

Integrable Hopf twists and marginal deformations

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Based on arXiv: 0811.3755 (with T. Månsson) and 1602.08061 (with H. Dlamini)







$\mathcal{N} = 4$ Super–Yang–Mills

- Fundamental role in the study of gauge and string theory
- Unique four–dimensional gauge theory with maximal global supersymmetry (16 supercharges)
- Extremely interesting properties:
 - ▶ Perturbative Finiteness
 - ▶ Nonperturbative Finiteness
 - ⇒ 4d Superconformal Field Theory
 - ▶ AdS/CFT Correspondence
 - ▶ Planar Integrability
 - ▶ Planar Amplitudes
- But is it the unique theory with these features?
- Are there theories which share only *some* of these features?
- How does the knowledge accumulated for $\mathcal{N} = 4$ SYM help to understand more realistic theories?

Marginal Deformations of $\mathcal{N} = 4$ SYM

- Look for theories as close as possible to $\mathcal{N} = 4$ SYM
- Preserve conformal invariance $\Rightarrow$ Marginal Deformations
- Focus on superpotential deformations $\Rightarrow \mathcal{N} = 1$ SUSY
- Write the $\mathcal{N} = 4$ SYM action in $\mathcal{N} = 1$ superspace

$$\mathcal{L} = \int d^4\theta \text{Tr} e^{gV} \bar{\Phi}_i e^{-gV} \Phi^i + \left(\int d^2\theta \mathcal{W} + \int d^2\bar{\theta} \bar{\mathcal{W}} \right) + \dots$$

- Chiral Superfields $\Phi^i = \phi^i + \theta^\alpha \psi_\alpha^i + \theta^2 F^i$, $i = 1, 2, 3$
- $\mathcal{N} = 4$ superpotential:

$$\mathcal{W} = g \text{Tr} \Phi^1 [\Phi^2, \Phi^3] = \frac{g}{3} \epsilon_{ijk} \text{Tr} \Phi^i \Phi^j \Phi^k$$

- Most general classically marginal deformation:

$$\delta \mathcal{W} = h_{ijk} \text{Tr} \Phi^i \Phi^j \Phi^k,$$

where h_{ijk} is a symmetric tensor

Exactly Marginal Deformations

- Not all of these deformations are *exactly marginal*
- Perturbative approaches [Parkes, West '84],[Jones,Mezincescu '84]
- Non-perturbative proof by Leigh and Strassler ('95) using the NSVZ β function
- The *Leigh–Strassler* superpotential:

$$\mathcal{W}_{LS} = \kappa \text{Tr} \left(\Phi^1 [\Phi^2, \Phi^3]_q + \frac{h}{3} \left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3 \right) \right)$$

- q -commutator $[X, Y]_q = XY - qYX$
- Recover $\mathcal{N} = 4$ SYM for $q = 1$, $h = 0$, $\kappa = g$
- Finite if $f(g, \kappa, q, h) = 0$, where f unknown in general
- 1-loop finiteness condition

$$2g^2 = \kappa \bar{\kappa} \left[\frac{2}{N^2} (1+q)(1+\bar{q}) + \left(1 - \frac{4}{N^2} \right) (1 + q\bar{q} + h\bar{h}) \right]$$

- An interesting case: $q = e^{i\beta}$, $h = 0$ (“Real β deformation”)

$\mathcal{N} = 4$ SYM vs. Leigh–Strassler?

- How do the LS theories compare with $\mathcal{N} = 4$ SYM?

	$\mathcal{N} = 4$ SYM	Leigh–Strassler
Conformally Invariant	✓	✓
AdS/CFT dual	✓	real β [Lunin, Maldacena'05]
Planar Integrability	✓	basically real β^*
Planar Amplitudes	✓	real β

*More soon

- What makes the real β deformation so special?
- Take a closer look at the symmetries
- Work at the level of the classical lagrangian

Symmetries: $\mathcal{N} = 4$ SYM

- 4d Superconformal group: $\text{PSU}(2, 2|4)$
- Focus on the R-symmetry subgroup $\text{SU}(4) \sim \text{SO}(6)$
- In $\mathcal{N} = 1$ superspace notation, the $\mathcal{N} = 4$ theory has manifest $\text{SU}(3) \times \text{U}(1)_R$ symmetry

$$\mathcal{W} = g \text{Tr} \Phi^1 [\Phi^2, \Phi^3] = \frac{g}{3} \epsilon_{ijk} \text{Tr} \Phi^i \Phi^j \Phi^k$$

- ϵ_{ijk} is the invariant tensor of $\text{SU}(3)$

$$\epsilon_{ijk} U^i_l U^j_m U^k_n = (\det U) \epsilon_{lmn} = \epsilon_{lmn}$$

- Transforming $\Phi^i \rightarrow U^i_j \Phi^j$ leaves the superpotential invariant

Symmetries: Leigh–Strassler

- The generic LS deformation breaks $SU(3)$ to a discrete subgroup

$$\mathcal{W}_{LS} = \kappa \text{Tr} \left(\Phi^1 [\Phi^2, \Phi^3]_q + \frac{h}{3} ((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3) \right)$$

- This superpotential has the following $\mathbb{Z}_3$ symmetries:

$$\mathbb{Z}_3^A : \Phi^1 \rightarrow \Phi^2, \quad \Phi^2 \rightarrow \Phi^3, \quad \Phi^3 \rightarrow \Phi^1$$

$$\mathbb{Z}_3^B : \Phi^1 \rightarrow \omega \Phi^1, \quad \Phi^2 \rightarrow \omega^2 \Phi^2, \quad \Phi^3 \rightarrow \Phi^3 \quad (\omega^3 = 1)$$

- Together with a third $\mathbb{Z}_3$ within $U(1)_R$ ($\Phi^i \rightarrow \omega \Phi^i$), they form a trihedral group known as Δ_{27} [Aharony et al. '02]
- For real β the symmetry group is enhanced to $U(1)^3$
- Is this all?

q-deforming SU(3)

- In 0811.3755 (with T. Månsson) it was claimed that the global SU(3) is not actually broken
- Rather, it is *deformed* to a quantum group.

