Nonlocal Electrodynamics in Graphene and Weyl Semi-Metals

Nov. 12, 2016, 9th String Workshop

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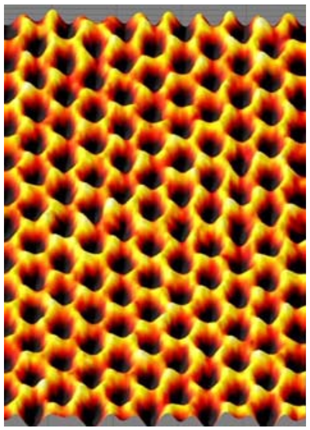
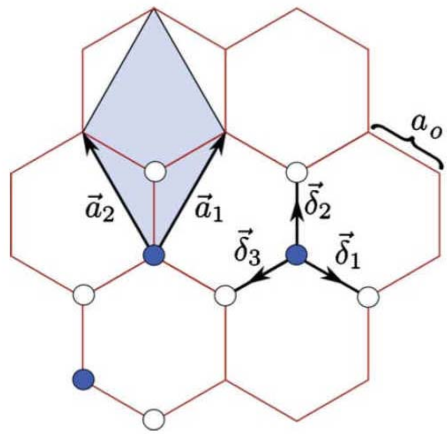
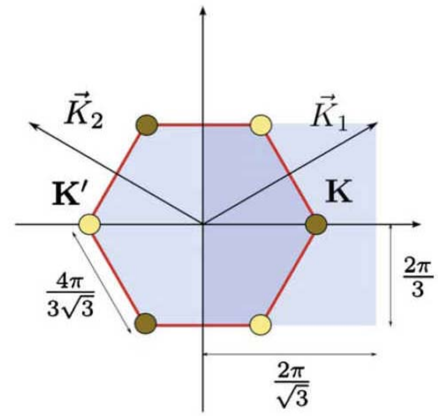
I. Introduction

Tight-binding model of graphene







$$\boldsymbol{a}_1 = \frac{a_0}{2}\left(\sqrt{3}, 3\right), \; \boldsymbol{a}_2 = \frac{a_0}{2}\left(-\sqrt{3}, 3\right), \; a_0 = 1.4 \text{ Å}; \tag{1}$$

$$\boldsymbol{K}_1 = \frac{2\pi}{3a_0}\left(\sqrt{3}, 1\right), \; \boldsymbol{K}_2 = \frac{2\pi}{3a_0}\left(-\sqrt{3}, 1\right); \tag{2}$$

$$\boldsymbol{\delta}_1 = \frac{a_0}{2}\left(\sqrt{3}, -1\right), \; \boldsymbol{\delta}_2 = a_0\left(0, 1\right), \; \boldsymbol{\delta}_3 = -\frac{a_0}{2}\left(\sqrt{3}, 1\right). \tag{3}$$

$$\hat{H} = \gamma \sum_{\text{sites}, \boldsymbol{r}, \alpha} \hat{a}_A^\dagger(\boldsymbol{r})\hat{a}_B(\boldsymbol{r} + \boldsymbol{\delta}_\alpha) + \text{c.c.}, \tag{4}$$

where $\gamma \approx 2.7$eV, the hopping energy.

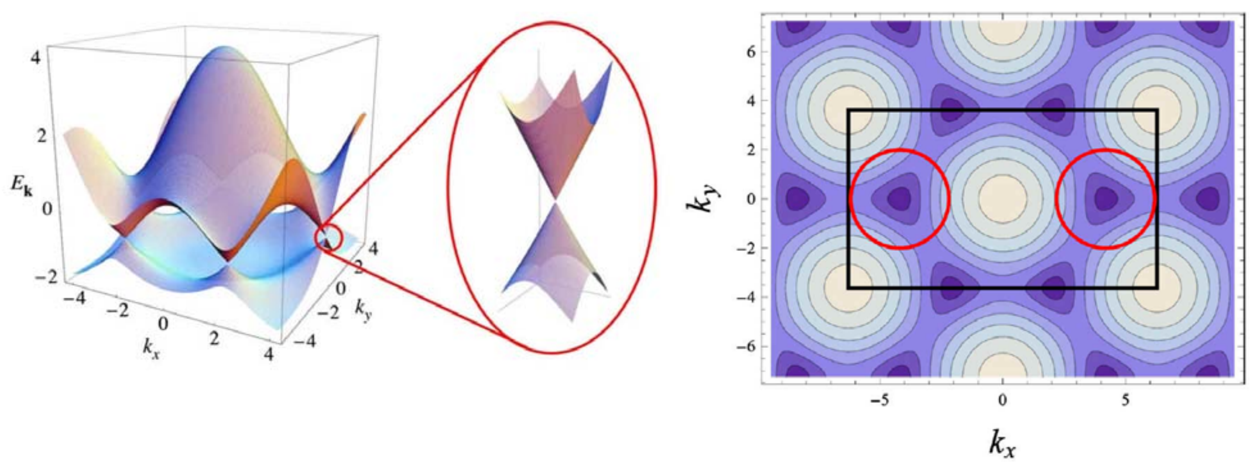
$$\text{In } \boldsymbol{k}-\text{space}, \ \hat{H} = \sum_{\text{BZ}} \left(\hat{a}^\dagger_A(\boldsymbol{k}), \hat{a}^\dagger_B(\boldsymbol{k})\right) \begin{pmatrix} 0 & h(\boldsymbol{k}) \\ h^*(\boldsymbol{k}) & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_A(\boldsymbol{k}) \\ \hat{a}_B(\boldsymbol{k}) \end{pmatrix}, \tag{5}$$

$$h(\boldsymbol{k}) = -\gamma \sum_\alpha e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}_\alpha} = -\gamma \left[e^{ik_y a_0} + 2e^{-ik_y a_0/2} \cos\left(\sqrt{3} k_x a_0/2\right)\right]. \tag{6}$$

$$\text{Dispersion relation}: \varepsilon(\boldsymbol{k}) = \pm |h(\boldsymbol{k})|. \tag{7}$$

$$\text{Dirac points(DP)}: h(\boldsymbol{k}) = 0 \Rightarrow \boldsymbol{K} = \frac{4\pi}{3\sqrt{3}a_0}(1,0), \ \boldsymbol{K}' = \frac{4\pi}{3\sqrt{3}a_0}(-1,0)$$

