{\zeta_{k,l}} + \text{BRST exact.}$$

where $\zeta_{kl} := \sqrt{\mu} \cosh\left[p\left[\tau + \pi i\left(\frac{k}{p} + \frac{l}{q}\right)\right]\right]$

$\zeta := \sqrt{\mu} \cosh(p\tau)$ $\quad \mu$ = cosmological const.

(r, s) FZZT- Cardy branes

= superposition of the analytical continued principle brane $|(1,1)\rangle_{\zeta}$

From the P-Q equation of the
elementary FZZT - Cardy Branes

To the $\mathfrak{B}$-Q equation of the
generalized Wronskian

P equation for $\psi^{(j)}$: $\zeta\psi^{(j)} = (4\partial^3 + u_2\partial + u_3)\psi^{(j)}$ $\quad j=1.2.3$

$$W_\Phi^{(2.1)} := \begin{vmatrix} \partial\psi^{(a)} & \partial\psi^{(b)} \\ \psi^{(a)} & \psi^{(b)} \end{vmatrix} \qquad a.b = 1.2.3$$

$$\partial W_\Phi^{(2,1)} = [\partial^2\psi^{(a)}]\,\psi^{(b)} - \psi^{(a)}[\partial^2\psi^{(b)}]$$

$$\partial^2 W_\Phi^{(2,1)} = -\frac{u_2}{4} W_\Phi^{(2,1)} + [\partial^2\psi^{(a)}][\partial\psi^{(b)}] - [\partial\psi^{(a)}][\partial^2\psi^{(b)}]$$

$$\partial^3 W_\Phi^{(2,1)} = -\frac{u_2}{4}[\partial W_\Phi^{(1,2)}] - \left(\frac{\zeta + u_2' - u_3}{4}\right) W_\Phi^{(1,2)}$$

Introducing the generalized Wronskian

$$W_{\phi}^{(2,1)} := \begin{vmatrix} \partial\psi^{(a)} & \partial\psi^{(b)} \\ \psi^{(a)} & \psi^{(b)} \end{vmatrix}, \quad W_{\Box}^{(2,1)} := \begin{vmatrix} \partial^2\psi^{(1)} & \partial^2\psi^{(2)} \\ \psi^{(1)} & \psi^{(2)} \end{vmatrix}$$

$$W_{\boxminus}^{(2,1)} := \begin{vmatrix} \partial^2\psi^{(a)} & \partial^2\psi^{(b)} \\ \partial\psi^{(a)} & \partial\psi^{(b)} \end{vmatrix}$$

$$\partial W_{\phi}^{(2,1)} = W_{\Box}^{(2,1)}$$

$$\partial W_{\Box}^{(2,1)} = -\frac{u_2}{4} W_{\phi}^{(2,1)} + W_{\boxminus}^{(2,1)}$$

$$\partial W_{\boxminus}^{(2,1)} = -\frac{\zeta - u_3}{4} W_{\phi}^{(2,1)}$$

Definition of the $\mathcal{B}^{(r,s)}$ equation

$$\vec{W}^{(2,1)} := \begin{pmatrix} W_{\phi}^{(2,1)} \\ W_{\square}^{(2,1)} \\ W_{\boxminus}^{(2,1)} \end{pmatrix}. \quad \partial \begin{pmatrix} W_{\phi}^{(2,1)} \\ W_{\square}^{(2,1)} \\ W_{\boxminus}^{(2,1)} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -\frac{u_2}{4} & 0 & 1 \\ -\frac{\zeta - u_3}{4} & 0 & 0 \end{pmatrix}}_{} \begin{pmatrix} W_{\phi}^{(2,1)} \\ W_{\square}^{(2,1)} \\ W_{\boxminus}^{(2,1)} \end{pmatrix}.$$

$$\partial \vec{W}^{(2,1)} = \mathcal{B}^{(2,1)} \vec{W}^{(2,1)}$$

Similarily, we can deduce

$$\frac{\partial}{\partial \zeta} \vec{W}^{(2,1)} = \mathcal{Q}^{(2,1)} \vec{W}^{(2,1)} \Rightarrow$$

isomonodromy system for the generalized Wronskian!