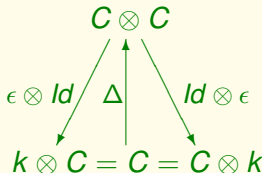
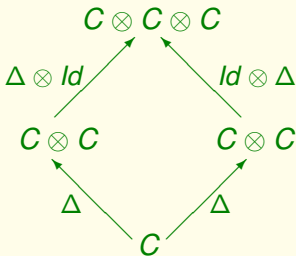
$$\Phi^i \rightarrow T^i_j \Phi^j \quad \text{where} \quad T = \begin{pmatrix} t^1_1 & t^1_2 & t^1_3 \\ t^2_1 & t^2_2 & t^2_3 \\ t^3_1 & t^3_2 & t^3_3 \end{pmatrix}$$

is a symmetry if the components of T satisfy:

$$\begin{aligned} \text{(a)} \quad & t^a_c t^{a+1}_c - q t^{a+1}_c t^a_c + h t^{a-1}_c t^{a-1}_c = h \left(t^a_{c+1} t^{a+1}_{c-1} - \bar{q} t^a_{c-1} t^{a+1}_{c+1} + \bar{h} t^a_c t^{a+1}_c \right) \\ \text{(b)} \quad & q[t^{a+1}_{c+1}, t^a_c] = -q^2 t^{a+1}_c t^a_{c+1} + h q t^{a-1}_c t^{a-1}_{c+1} + h t^{a-1}_{c+1} t^{a-1}_c + t^a_{c+1} t^{a+1}_c \\ \text{(c)} \quad & -q t^{a+1}_c t^a_{c+1} + \bar{q} t^a_{c+1} t^{a+1}_c = \bar{h} t^a_{c-1} t^{a+1}_{c-1} - h t^{a-1}_c t^{a-1}_{c+1} \\ \text{(d)} \quad & h(t^{a+1}_{c+1} t^{a-1}_{c-1} - \bar{q} t^a_{c-1} t^{a+1}_{c+1}) = \bar{h}(t^{a+1}_c t^{a-1}_c - q t^{a-1}_c t^{a+1}_c) \end{aligned}$$

Quantum Groups

- What are quantum groups? Some definitions
- Recall an *algebra* $(\mathcal{C}, +, \cdot, \eta; k)$ is a vector space together with a product $\cdot : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$ and a unit map $\eta : k \rightarrow \mathcal{C}$
- A *coalgebra* $(\mathcal{C}, +, \Delta, \epsilon; k)$ is instead equipped with a coproduct $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ and a counit $\epsilon : \mathcal{C} \rightarrow k$



- A *bialgebra* $(H, +, \cdot, \eta, \Delta, \epsilon; k)$ is both an algebra and a coalgebra in a compatible way

Hopf Algebras

- A Hopf Algebra is a bialgebra equipped with an antipode $S : \mathcal{C} \rightarrow \mathcal{C}$

$$\cdot (S \otimes \text{id}) \circ \Delta = \cdot (\text{id} \otimes S) \circ \Delta = \eta \circ \epsilon \cdot$$

- One can construct Hopf algebras using the RTT relations

[Faddeev, Reshetikhin, Takhtajan]

$$R_{ab}^{ik} t_j^a t_l^b = t_b^k t_a^i R_{j\ l}^{ab},$$

where $R : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ is called an R -matrix.

- If the R -matrix satisfies the Quantum Yang–Baxter Equation (QYBE) the Hopf algebra is called *quasitriangular*

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad \left(R_{sr}^{ij} R_{lp}^{sk} R_{mn}^{rp} = R_{sp}^{jk} R_{rn}^{ip} R_{lm}^{rs} \right)$$

- The YBE guarantees the resulting algebra is not too trivial
- R controls Δ non-commutativity: $\tau \circ \Delta(h) = R(\Delta(h))R^{-1}$
- We call a quasitriangular Hopf algebra a *quantum group*

R-matrix for the general LS deformation

- $H_{kl}^{ij} = E_{kl} F^{nij}$ is the 1-loop spin chain Hamiltonian [Roiban '03]

$$H_{l,l+1} = \frac{1}{1 + q\bar{q} + h\bar{h}} \begin{pmatrix} h\bar{h} & 0 & 0 & 0 & 0 & \bar{h} & 0 & -\bar{h}q & 0 \\ 0 & 1 & 0 & -q & 0 & 0 & 0 & 0 & h \\ 0 & 0 & q\bar{q} & 0 & -h\bar{q} & 0 & -\bar{q} & 0 & 0 \\ 0 & -\bar{q} & 0 & q\bar{q} & 0 & 0 & 0 & 0 & -h\bar{q} \\ 0 & 0 & -\bar{h}q & 0 & h\bar{h} & 0 & \bar{h} & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 1 & 0 & -q & 0 \\ 0 & 0 & -q & 0 & h & 0 & 1 & 0 & 0 \\ -h\bar{q} & 0 & 0 & 0 & 0 & -\bar{q} & 0 & q\bar{q} & 0 \\ 0 & \bar{h} & 0 & -\bar{h}q & 0 & 0 & 0 & 0 & h\bar{h} \end{pmatrix}$$

- Define $\hat{R}_{kl}^{ij} = \delta_k^i \delta_l^j - H_{kl}^{ij} \implies R_{kl}^{ij} = \hat{R}_{kl}^{ij}$

$$R = \frac{1}{2d^2} \begin{pmatrix} 1+q\bar{q}-h\bar{h} & 0 & 0 & 0 & 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 \\ 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 & 0 & 0 & 0 & 2h\bar{q} \\ 0 & 0 & 2q & 0 & -2h & 0 & q\bar{q}+h\bar{h}-1 & 0 & 0 \\ 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 & 0 & 0 & 0 & -2h \\ 0 & 0 & 2\bar{h}q & 0 & 1+q\bar{q}-h\bar{h} & 0 & -2\bar{h} & 0 & 0 \\ 2h\bar{q} & 0 & 0 & 0 & 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 \\ 0 & 0 & 1-q\bar{q}+h\bar{h} & 0 & 2h\bar{q} & 0 & 2\bar{q} & 0 & 0 \\ -2h & 0 & 0 & 0 & 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 \\ 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 & 0 & 0 & 0 & 1+q\bar{q}-h\bar{h} \end{pmatrix}$$

$$(2d^2 = 1 + q\bar{q} + h\bar{h})$$

General LS deformation

- R does not satisfy YBE in general
- $\text{YBE} \iff$ known integrable deformations
- However, generically we do not have a quasitriangular Hopf Algebra. Is it still a Hopf Algebra?
- Showed that this is indeed the case by explicitly constructing the antipode
- Consistent up to cubic level
- Associativity will imply higher relations $\implies$ danger of trivialising the algebra
- Has not yet been analysed in detail
- Hope is to learn about the limits of integrability
- In this talk I'll take a different approach