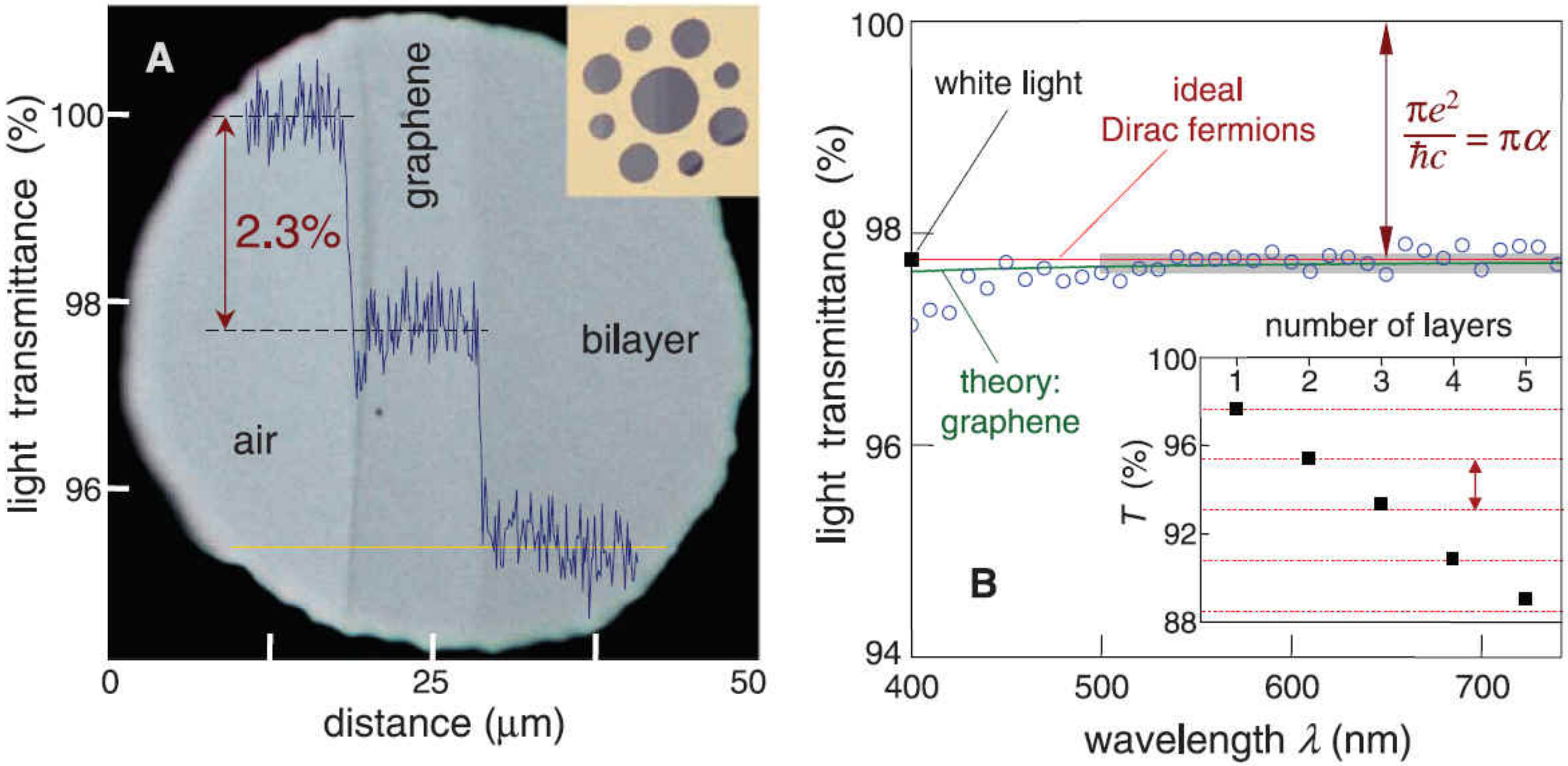
$$\text{Around DP}: \varepsilon(\boldsymbol{k}) = v_g|\boldsymbol{k}|, \quad v_g = \frac{3\gamma a_0}{2\hbar} \approx \frac{c}{300}. \tag{8}$$



Optical conductivity of graphene

$$\sigma(\omega) = \sigma_0 = \frac{1}{4}\frac{e^2}{\hbar}; \tag{9}$$

$$T = 1 - \frac{4\pi\sigma(\omega)}{c} = 1 - \pi\alpha_{\text{QED}} \approx 1 - 2.3\%. \tag{10}$$



Nair et al, Science 102 10451 (08).

Time scale for pure graphene: $t_\gamma = \hbar/\gamma = 0.24 \times 10^{-15}$(sec).

By breaking $\mathcal{T}, \mathcal{P}$ we may introduce a mass term m, m' near $\boldsymbol{K}, \boldsymbol{K}'$.

Bloch Hamiltonian : $\mathcal{H} = \boldsymbol{d}(\boldsymbol{k}) \cdot \boldsymbol{\sigma},\ \boldsymbol{d} = (d_x, d_y, d_z),\ \boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$(11)

The Berry flux is related to the solid angle subtended by $\hat{\boldsymbol{d}}$:

$$\mathcal{C} = \frac{1}{4\pi} \int d^2\boldsymbol{k} \left\{ \partial_{k_x} \hat{\boldsymbol{d}}) \times (\partial_{k_y} \hat{\boldsymbol{d}} \right\} \cdot \hat{\boldsymbol{d}}. \tag{12}$$

Each Dirac point with $\boldsymbol{d}(\boldsymbol{q}) = (v_x q_x, v_y q_y, m)$ contributes

$$\mathcal{C} = \frac{1}{2}\mathrm{sgn}(v_x v_y m). \tag{13}$$

- Semenoff mass term: $\mathcal{T}$ inv., $m' = m \Rightarrow \mathcal{C} = 0$.
- Haldane mass term: $\mathcal{P}$ inv., $m' = -m \Rightarrow \mathcal{C} = 1$.