Isomonodromy Douglas equation for the generalized Wronskian

= integrability condition of the $\mathcal{B}$-$\mathcal{Q}$ system

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$$[\partial_t - \mathcal{B} \, . \, \partial_\zeta - \mathcal{Q}] = 0$$

$$\Leftrightarrow \; [P(t,\partial) \, . \, Q(t,\partial)] = 1 \quad \text{string equation}$$

↳ Painleve equation and their generalization

Same equation for $u(t)$

BUT different spectral curve !!

$$0 = \det [Q \, \mathbb{I} - \mathcal{Q}(t,\zeta)]$$

Kac table and the duality symmetry

for $(p,q) = (3,4)$ M.M.

(1,3)	(2,3)
(1,2)	(2,2)
(1,1)	(2,1)

(r,s) FZZT-Cardy Brane

$1 \leq r \leq p-1, \quad 1 \leq s \leq q-1$

$$qr - ps > 0$$

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(1) Reflection relation

$$W^{(1,1)} = W^{(2,3)}, \quad W^{(1,2)} = W^{(2,2)}, \quad W^{(1,3)} = W^{(2,1)}$$

(2) Charge Conjugation of the isomonodromy system

There exists a matrix C such that

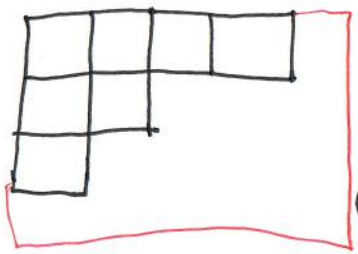
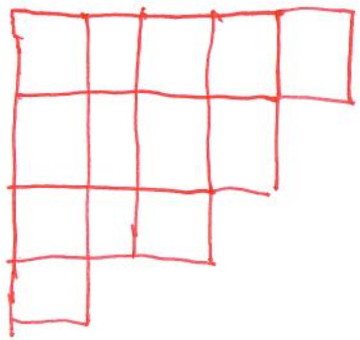
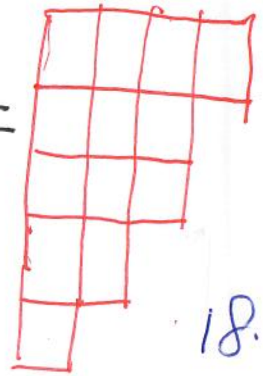
$$\mathcal{B}^{(p-r,1)} = -C[\mathcal{B}^{(r,1)}]^{T} C^{-1}$$

$$\mathcal{Q}^{(p-r,1)} = -C[\mathcal{Q}^{(r,1)}]^{T} C^{-1}$$

$$C[W_{\lambda}^{(r,1)}] = W_{c(\lambda)}^{(p-r,1)}$$

where $c(\lambda) = (-1)^{|\lambda|} \cdot \lambda^{cc}$

$|\lambda|$: size of the Young diagram

$\lambda =$  $\Rightarrow \lambda^{\perp} =$  $\Rightarrow \lambda^{cc} := (\lambda^{\perp})^{\vee} =$ 

complement

complementary conjugation

Summary

resolvent $\det(x-M)$ $\xRightarrow[\text{polynomials}]{\text{orthogonal}}$ determinant $\det(P_{N-i+1}(x_j))$ $\xRightarrow[\text{scaling}]{\text{doubling}}$ wronskian $W_{\Phi}^{[j]}=\det(\Psi^{(i)}(x_j))$ $\xRightarrow[\text{fermion}]{\text{twist}}$ genealized wronskian $W_{\lambda}^{[j]}(X)$

To do list :

1. complete the Kac table!
2. quantum correction of the spectral curve.
3. Stoke's phenomena associated with $\{\mathcal{B}.\,\mathcal{Q}\}$ isomonodromy system!

Thanks for Your Patience!