Our goals

- We are claiming that the LS theories have more global symmetry than is naively visible
- Lie algebra $\rightarrow$ Hopf algebra
- How do we work with such symmetries?
- Is this quantum symmetry visible on the gravity side?
- If so, can it be turned into a solution-generating technique for AdS/CFT duals?
- First step: Focus on the integrable cases

Our goals

- We are claiming that the LS theories have more global symmetry than is naively visible
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- How do we work with such symmetries?
- Is this quantum symmetry visible on the gravity side?
- If so, can it be turned into a solution-generating technique for AdS/CFT duals?
- First step: Focus on the integrable cases
- Derive them as *twists* of the $\mathcal{N} = 4$ structure
- Drinfeld (Hopf) twist

$$\Delta_F(h) = F(\Delta(h))F^{-1} , \qquad R_F = F_{21} \cdot R \cdot F^{-1}$$

- Cocycle condition: $(1 \otimes F)(\text{id} \otimes \Delta)F = (F \otimes 1)(\Delta \otimes \text{id})F$
- For $\mathcal{N} = 4$ SYM, $R = I \otimes I$

The β -deformation

- Gauge theory has $U(1)^3$ symmetry
- $q = e^{i\beta}$, $h = 0$. For β real, the R -matrix is

$$R_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Satisfies YBE — quasitriangular case
- Dual background constructed by Lunin and Maldacena using TsT transformations
- Obtained as a twist of the $\mathcal{N} = 4$ case [Beisert, Roiban '05]
- Too simple. Would like a theory without the extra $U(1)$'s

The w -deformation

- Set $q = 1 + w, h = w$
- Equivalent to real- β case by unitary transformation
- We will not take advantage of this fact
- w -deformed R-matrix satisfies YBE

[Bundzik, Mansson '05]

$$R_w = \frac{1+w}{1+w+w^2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{w}{1+w} & 0 & w & 0 \\ 0 & 1 & 0 & -\frac{w}{1+w} & 0 & 0 & 0 & 0 & w \\ 0 & 0 & 1 & 0 & -\frac{w}{1+w} & 0 & w & 0 & 0 \\ 0 & w & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{w}{1+w} \\ 0 & 0 & w & 0 & 1 & 0 & -\frac{w}{1+w} & 0 & 0 \\ w & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{w}{1+w} & 0 \\ 0 & 0 & -\frac{w}{1+w} & 0 & w & 0 & 1 & 0 & 0 \\ -\frac{w}{1+w} & 0 & 0 & 0 & 0 & w & 0 & 1 & 0 \\ 0 & -\frac{w}{1+w} & 0 & w & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- More compactly:

$$R_w = \frac{1+w}{1+w+w^2} \left[I \otimes I + w U \otimes V - \frac{w}{1+w} V \otimes U \right].$$

with

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V := U^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad U^3 = I.$$

β -deformation from a twist

- The appropriate twist is simply

$$F_q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

- Can be expressed as

$$F_q = e^{i\frac{\beta}{2} H_1 \wedge H_2} , \quad \text{where } H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

- Depends on Cartan elements $\implies$ Abelian

w -deformation from a twist

- The twist is a bit more complicated

$$F_w = \tilde{C} \begin{pmatrix} 1+w & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 \\ 0 & 1+w & 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+w & 0 & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+w & 0 & 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 & 1+w & 0 & w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+w & 0 & w & 0 \\ 0 & 0 & w & 0 & 0 & 0 & 1+w & 0 & 0 \\ w & 0 & 0 & 0 & 0 & 0 & 0 & 1+w & 0 \\ 0 & w & 0 & 0 & 0 & 0 & 0 & 0 & 1+w \end{pmatrix}$$

- More compact form:

$$F_w = \tilde{C}[(1+w)I \otimes I + wV \otimes U].$$

- $[V, U] = 0 \implies$ Also abelian
- Inverse twist

$$F_w^{-1} = \frac{(1+w)^2}{\tilde{C}(1+2w)(1+w+w^2)} \left[I \otimes I - \frac{w}{1+w} V \otimes U + \frac{w^2}{(1+w)^2} U \otimes V \right].$$

Star product

- The twist can be used to define a star product
- Hopf algebra acts as derivation

$$g \triangleright (x \cdot y) = (g \triangleright x) \cdot y + x \cdot (g \triangleright y) .$$

- Compatibility of algebra and module product

$$\begin{aligned} g \triangleright (x \cdot y) &= m(\Delta(g) \triangleright [x \otimes y]) = m([g \otimes 1 + 1 \otimes g] \triangleright [x \otimes y]) \\ &= m((g \triangleright x) \otimes y + x \otimes (g \triangleright y)) = (g \triangleright x) \cdot y + x \cdot (g \triangleright y) \end{aligned}$$

- Twisting Δ also twists the module product

$$m_F(x \otimes y) = m(F^{-1} \triangleright x \otimes y) = (F_{(1)}^{-1} \triangleright x) \cdot (F_{(2)}^{-1} \triangleright y)$$