There are (at least) two kind of insulators: NI and TI.

Zero Energy edge states

Whenever we have an interface between the two kind of insulators, there is a gapless edge mode. [Hasan and Kane (2010)]

$$\mathcal{H}_{\text{eff}} = v_F \left\{ \sigma_x(-i\partial_x) + \sigma_y(k_y) \right\} + \text{sgn}(x) m v_F^2 \sigma_z. \tag{14}$$

Right-moving zero energy edge:

$$\omega = v_F k_y,\ \psi = e^{ik_y y - m v_F |x|} \begin{pmatrix} 1 \\ i \end{pmatrix}. \tag{15}$$

- Tuning parameters so that there is a transition from NI to TI, there is always a gap closing point.
- Weyl Semi-Metal(WSM) lies exactly on the critical point.

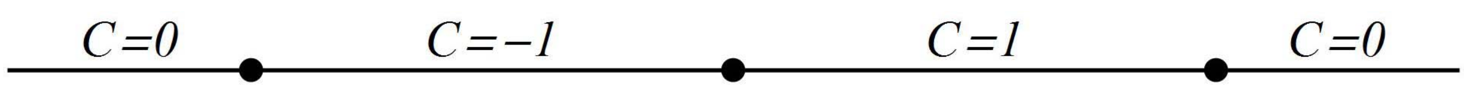
Lattice Model of TI

A simple lattice model of the Chern insulator:

$$\boldsymbol{d} = (\sin k_x, \sin k_y, -\mu - 2\cos \kappa_x - 2\cos k_y). \tag{16}$$

- Around $\boldsymbol{k} = (0,0), m = -\mu - 4, d_x + id_y = q_x + iq_y$.
- Around $\boldsymbol{k} = (\pi,\pi), m = -\mu + 4, d_x + id_y = -(q_x + iq_y)$.
- Around $\boldsymbol{k} = (\pi,0),(0,\pi), m = -\mu, d_x + id_y = \mp(q_x - iq_y)$.

$$\mathcal{C} = \frac{1}{2}\{\text{sgn}(-\mu - 4) + \text{sgn}(-\mu + 4) + 2\text{sgn}(\mu)\}\,. \tag{17}$$



Weyl Semi-Metal(WSM)

$$\boldsymbol{d} = (\sin k_x, \sin k_y, -\mu - 2\cos \kappa_x - 2\cos k_y - 2\cos k_z),\ -6 < \mu < 6; \quad (18)$$

$$\boldsymbol{d} = (\sin k_x, \sin k_y, \sin k_z). \quad (19)$$

$\sim$ Wislon fermion.

Weyl points:

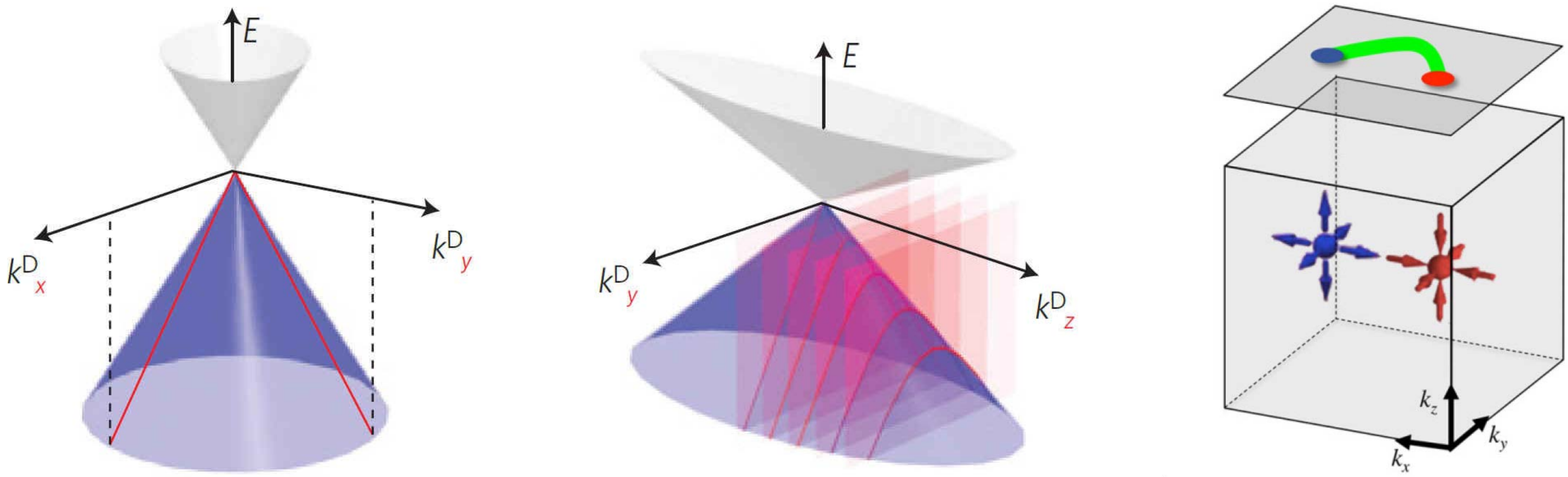
(i) $\boldsymbol{k} = (k_z)_a \hat{\boldsymbol{z}},\ (k_z)_a = \pm \cos^{-1}(-\mu/2 - 2)$.

(ii) $\boldsymbol{k} = \pi \hat{\boldsymbol{x}} + (k_z)_b \hat{\boldsymbol{z}}, \pi \hat{\boldsymbol{y}} + (k_z)_b \hat{\boldsymbol{z}},\ (k_z)_b = \pm \cos^{-1}(-\mu/2)$.