- Star product

$$x \star y = m_F(x \otimes y) .$$

Details of the star product

- Choose basis z^i , $i = 1, 2, 3$, use index notation

$$z^i \star z^j = (F^{-1})^j_{l}{}^i_{k} z^l z^k$$

- Quadratic products for w -deformation

$$z^1 \star z^2 = z^1 z^2 - \frac{w}{1+w} (z^3)^2 + \frac{w^2}{(1+w)^2} z^1 z^2 = \left(1 + \frac{w^2}{(1+w)^2}\right) z^1 z^2 - \frac{w}{1+w} (z^3)^2$$

$$z^2 \star z^1 = z^1 z^2 - \frac{w}{1+w} z^1 z^2 + \frac{w^2}{(1+w)^2} (z^3)^2 = \left(1 - \frac{w}{1+w}\right) z^1 z^2 + \frac{w^2}{(1+w)^2} (z^3)^2$$

$$z^3 \star z^3 = (z^3)^2 - \frac{w}{1+w} z^1 z^2 = \frac{w^2}{(1+w)^2} z^1 z^2 = -\frac{w}{1+w} \left(1 - \frac{w}{1+w}\right) z^1 z^2 + (z^3)^2$$

- Cubic products — careful with associativity!

$$(z^1 \star z^2) \star z^3 = (F^{-1})^{32}_{k'l'} (F^{-1})^{k'l}{}_{ni'} (F^{-1})^{j'i'}_{ml} z^l z^m z^n$$

$$z^1 \star (z^2 \star z^3) = (F^{-1})^{21}_{j'i'} (F^{-1})^{3i'}_{k'l} (F^{-1})^{k'l}{}_{nm} z^l z^m z^n .$$

- Integrable twist $\implies$ associative product

$$z^1 \star z^2 \star z^3 = (F^{-1})^{21}_{j'i'} (F^{-1})^{3i'}_{k'l} (F^{-1})^{k'l}{}_{nm} z^l z^m z^n .$$

More details of the star product

- By explicit computation

$$F_{12}^{-1} F_{13}^{-1} F_{23}^{-1} = \frac{1+w}{(1+2w)^2(1+w+w^2)} [(1+4w+5w^2+w^3)I \otimes I \otimes I \\ + w^2(1+w)(I \otimes U \otimes V + U \otimes V \otimes I + U \otimes I \otimes V + U \otimes U \otimes U + V \otimes V \otimes V) \\ - w(1+3w+3w^2)(I \otimes V \otimes U + V \otimes U \otimes I) - \frac{w(1+3w+2w^2-w^3)}{1+w} V \otimes I \otimes U]$$

- Apply to $\mathcal{N} = 4$ superfields

$$\Phi^1 \star \Phi^2 \star \Phi^3 + \Phi^2 \star \Phi^3 \star \Phi^1 + \Phi^3 \star \Phi^1 \star \Phi^2 - (1+w) (\Phi^1 \star \Phi^3 \star \Phi^2 + \Phi^3 \star \Phi^2 \star \Phi^1 + \Phi^2 \star \Phi^1 \star \Phi^3) \\ + w (\Phi^1 \star \Phi^1 \star \Phi^1 + \Phi^2 \star \Phi^2 \star \Phi^2 + \Phi^3 \star \Phi^3 \star \Phi^3) \\ = (1+2w) [\Phi^1 \Phi^2 \Phi^3 + \Phi^2 \Phi^3 \Phi^1 + \Phi^3 \Phi^1 \Phi^2 - \Phi^1 \Phi^3 \Phi^2 - \Phi^3 \Phi^2 \Phi^1 - \Phi^2 \Phi^1 \Phi^3]$$

- Take gauge theory trace

$$\kappa \text{Tr} \left(\Phi^1 \star \Phi^2 \star \Phi^3 - (1+w) \Phi^1 \star \Phi^3 \star \Phi^2 + \frac{w}{3} \left((\Phi^1)_\star^3 + (\Phi^2)_\star^3 + (\Phi^3)_\star^3 \right) \right) = \text{Tr} \left(\Phi^1 [\Phi^2, \Phi^3] \right)$$

Inverse star product

- Would like to generate the LS superpotential from the $\mathcal{N} = 4$ one

$$x * y = m(F \triangleright x \otimes y) \quad \left(\text{in indices: } z^i * z^j = F_{lk}^{ji} z^k z^l \right)$$

- Obtain:

$$\begin{aligned} & \phi^1 * \phi^2 * \phi^3 + \phi^2 * \phi^3 * \phi^1 + \phi^3 * \phi^1 * \phi^2 - \phi^1 * \phi^3 * \phi^2 - \phi^3 * \phi^2 * \phi^1 - \phi^2 * \phi^1 * \phi^3 \\ &= \frac{1}{1+2w} \left[\phi^1 \phi^2 \phi^3 + \phi^2 \phi^3 \phi^1 + \phi^3 \phi^1 \phi^2 - (1+w) \left[\phi^1 \phi^3 \phi^2 - \phi^3 \phi^2 \phi^1 - \phi^2 \phi^1 \phi^3 \right] \right. \\ & \quad \left. + w \left((\phi^1)^3 + (\phi^2)^3 + (\phi^3)^3 \right) \right] \end{aligned}$$

- Take the gauge theory trace

$$\begin{aligned} & \text{Tr} \left(\phi^1 * \phi^2 * \phi^3 - \phi^1 * \phi^3 * \phi^2 \right) \\ &= \frac{3}{1+2w} \text{Tr} \left(\phi^1 \phi^2 \phi^3 - (1+w) \phi^1 \phi^3 \phi^2 + \frac{w}{3} \left((\phi^1)^3 + (\phi^2)^3 + (\phi^3)^3 \right) \right) \end{aligned}$$

First-order star product

- Expand the full star product to first order

$$F_{k l}^{i j} = \delta_k^i \delta_l^j - \frac{i\beta}{2} r_{k l}^{i j}$$

- r is the classical R -matrix
- Deformed commutator

$$z^i * z^j - z^j * z^i = \Theta^{ij}$$

- Noncommutativity matrices

$$\Theta_{\beta}^{ij} = i\beta \begin{pmatrix} 0 & -z^1 z^2 & z^1 z^3 \\ z^1 z^2 & 0 & -z^2 z^3 \\ -z^1 z^3 & z^2 z^3 & 0 \end{pmatrix}$$

[Kulaxizi '06]

$$\Theta_w^{ij} = w \begin{pmatrix} 0 & (z^3)^2 - z^1 z^2 & z^1 z^3 - (z^2)^2 \\ z^1 z^2 - (z^3)^2 & 0 & (z^1)^2 - z^2 z^3 \\ (z^2)^2 - z^1 z^3 & z^2 z^3 - (z^1)^2 & 0 \end{pmatrix}$$

So far...

- Understood how to obtain the Hopf algebra structure via a twist
- The twist defines an all-orders star product
- Associative for the integrable cases
- Expanding to first order reproduces known results
- How about the gravity side?
- Can we see the quantum symmetry?
- A convenient framework is that of generalised complex geometry

Generalised geometry

[Hitchin '02][Gualtieri '04][Grana, Minasian, Petrini, Tomasiello, Zabzine, Zaffaroni...]

- Generalises and unifies complex and symplectic geometry
- Metric and B-field treated on equal footing
- Very natural from string theory perspective
- Consider structures on $T \oplus T^*$
- 6d space transverse to D3 branes $\longrightarrow$ 12d $T \oplus T^*$

$$X = \{\iota_I, dz^I\} = \{\iota_1, \bar{\iota}_1, \iota_2, \dots, d\bar{z}^2, dz^3, d\bar{z}^3\}$$

- Gamma matrices: $\Gamma_I = \iota_I, \Gamma_{I+6} = dz^I$

Generalised complex structures

- Holomorphic and Kähler forms

$$\Omega = dz^1 \wedge dz^2 \wedge dz^3 \quad \text{and} \quad J = \frac{i}{2} \sum_{i=1}^3 dz^i \wedge d\bar{z}^{\bar{i}}$$

- Define pure spinors (here for flat space)

$$\Phi_-^0 = \Omega = dz^1 \wedge dz^2 \wedge dz^3 \quad \text{and}$$

$$\Phi_+^0 = e^{-iJ} = 1 + \frac{1}{2} \sum_i dz^i \wedge d\bar{z}^{\bar{i}} + \frac{1}{4} \sum_i dz^i \wedge d\bar{z}^{\bar{i}} \wedge dz^{i+1} \wedge d\bar{z}^{\bar{i}+1} + \frac{1}{8} dz^1 \wedge d\bar{z}^{\bar{1}} \wedge dz^2 \wedge d\bar{z}^{\bar{2}} \wedge dz^3 \wedge d\bar{z}^{\bar{3}}$$

- For IIB $\mathcal{N} = 2$ backgrounds

$$d\Phi_{\pm} = 0$$

$$\langle \Phi_-, X\Phi_+ \rangle = 0, \quad \langle \bar{\Phi}_-, X\Phi_+ \rangle = 0 \quad \langle \bar{\Phi}_+, \Phi_+ \rangle = \langle \bar{\Phi}_-, \Phi_- \rangle$$

$$(\text{Mukai pairing } \langle A, B \rangle = \sum (-1)^{[n/2]} A_n \wedge B_{6-n})$$

- Generalised complex structures

$$\mathcal{J}_{\pm MN} = \langle \bar{\Phi}_{\pm}, \Gamma_{MN} \Phi_{\pm} \rangle \quad \mathcal{J}^2 = -1$$

Generalised metric

- Constructed from the two generalised complex structures

$$\mathcal{G}_{MN} = -\mathcal{J}_{+ML}\mathcal{J}_{-}^L{}_N.$$

- Can just read off the NS-NS fields

$$\mathcal{G}_N^M = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}, \quad e^{2\Phi} = \sqrt{|\det g|}$$

- Example: Flat space

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}, \quad \text{where} \quad g = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Twisting the pure spinors

- Use gauge-theory star product to twist flat-space pure spinors

$$dz^I \wedge_* dz^J = \left(1 - \frac{i}{2} \Theta^{KL} \iota_K \wedge \iota_L\right) dz^I \wedge dz^J = dz^I \wedge dz^J - i \Theta^{IJ}$$

- Equivalent to a bivector deformation (for our case!)

$$\Phi_{\pm} = e^{\Theta^{IJ} \iota_I \wedge \iota_J} \Phi_{\pm}^0$$

For real- β : [Halmagyi-Tomasiello '07]

- Claim: This leads to the deformed background!

$$\Phi_-^* = dz^1 \wedge_* dz^2 \wedge_* dz^3 \text{ and}$$

$$\begin{aligned} \Phi_+^* = & 1 + \frac{1}{2} \sum_{i=1}^3 dz^i \wedge_* d\bar{z}^i + \frac{1}{4} \sum_{i=1}^3 dz^i \wedge_* d\bar{z}^i \wedge_* dz^{i+1} \wedge_* d\bar{z}^{i+1} \\ & + \frac{1}{8} dz^1 \wedge_* d\bar{z}^1 \wedge_* dz^2 \wedge_* d\bar{z}^2 \wedge_* dz^3 \wedge_* d\bar{z}^3 \end{aligned}$$

- Geometric manifestation of the CFT quantum symmetry

Results

- Using the real- β $*$ -product we obtain the pure spinors of the $\mathcal{N} = 2$ LM precursor background [Lunin-Maldacena '05]
- The w -deformed $*$ -product leads to

$$\Phi_-^w = dz^1 \wedge dz^2 \wedge dz^3 - iw[(z^2 z^3 - (z^1)^2)dz^1 + (z^3 z^1 - (z^2)^2)dz^2 + (z^1 z^2 - (z^3)^2)dz^3] ,$$

$$\begin{aligned} \Phi_+^w = & \Phi_+^0 + \frac{iw}{4} \left[(z^3 \bar{z}^3 - z^2 \bar{z}^2)dz^1 \wedge d\bar{z}^1 + ((\bar{z}^3)^2 - \bar{z}^1 \bar{z}^2)dz^1 \wedge dz^2 + (z^1 \bar{z}^3 - \bar{z}^2 z^3)dz^1 \wedge d\bar{z}^2 \right. \\ & \left. + (z^2 \bar{z}^3 - z^1 \bar{z}^2)dz^1 \wedge d\bar{z}^3 + (z^1 z^2 - (z^3)^2)d\bar{z}^1 \wedge d\bar{z}^2 + \text{cyclic} \right] \\ & + \frac{iw}{8} \left[(z^1 \bar{z}^1 - z^2 \bar{z}^2)dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 - (\bar{z}^2 \bar{z}^3 - (\bar{z}^1)^2)dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge dz^3 \right. \\ & \left. + (\bar{z}^1 z^3 - z^1 \bar{z}^2)dz^1 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 - (z^1 \bar{z}^3 - \bar{z}^1 z^2)dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^3 \right. \\ & \left. + (z^2 z^3 - (z^1)^2)dz^1 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 + \text{cyclic} \right] \end{aligned}$$