(iii) $\boldsymbol{k} = \pi \hat{\boldsymbol{x}} + \pi \hat{\boldsymbol{y}} + (k_z)_c \hat{\boldsymbol{z}},\ (k_z)_c = \pm \cos^{-1}(-\mu/2 + 2)$.

- Around a Weyl point, low-energy excitation $\sim$ a Weyl fermion.
- Weyl points always show up in L-R pairs(Nielsen-Ninomiya).
- Weyl points $\sim$ magnetic monopole in k space.
- Fermi arc linking Weyl points projects to the edge.

3D Dirac semi-metal(Cd_3As_2, Na_3Bi):



Liu et al, Nature Meterial (2014); Xu et al, Science (2015).

II. Diagramatic calculation

General formulation

Coulomb interaction: $L_I = -\frac{e^2}{2}\int d^3\mathbf{x}d^3\mathbf{y}\frac{\rho(x)\rho(y)}{|\mathbf{x}-\mathbf{y}|}$,

$\rho(x) = \psi^\dagger(x)\psi(x)$.

Propagators for electrons: $G_{\boldsymbol{k},\omega} = (i\omega - \boldsymbol{k}\cdot\boldsymbol{\sigma})^{-1}$.

"Propagators for photon": $g_{\boldsymbol{k}} = \frac{4\pi}{4\sin(k_i/2)\sin(k_i/2)}$.

Here, $(\widehat{\boldsymbol{k}})_i = \sin(k_i), \widehat{k} \equiv \sqrt{\widehat{\boldsymbol{k}}\cdot\widehat{\boldsymbol{k}}}$.

Matsubara(Euclidean) formalism is used.

One need to let $\omega \to -i\omega$ to get the physical conductivity.

The $\rho - \rho$ and $J - J$ correlators and Kubo formula:

$$\chi(\omega, \boldsymbol{k}) = \langle \rho(\omega, \boldsymbol{k}) \rho(-\omega, -\boldsymbol{k}) \rangle, \; K_{ij}(\omega, \boldsymbol{k}) = \langle J_i(\omega, \boldsymbol{k}) J_j(-\omega, -\boldsymbol{k}) \rangle, \tag{20}$$

$\omega^2 \chi(\omega, \boldsymbol{k}) = k_i k_j K_{ij}$. (gauge invariance)

$$\varepsilon(\omega, \boldsymbol{k}) = \frac{4\pi e^2}{k^2} \chi(\omega, \boldsymbol{k}), \; \sigma_{ij}(\omega, \boldsymbol{k}) = \frac{e^2 \left\{ K_{ij}(\omega, \boldsymbol{k}) - K_{ij}(0, \boldsymbol{k}) \right\}}{\omega}. \tag{21}$$

$$\sigma_{ij}(\omega, \boldsymbol{k}) = \sigma_T(\omega, \boldsymbol{k}) \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) + \sigma_L(\omega, \boldsymbol{k}) \frac{k_i k_j}{k^2}, \tag{22}$$

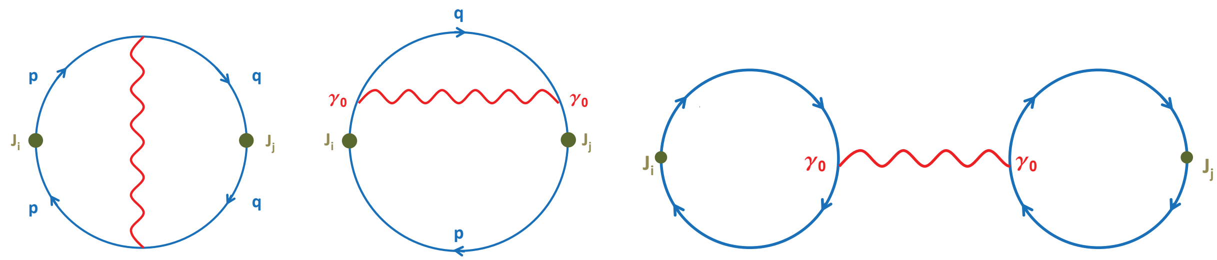
Define $\varepsilon(\omega) \equiv \lim_{k \to 0} \varepsilon(\omega, \boldsymbol{k}), \; \sigma_{ij}(\omega) \equiv \lim_{k \to 0} \sigma_{ij}(\omega, \boldsymbol{k})$.

To two – loop order : $\varepsilon(\omega, \boldsymbol{k}) = 1 + e^2 \varepsilon^{(1)}(\omega, \boldsymbol{k}) + e^4 \varepsilon^{(2)}(\omega, \boldsymbol{k}).$ (23)

$$\varepsilon^{(1)}(\omega, \boldsymbol{k}) = -g_{\boldsymbol{k}} \sum_{\boldsymbol{p},\nu} \text{Tr} \left(G_{\boldsymbol{p},\nu} G_{\boldsymbol{p}+\boldsymbol{k},\nu+\omega} \right); \tag{24}$$

$$\begin{aligned}\varepsilon^{(2)}(\omega, \boldsymbol{k}) = {} & g_{\boldsymbol{k}} \sum_{\boldsymbol{p},\nu,\boldsymbol{q},\rho} g_{\boldsymbol{q}-\boldsymbol{p}} \text{Tr} \left(G_{\boldsymbol{p}+\boldsymbol{k},\nu+\omega} G_{\boldsymbol{p},\nu} G_{\boldsymbol{q},\rho} G_{\boldsymbol{q}+\boldsymbol{k},\rho+\omega} \right) \\ & + 2 g_{\boldsymbol{k}} \sum_{\boldsymbol{p},\nu,\boldsymbol{q},\rho} g_{\boldsymbol{q}-\boldsymbol{p}} \text{Tr} \left(G_{\boldsymbol{p},\nu} G_{\boldsymbol{p}+\boldsymbol{k},\nu+\omega} G_{\boldsymbol{p},\nu} G_{\boldsymbol{q},\nu+\rho} \right) \\ & + \left\{ g_{\boldsymbol{k}} \sum_{\boldsymbol{p},\nu} \text{Tr} \left(G_{\boldsymbol{p},\nu} G_{\boldsymbol{p}+\boldsymbol{k},\nu+\omega} \right) \right\}^2 . \end{aligned} \tag{25}$$

They coresspond to the egg, self-enegy, and glasses diagrams, repectively.