- Compatible pair of pure spinors $\implies \mathcal{N} = 2$ background!
- Put D3-brane at origin, near-horizon limit $\implies \text{AdS}_5 \times \tilde{S}^5$
- Purely NS-NS. IIB equations of motion $\implies$ R-R fields.

The w -deformed background

$$e^{2\Phi} = G . \quad g_{\mu\nu} = G \tilde{g}_{\mu\nu} ,$$

$$G^{-1} = 1 + w^2 R^4 (1 - s_\alpha^2 c_\alpha^2 - s_\alpha^4 s_\theta^2 c_\theta^2 - 2c_\alpha s_\alpha^3 c_\theta s_\theta^2 C_2 - 2s_\alpha^2 c_\alpha^2 s_\theta c_\theta C_1 - 2c_\alpha s_\alpha^3 c_\theta^2 s_\theta C_3)$$

$$\tilde{g}_{\alpha\alpha} = R^2 + \frac{w^2 R^6}{2} \left[1 - 2c_\alpha^2 + 2c_\alpha^2 s_\alpha^2 c_\theta^2 s_\theta^2 + 2c_\alpha^4 + (4c_\alpha^2 - 3)c_\alpha s_\alpha c_\theta s_\theta^2 C_2 - c_\alpha s_\alpha c_\theta^3 C_{3,1} \right. \\ \left. - c_\alpha s_\alpha s_\theta^3 C_{2,1} + (4c_\alpha^4 - 3c_\alpha^2 + 1)c_\theta s_\theta C_1 + (4c_\alpha^3 - 3c_\alpha)s_\alpha c_\theta^2 s_\theta C_3 \right] ,$$

$$\tilde{g}_{\alpha\theta} = \frac{w^2 R^6}{2} \left[c_\alpha s_\alpha^3 c_\theta s_\theta (c_\theta^2 - s_\theta^2) + s_\alpha^2 c_\alpha^2 s_\theta c_\theta^2 C_{3,1} + (2s_\alpha^2 s_\theta^2 - s_\theta^2 - s_\alpha^2)s_\alpha^2 c_\theta C_3 \right. \\ \left. - s_\alpha^2 c_\alpha^2 c_\theta s_\theta^2 C_{2,1} + c_\alpha^3 s_\alpha (1 - 2s_\theta^2)C_1 + (2s_\alpha^4 s_\theta^3 - s_\alpha^2 s_\theta^3 + s_\theta s_\alpha^2 c_\alpha^2)C_2 \right] ,$$

$$\tilde{g}_{\theta\theta} = R^2 \left[s_\alpha^2 + \frac{w^2 R^4}{2} \left[s_\alpha^2 - 2s_\alpha^4 s_\theta^2 + 2s_\alpha^4 s_\theta^4 + c_\alpha s_\alpha^3 s_\theta (s_\alpha^2 s_\theta^2 + s_\theta^2 - s_\alpha^2)C_3 - c_\alpha^3 s_\alpha^3 s_\theta c_\theta^3 C_{2,1} \right. \right. \\ \left. \left. - c_\alpha^3 s_\alpha^3 c_\theta s_\theta^2 C_{3,1} - c_\alpha s_\alpha^3 c_\theta (s_\theta^2 s_\alpha^2 - c_\theta^2)C_2 + (s_\alpha^6 + s_\alpha^4 - 2s_\alpha^2)c_\theta s_\theta C_1 - s_\alpha^6 c_\theta s_\theta C_{3,2} \right] \right] ,$$

$$\tilde{g}_{\alpha\phi_1} = \frac{w^2 R^6}{2} \left[c_\alpha^2 s_\alpha^2 s_\theta c_\theta^2 S_3 + c_\alpha^2 s_\alpha^2 s_\theta^3 S_{2,1} + c_\alpha (1 + c_\alpha^2)s_\alpha c_\theta s_\theta S_1 + c_\alpha^2 s_\alpha^2 c_\theta^3 S_{3,1} + c_\alpha^2 s_\alpha^2 c_\theta s_\theta^2 S_2 \right] ,$$

$$\tilde{g}_{\alpha\phi_2} = \frac{w^2 R^6}{2} \left[-s_\alpha^4 c_\theta s_\theta^4 S_2 - c_\alpha s_\alpha^3 c_\theta s_\theta^3 S_1 + s_\alpha^2 c_\alpha^2 s_\theta^3 S_{2,1} - s_\alpha^2 c_\alpha^2 c_\theta s_\theta^2 S_2 + c_\alpha^3 s_\alpha c_\theta s_\theta S_1 - s_\alpha^4 c_\theta^4 s_\theta S_3 \right] ,$$

$$\tilde{g}_{\alpha\phi_3} = \frac{w^2 R^6}{2} \left[-s_\alpha^4 c_\theta s_\theta^4 S_2 - s_\alpha^4 c_\theta^4 s_\theta S_3 - c_\alpha s_\alpha^3 c_\theta^3 s_\theta S_1 - s_\alpha^2 c_\alpha^2 c_\theta^2 s_\theta S_3 + c_\alpha^3 s_\alpha c_\theta s_\theta S_1 + s_\alpha^2 c_\alpha^2 c_\theta^3 S_{3,1} \right] ,$$

The w -deformed background

$$\tilde{g}_{\theta\phi_1} = \frac{w^2 R^6}{2} \left[c_\alpha^3 s_\alpha^3 s_\theta^3 S_{3,1} + c_\alpha s_\alpha^3 s_\theta (c_\alpha^2 c_\theta^2 - s_\alpha^2 s_\theta^2) S_2 - c_\alpha s_\alpha^3 c_\theta s_\theta^2 S_3 + c_\alpha^3 s_\alpha^3 c_\theta s_\theta^2 S_{2,1} \right. \\ \left. - 2c_\alpha^4 s_\alpha^2 s_\theta^2 S_1 - c_\alpha^3 s_\alpha^3 s_\theta S_{3,1} + c_\alpha s_\alpha^5 c_\theta S_3 + c_\alpha^4 s_\alpha^2 S_1 \right] ,$$

$$\tilde{g}_{\theta\phi_2} = \frac{w^2 R^6}{2} \left[-c_\alpha s_\alpha^3 s_\theta (2s_\alpha^2 s_\theta^2 + c_\theta^2 - s_\alpha^2 s_\theta^4) S_2 - s_\alpha^2 s_\theta^2 (c_\alpha^2 - s_\alpha^2 s_\theta^2 c_\alpha^2) S_1 \right. \\ \left. - c_\alpha s_\alpha^5 c_\theta s_\theta^4 S_3 + c_\alpha^3 s_\alpha^3 c_\theta s_\theta^2 S_{2,1} - s_\alpha^6 s_\theta^2 c_\theta^2 S_{3,2} \right] ,$$

$$\tilde{g}_{\theta\phi_3} = \frac{w^2 R^6}{2} \left[c_\alpha s_\alpha^5 c_\theta^4 s_\theta S_2 - c_\alpha^3 s_\alpha^3 c_\theta^2 s_\theta S_{3,1} - c_\alpha s_\alpha^5 c_\theta^5 S_3 - s_\alpha^4 c_\alpha^2 c_\theta^4 S_1 - s_\alpha^6 c_\theta^2 s_\theta^2 S_{3,2} \right. \\ \left. - c_\alpha s_\alpha^3 (1 - 2s_\alpha^2) c_\theta^3 S_3 + s_\alpha^2 c_\alpha^2 c_\theta^2 S_1 + c_\alpha s_\alpha^3 c_\theta S_3 \right] ,$$

$$\tilde{g}_{\phi_1\phi_1} = R^2 \left[c_\alpha^2 + \frac{w^2 R^4}{2} \left[-c_\alpha^3 s_\alpha^3 c_\theta^2 s_\theta C_3 + c_\alpha^3 s_\alpha^3 s_\theta^3 C_{2,1} - (c_\alpha^2 s_\alpha^4 + 6c_\alpha^4 s_\alpha^2) c_\theta s_\theta C_1 \right. \right. \\ \left. \left. + c_\alpha^3 s_\alpha^3 c_\theta^3 C_{3,1} - c_\alpha^3 s_\alpha^3 c_\theta s_\theta^2 C_2 + c_\alpha^2 s_\alpha^4 + c_\alpha^4 s_\alpha^2 + 2c_\alpha^6 \right] \right] ,$$

$$\tilde{g}_{\phi_1\phi_2} = -\frac{w^2 R^6}{2} \left[c_\alpha^4 s_\alpha^2 s_\theta^2 + c_\alpha^2 s_\alpha^4 s_\theta^4 - 2c_\alpha^2 s_\alpha^4 c_\theta^2 s_\theta^2 + (c_\alpha s_\alpha^5 c_\theta s_\theta^4 - c_\alpha^3 s_\alpha^3 s_\theta^2 c_\theta) C_2 \right. \\ \left. + (c_\alpha^4 s_\alpha^2 c_\theta s_\theta - c_\alpha^2 s_\alpha^4 c_\theta s_\theta^3) C_1 - c_\alpha^3 s_\alpha^3 s_\theta^3 C_{2,1} + c_\alpha s_\alpha^5 c_\theta^4 s_\theta C_3 \right] ,$$

$$\tilde{g}_{\phi_1\phi_3} = -\frac{w^2 R^6}{2} \left[c_\alpha s_\alpha^5 c_\theta s_\theta^4 C_2 - 2c_\alpha^2 s_\alpha^4 c_\theta^2 s_\theta^2 + c_\alpha s_\alpha^5 c_\theta^4 s_\theta C_3 - c_\alpha^2 s_\alpha^4 c_\theta^3 s_\theta C_1 - c_\alpha^3 s_\alpha^3 c_\theta^2 s_\theta C_3 \right. \\ \left. + c_\alpha^4 s_\alpha^2 c_\theta s_\theta C_1 + c_\alpha^2 s_\alpha^4 c_\theta^4 - c_\alpha^3 s_\alpha^3 c_\theta^3 C_{3,1} + c_\alpha^4 s_\alpha^2 c_\theta^2 \right] ,$$

$$\tilde{g}_{\phi_2\phi_2} = R^2 \left[s_\alpha^2 s_\theta^2 + \frac{w^2 R^4}{2} \left[s_\alpha^2 s_\theta^2 (1 - s_\alpha^2 s_\theta^2 + 2s_\alpha^4 s_\theta^4) + s_\alpha^6 c_\theta^3 s_\theta^3 C_{3,2} - c_\alpha s_\alpha^5 s_\theta^3 c_\theta^2 C_3 \right. \right. \\ \left. \left. - c_\alpha^2 s_\alpha^4 c_\theta s_\theta^3 C_1 + c_\alpha^3 s_\alpha^3 s_\theta^3 C_{2,1} - c_\alpha s_\alpha^3 c_\theta s_\theta^2 (6s_\alpha^2 s_\theta^2 + s_\alpha^2 c_\theta^2 + c_\alpha^2) C_2 \right] \right] ,$$