To two-loop order:

$$\bar{K}_{ii}^{(1)}(\omega) = \frac{1}{3} \sum_{\boldsymbol{p},\nu,i} \cos^2 p_i \, \mathrm{Tr} \left(\sigma_i G_{\boldsymbol{p},\nu} \sigma_i G_{\boldsymbol{p},\nu+\omega} \right); \tag{26}$$

$$\begin{aligned} \bar{K}_{ii}^{(2)}(\omega, \boldsymbol{k}) = & \frac{1}{3} \sum_{\boldsymbol{p},\nu,\boldsymbol{q},\rho} \tilde{\boldsymbol{p}} \cdot \tilde{\boldsymbol{q}} \, \mathrm{Tr} \left(g_{\boldsymbol{p}-\boldsymbol{q}} G_{\boldsymbol{p},\nu} G_{\boldsymbol{q},\rho} \sigma_i G_{\boldsymbol{q},\rho+\omega} G_{\boldsymbol{p},\nu} \sigma_i \right) \\ & + \frac{2}{3} \sum_{\boldsymbol{p},\nu,\boldsymbol{q},\rho} \tilde{\boldsymbol{p}} \cdot \tilde{\boldsymbol{p}} \, \mathrm{Tr} \left(g_{\boldsymbol{p}-\boldsymbol{q}} G_{\boldsymbol{p},\nu} G_{\boldsymbol{q},\rho} G_{\boldsymbol{p},\nu} \sigma_i G_{\boldsymbol{p},\nu+\omega} \sigma_i \right) \\ & + \frac{g_{\boldsymbol{k}}}{3} \left\{ \sum_{\boldsymbol{p},\nu} \mathrm{Tr} \left(\sigma_i G_{\boldsymbol{p},\nu} G_{\boldsymbol{p}+\boldsymbol{k},\nu+\omega} \right) \right\}^2 . \end{aligned} \tag{27}$$

Here, $(\tilde{\boldsymbol{p}})_i = \cos(p_i)$.

They coresspond to the egg, self-enegy, and glasses diagrams, repectively.

RPA calculation

After simplification and integration over ν:

$$\varepsilon^{(1)}(\omega) = \frac{4\pi}{3} \sum_{\boldsymbol{p}} \frac{1}{\widehat{p}^3 \left(\omega^2 + 4\widehat{p}^2\right)} \left\{ 3\widehat{p}^2 - \widehat{p}^4 - \left(\widehat{2p}\right)^2 / 4 \right\}. \tag{28}$$

In the contunuum limit,

$$\varepsilon^{(1)}(\omega) = 8 \times \int \frac{d^3 p}{(2\pi)^3} \frac{8\pi}{3p \left(\omega^2 + 4p^2\right)}. \tag{29}$$

$$\sigma_{ij}(\omega, \boldsymbol{k}) = \sigma_T(\omega, \boldsymbol{k}) \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) + \sigma_L(\omega, \boldsymbol{k}) \frac{k_i k_j}{k^2};$$

$$-i\varepsilon(\omega, \boldsymbol{k}) k_i E_i = 4\pi\rho; \quad j_i = \sigma_{ij} E_j. \tag{30}$$

From the Ward indentity:

$$\sigma_L^{(1)}(\omega) = \frac{e^2 \omega}{4\pi} \varepsilon^{(1)}(\omega) = \frac{e^2 \omega}{24\pi^2} \ln \left[4\Lambda^2 / \omega^2 \right]. \tag{31}$$

$j - j$ correlator:

$$\bar{K}^{(1)}_{ii}(\omega) = \frac{1}{3} \sum_{\boldsymbol{p},\nu,i} \cos^2 p_i \, \mathrm{Tr}\left(\sigma_i G_{\boldsymbol{p},\nu} \sigma_i G_{\boldsymbol{p},\nu+\omega}\right). \tag{32}$$

After simplification and integration over ν:

$$\bar{K}^{(1)}_{ii}(\omega) = \sum_{\boldsymbol{p}} \frac{4}{3\hat{p}\left(\omega^2 + 4\hat{p}^2\right)} \left\{-2\hat{p}^2 - 2\left[(\hat{\boldsymbol{p}} * \hat{\boldsymbol{p}}) \cdot (\hat{\boldsymbol{p}} * \hat{\boldsymbol{p}})\right] + \hat{p}^4\right\}. \tag{33}$$

$(\hat{\boldsymbol{q}} * \hat{\boldsymbol{p}})_i = \sin(q_i)\sin(p_i)$.

In the contunuum limit, $\bar{K}^{(1)}_{ii}(\omega) = 8 \times \int \frac{d^3p}{(2\pi)^3} \frac{-8p}{3\left(\omega^2 + 4p^2\right)}$. (34)

Kubo formula: $\bar{\sigma}^{(1)}_{ii}(\omega) = (2/3)\sigma^{(1)}_T + (1/3)\sigma^{(1)}_L = \frac{\left[\bar{K}^{(1)}_{ii}(\omega) - \bar{K}^{(1)}_{ii}(0)\right]}{\omega}$.

It can be verified that $\sigma^{(1)}_T(\omega) = \sigma^{(1)}_L(\omega)$.

$\lim_{\omega\to 0} \sigma^{(1)}(\omega) = 0$ in 3D.

Two-loop calculation

$$\varepsilon^{(2)\mathrm{egg}}(\omega) = \frac{\pi}{6}\sum_{\boldsymbol{p},\boldsymbol{q}} g_{\boldsymbol{q}-\boldsymbol{p}} \frac{-1}{\widehat{p}\widehat{q}} \left\{ \frac{\left[4\widehat{p}^2\widehat{q}^2 - \omega^2(\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{q}})\right]\left[(\widehat{2\boldsymbol{p}})\cdot(\widehat{2\boldsymbol{q}})\right]}{\widehat{p}^2\widehat{q}^2(\omega^2+4\widehat{p}^2)(\omega^2+4\widehat{q}^2)} + \frac{4\left[4\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{q}} - \omega^2\right](\tilde{\boldsymbol{p}}\cdot\tilde{\boldsymbol{q}})}{(\omega^2+4\widehat{p}^2)(\omega^2+4\widehat{q}^2)} - \frac{4\widehat{q}^2\left[(\widehat{2\boldsymbol{p}})\cdot(\widehat{2\boldsymbol{q}})\right] - 2\omega^2\left[(\widehat{2\boldsymbol{q}})\cdot(\widehat{\boldsymbol{q}}*\tilde{\boldsymbol{p}})\right]}{\widehat{q}^2(\omega^2+4\widehat{p}^2)(\omega^2+4\widehat{q}^2)} - \frac{4\widehat{p}^2\left[(\widehat{2\boldsymbol{p}})\cdot(\widehat{2\boldsymbol{q}})\right] - 2\omega^2\left[(\widehat{2\boldsymbol{p}})\cdot(\widehat{\boldsymbol{p}}*\tilde{\boldsymbol{q}})\right]}{\widehat{p}^2(\omega^2+4\widehat{p}^2)(\omega^2+4\widehat{q}^2)} \right\}. \quad (35)$$

$(\tilde{\boldsymbol{p}})_i = \cos(p_i)$. In the continuum limit,

$$\varepsilon^{(2)\mathrm{egg}}(\omega) = 8 \times \frac{-\pi}{6}\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} g_{\boldsymbol{q}-\boldsymbol{p}} \left\{ \frac{-\omega^2(\boldsymbol{p}\cdot\boldsymbol{q})^2}{p^3q^3(\omega^2+4p^2)(\omega^2+4q^2)} + \frac{\left[8(\boldsymbol{p}\cdot\boldsymbol{q}) - \omega^2\right]}{pq(\omega^2+4p^2)(\omega^2+4q^2)} \right\}. \quad (36)$$

IR divergent?

$$\varepsilon^{(2)\mathrm{se}}(\omega) = \frac{\pi}{6}\sum_{\boldsymbol{p},\boldsymbol{q}} g_{\boldsymbol{q}-\boldsymbol{p}} \frac{1}{\widehat{p}\widehat{q}} \Bigg\{ \frac{\left[4(3-\widehat{p}^2)(\omega^2-4\widehat{p}^2)\right](\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{q}})}{\widehat{p}^2(\omega^2+4\widehat{p}^2)^2} - \frac{\left[(3\omega^2+20\widehat{p}^2)\left(\widehat{2p}\right)^2\right](\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{q}})}{\widehat{p}^4(\omega^2+4\widehat{p}^2)^2} + \frac{\left[8(\omega^2+12\widehat{p}^2)\right]\left[(\widehat{\boldsymbol{p}}*\tilde{\boldsymbol{p}}*\tilde{\boldsymbol{p}})\cdot\widehat{\boldsymbol{q}}\right]}{\widehat{p}^2(\omega^2+4\widehat{p}^2)^2} \Bigg\}. \tag{37}$$

In the continuum limit,

$$\varepsilon^{(2)\mathrm{se}}(\omega) = 8\times\frac{4\pi}{3}\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} g_{\boldsymbol{q}-\boldsymbol{p}} \left\{ \frac{(\boldsymbol{p}\cdot\boldsymbol{q})\left[\omega^2-4p^2\right]}{p^3 q(\omega^2+4p^2)^2} \right\}. \tag{38}$$

IR divergence cancels!

$$\varepsilon^{(2)\mathrm{gl}}(\omega) = \left\{ -\frac{4\pi}{3} \sum_{\boldsymbol{p}} \frac{1}{\hat{p}^3 \left(\omega^2 + 4\hat{p}^2\right)} \left[3\hat{p}^2 - \hat{p}^4 - \left(\widehat{2p}\right)^2 / 4 \right] \right\}^2 . \tag{39}$$

In the continuum limit,

$$\varepsilon^{(2)\mathrm{gl}}(\omega) = \left\{ -8 \times \int \frac{d^3p}{(2\pi)^3} \frac{8\pi}{3p\left(\omega^2 + 4p^2\right)} \right\}^2 . \tag{40}$$

Combining the egg,self energy, and glasses diagrams, we achieve

$$\sigma_L^{(2)}(\omega) = \frac{e^2\omega}{432\pi^3} \left\{ 6 \left[\ln\left(\frac{4\Lambda^2}{\omega^2}\right) \right]^2 - 4(5 - 3\ln 2) \ln\left(\frac{4\Lambda^2}{\omega^2}\right) \right.$$
$$\left. -133/3 - 24\ln 2 + 12(\ln 2)^2 + 63\zeta(3) \right\} . \tag{41}$$

$$\bar{K}_{ii}^{(2)\mathrm{egg}}(\omega) = \frac{1}{3}\sum_{\boldsymbol{p},\boldsymbol{q}} g_{\boldsymbol{q}-\boldsymbol{p}} \frac{1}{2\hat{p}\hat{q}(\omega^2+4\hat{p}^2)(\omega^2+4\hat{q}^2)}$$
$$\times\Big\{-\omega^2\left[4(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{q}})(\tilde{\boldsymbol{p}}\cdot\tilde{\boldsymbol{q}}) - (\widehat{2\boldsymbol{p}})\cdot(\widehat{2\boldsymbol{q}})\right] + 4(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{q}})\left[(\widehat{2\boldsymbol{p}})\cdot(\widehat{2\boldsymbol{q}})\right]$$
$$+16\left(3-2\tilde{p}^2-2\tilde{q}^2+\tilde{p}^2\tilde{q}^2\right)(\tilde{\boldsymbol{p}}\cdot\tilde{\boldsymbol{q}})$$
$$+16\left(3-\tilde{p}^2\right)[\tilde{\boldsymbol{p}}\cdot(\tilde{\boldsymbol{q}}*\tilde{\boldsymbol{q}}*\tilde{\boldsymbol{q}})] + 16\left(3-\tilde{q}^2\right)[\tilde{\boldsymbol{q}}\cdot(\tilde{\boldsymbol{p}}*\tilde{\boldsymbol{p}}*\tilde{\boldsymbol{p}})]\Big\}. \tag{42}$$