The w -deformed background

$$\tilde{g}_{\phi_2\phi_3} = \frac{w^2 R^6}{2} \left[2c_\alpha^2 s_\alpha^4 c_\theta^2 s_\theta^2 - s_\alpha^6 c_\theta^2 s_\theta^2 + c_\alpha s_\alpha^5 c_\theta s_\theta^2 (c_\theta^2 - s_\theta^2) C_2 + s_\alpha^6 c_\theta^3 s_\theta^3 C_{3,2} \right. \\ \left. + c_\alpha s_\alpha^5 c_\theta^2 s_\theta (s_\theta^2 - c_\theta^2) C_3 - c_\alpha^4 s_\alpha^2 c_\theta s_\theta C_1 \right] ,$$

$$\tilde{g}_{\phi_3\phi_3} = R^2 \left[s_\alpha^2 c_\theta^2 + \frac{w^2 R^4}{2} \left[s_\alpha^2 c_\theta^2 (1 - s_\alpha^2 c_\theta^2 + 2s_\alpha^4 c_\theta^4) + s_\alpha^6 c_\theta^3 s_\theta^3 C_{3,2} - s_\alpha^4 c_\alpha^2 c_\theta^3 s_\theta C_1 \right. \right. \\ \left. \left. - 5c_\alpha s_\alpha^5 c_\theta^4 s_\theta C_3 - c_\alpha s_\alpha^3 c_\theta^2 s_\theta C_3 - c_\alpha s_\alpha^5 c_\theta^3 s_\theta^2 C_2 + c_\alpha^3 s_\alpha^3 c_\theta^3 C_{3,1} \right] \right] .$$

$$B_{\alpha\theta} = wR^4 G \left(s_\alpha^2 s_\theta S_2 + s_\alpha^2 c_\theta S_3 + c_\alpha s_\alpha S_1 \right) / 2 ,$$

$$B_{\alpha\phi_1} = wR^4 G \left(2c_\alpha s_\alpha s_\theta^2 - c_\alpha s_\alpha \right) / 2 ,$$

$$B_{\alpha\phi_2} = wR^4 G \left(s_\alpha^2 c_\theta s_\theta^2 C_2 + c_\alpha s_\alpha s_\theta^2 - s_\alpha^2 c_\theta^2 s_\theta C_3 - c_\alpha s_\alpha c_\theta s_\theta C_1 \right) / 2 ,$$

$$B_{\alpha\phi_3} = wR^4 G \left(s_\alpha^2 c_\theta s_\theta^2 C_2 + c_\alpha s_\alpha s_\theta c_\theta C_1 - s_\alpha^2 c_\theta^2 s_\theta C_3 - c_\alpha s_\alpha c_\theta^2 \right) / 2 ,$$

$$B_{\theta\phi_1} = wR^4 G \left(2c_\alpha^2 s_\alpha^2 c_\theta s_\theta - c_\alpha s_\alpha^3 s_\theta C_2 - c_\alpha s_\alpha^3 c_\theta C_3 + c_\alpha^2 s_\alpha^2 c_1 \right) / 2 ,$$

$$B_{\theta\phi_2} = wR^4 G \left(-c_\alpha s_\alpha^3 s_\theta^3 C_2 + c_\alpha s_\alpha^3 c_\theta s_\theta^2 C_3 + s_\alpha^2 c_\alpha^2 s_\theta^2 C_1 + s_\alpha^2 (1 - 2s_\alpha^2) c_\theta s_\theta \right) / 2 ,$$

$$B_{\theta\phi_3} = wR^4 G \left(c_\alpha s_\alpha^3 c_\theta^2 s_\theta C_2 + s_\alpha^2 (1 - 2s_\alpha^2) c_\theta s_\theta - c_\alpha s_\alpha^3 c_\theta^3 C_3 + s_\alpha^2 c_\alpha^2 c_\theta^2 C_1 \right) / 2 ,$$

$$B_{\phi_1\phi_2} = wR^4 G \left(-c_\alpha s_\alpha^3 c_\theta s_\theta^2 S_2 + c_\alpha s_\alpha^3 c_\theta^2 s_\theta S_3 - c_\alpha^2 s_\alpha^2 c_\theta s_\theta S_1 \right) / 2 ,$$

$$B_{\phi_1\phi_3} = wR^4 G \left(-c_\alpha s_\alpha^3 c_\theta s_\theta^2 S_2 + c_\alpha s_\alpha^3 c_\theta^2 s_\theta S_3 + c_\alpha^2 s_\alpha^2 c_\theta s_\theta S_1 \right) / 2 ,$$

$$B_{\phi_2\phi_3} = wR^4 G \left(-c_\alpha s_\alpha^3 c_\theta s_\theta^2 S_2 - c_\alpha s_\alpha^3 c_\theta^2 s_\theta S_3 + c_\alpha^2 s_\alpha^2 c_\theta s_\theta S_1 \right) / 2 .$$

+ RR fields

The w -deformed background

- Diffeomorphic to the LM background — only appears to be more complicated
- Deformed sphere, depending on angles $\alpha, \theta, \phi_1, \phi_2, \phi_3$
- Visualise by taking $\phi_i = 0$ and plotting the scalar curvature as a function of α, θ . Here for $wR = 0.3$

Summary

- Uncovered a quantum algebra structure underlying the general Leigh–Strassler deformation
- The $SU(3) \times U(1)$ R–symmetry of $\mathcal{N} = 4$ SYM is not broken, just q –deformed
- This quantum symmetry appears at the level of the classical Lagrangian
- Also a symmetry of the 1–loop spin chain Hamiltonian
- Focused on an integrable deformation: $q = 1 + w$, $h = w$
- Superficially more complicated than $q = e^{i\beta}$, $h = 0$, though unitarily equivalent
- Exhibited the star product leading to this deformation
- Showed how it can be applied on the supergravity side to obtain the dual $\mathcal{N} = 1$ background

Outlook

- ▶ Extend to non-integrable cases $\longrightarrow$ non-associativity?
- ▶ Could the appropriate structure be a quasi-Hopf algebra?
[Drinfel'd '89], [Mack, Schomerus '92]
- ▶ Construction of more general dual backgrounds
- ▶ Relation to other solution generating techniques (CYBE, λ -deformations)
- ▶ Higher loops? Non-planar level?
- ▶ Relation between quantum symmetry and perturbative finiteness ?