In the continuum limit,

$$\bar{K}_{ii}^{(2)\mathrm{egg}}(\omega) = 8\times\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} g_{\boldsymbol{q}-\boldsymbol{p}}\left\{\frac{4\left[-\omega^2(\boldsymbol{p}\cdot\boldsymbol{q})+2p^2q^2+2(\boldsymbol{p}\cdot\boldsymbol{q})^2\right]}{3pq(\omega^2+4p^2)(\omega^2+4q^2)}\right\}. \tag{43}$$

IR divergence also cancels here.

$$\bar{K}_{ii}^{(2)\mathrm{se}}(\omega) = \frac{1}{3}\sum_{\boldsymbol{p},\boldsymbol{q}} g_{\boldsymbol{q}-\boldsymbol{p}} \frac{2}{\widehat{p}^3\widehat{q}(\omega^2+4\widehat{p}^2)^2}
\times\Big\{\omega^2\left\{\left[2\widehat{p}^2-\widehat{p}^4-(\widehat{\boldsymbol{p}}*\widehat{\boldsymbol{p}})\cdot(\widehat{\boldsymbol{p}}*\widehat{\boldsymbol{p}})\right](\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{q}})+2\widehat{p}^2\left[(\widehat{\boldsymbol{p}}*\widehat{\boldsymbol{p}}*\widehat{\boldsymbol{p}})\cdot\widehat{\boldsymbol{q}}\right]\right\}
+4\widehat{p}^2\left\{\left[-2\widehat{p}^2+\widehat{p}^4-3(\widehat{\boldsymbol{p}}*\widehat{\boldsymbol{p}})\cdot(\widehat{\boldsymbol{p}}*\widehat{\boldsymbol{p}})\right](\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{q}})+2\widehat{p}^2\left[(\widehat{\boldsymbol{p}}*\widehat{\boldsymbol{p}}*\widehat{\boldsymbol{p}})\cdot\widehat{\boldsymbol{q}}\right]\right\}\Big\}. \tag{44}$$

In the continuum limit,

$$\bar{K}_{ii}^{(2)\mathrm{se}}(\omega) = 8\times\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}g_{\boldsymbol{q}-\boldsymbol{p}}\left\{\frac{4\left[\omega^2-4p^2\right](\boldsymbol{p}\cdot\boldsymbol{q})}{3p^3q(\omega^2+4p^2)^2}\right\}. \tag{45}$$

$\bar{K}_{ii}^{(2)\mathrm{gl}}(\omega)=0$. In the end, we achieve

$$\bar{\sigma}_{ii}^{(2)}(\omega) = \frac{e^2\omega}{432\pi^3}\left\{3\left[\ln\left(\frac{4\Lambda^2}{\omega^2}\right)\right]^2-4(8-3\ln 2)\ln\left(\frac{4\Lambda^2}{\omega^2}\right)
-175/3-24\ln 2+12(\ln 2)^2+63\zeta(3)\right\}. \tag{46}$$

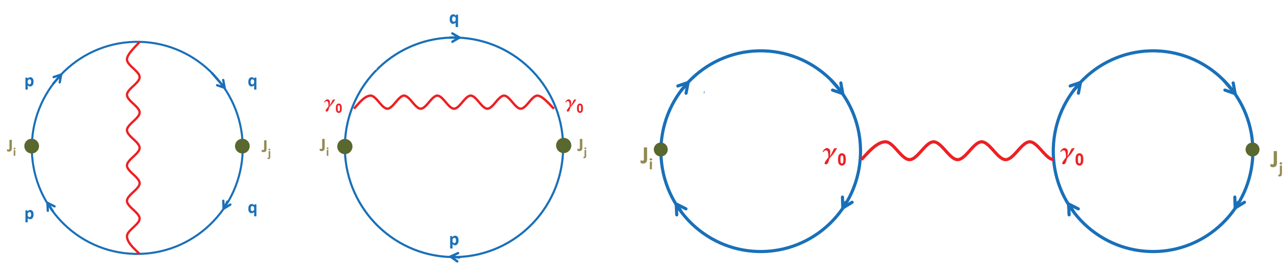
III. Nonlocality in Graphene and WSM

Nonlocality effect

Condition for locality:

- Gapped excitation (insulators)
- Screening (no long-range interaction)
- Disorder (impurity)

In graphene or WSM all these arguments fail and the two-loop contributions do give rise to nonlocality.



$$\sigma_L^{(2)}(\omega) = C_L \sigma_0 \alpha_g, \tag{47}$$

$$\sigma_T^{(2)}(\omega) = C_T \sigma_0 \alpha_g. \tag{48}$$

Here, $C_L = 19/12 - \pi/2 \approx 0.01 << 1$, and $C_T = 31/12 - \pi/2 \approx 1.01$.

$\sigma_0 = e^2/(4\hbar), \alpha_g = e^2/(\epsilon \hbar v_g)$.

$$\rho_{ij}(\omega, \boldsymbol{k}) = \rho_T(\omega, \boldsymbol{k}) \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) + \rho_L(\omega, \boldsymbol{k}) \frac{k_i k_j}{k^2}; \tag{49}$$

$$\rho_{ik}(\omega, \boldsymbol{k}) \sigma_{kj}(\omega, \boldsymbol{k}) = \delta_{ij},$$

$$\rho_T(\omega, \boldsymbol{k}) = 1/\sigma_T(\omega, \boldsymbol{k}), \quad \rho_L(\omega, \boldsymbol{k}) = 1/\sigma_L(\omega, \boldsymbol{k}),$$

$$\sigma_{ij}(\omega, \boldsymbol{k}) = \sigma_T(\omega, \boldsymbol{k}) \delta_{ij} + \sigma_{nl}(\omega, \boldsymbol{k}) \frac{k_i k_j}{k^2}, \quad \sigma_{\mathrm{nl}} = \sigma_L - \sigma_T = -\sigma_0 \alpha_g;$$

$$\rho_{ij}(\omega, \boldsymbol{k}) = \rho_T(\omega, \boldsymbol{k}) \delta_{ij} + \rho_{nl}(\omega, \boldsymbol{k}) \frac{k_i k_j}{k^2}, \quad \rho_{\mathrm{nl}} = \rho_L - \rho_T.$$

Nonlocality effect in electric transport

$$J_i(\boldsymbol{r}) = J_i^{\text{irrot}} + J_i^{\text{sol}} = \partial_i u(\boldsymbol{r}) + \varepsilon_{ij}\partial_j h(\boldsymbol{r}), \tag{50}$$

$$E_i = \rho_{ij} J_j = \rho_L J_i^{\text{irrot}} + \rho_T J_i^{\text{sol}} = \rho_L \partial_i u + \rho_T \varepsilon_{ij}\partial_j h; \tag{51}$$

$$\partial_i E_i = \rho_L \nabla^2 u, \quad \varepsilon_{ij}\partial_i E_j = -\rho_T \nabla^2 h.$$

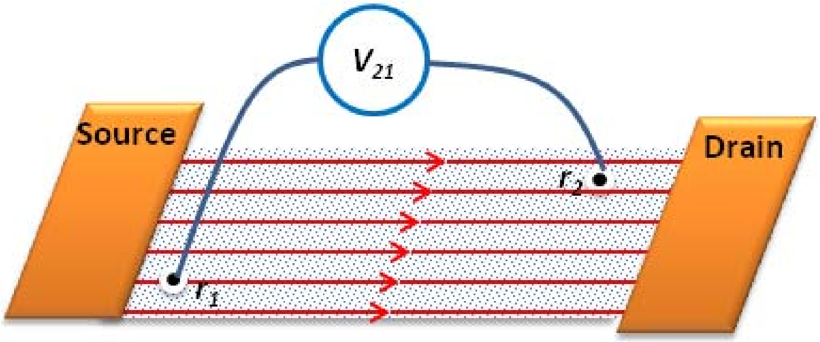
$$\nabla^2 u(\boldsymbol{r}) = \partial_i J_i(\boldsymbol{r}) \equiv s(\boldsymbol{r}),$$

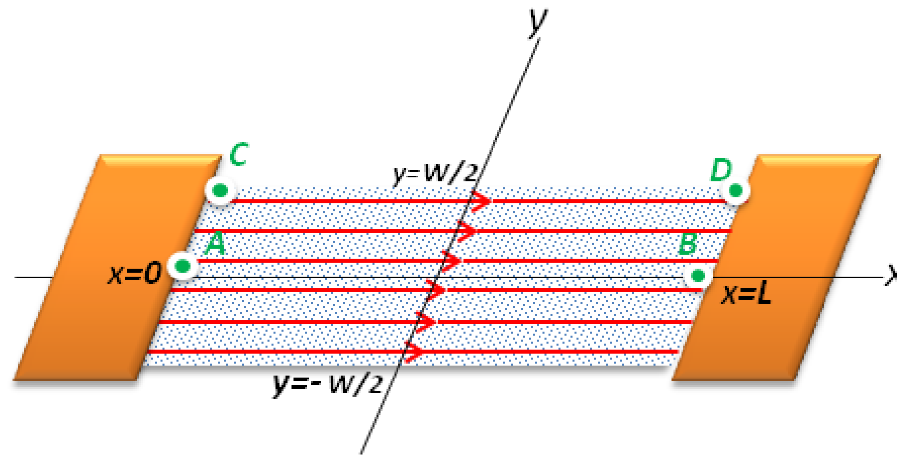
$$u(\boldsymbol{r}) = -\frac{1}{2\pi}\int_{\boldsymbol{r}'\in\text{flake}} \log\left|\boldsymbol{r}-\boldsymbol{r}'\right| s\left(\boldsymbol{r}'\right). \tag{52}$$

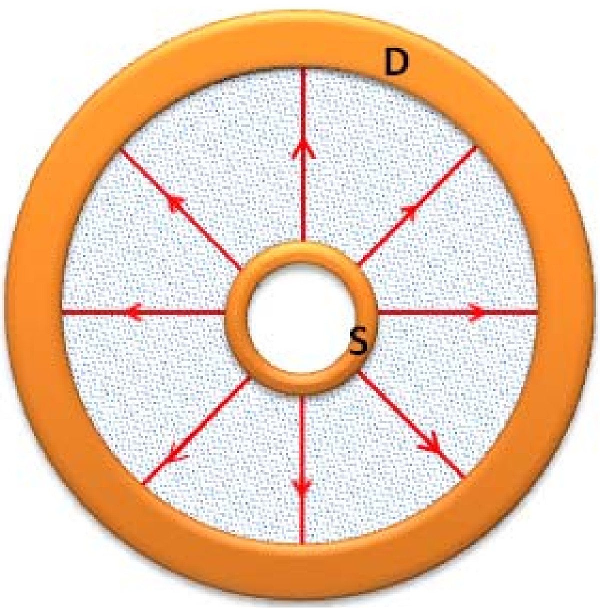
$$\nabla^2 h = -\varepsilon_{ij}\partial_i J_j; \quad \nabla \times \boldsymbol{E} + \dot{\boldsymbol{B}}/c = 0.$$

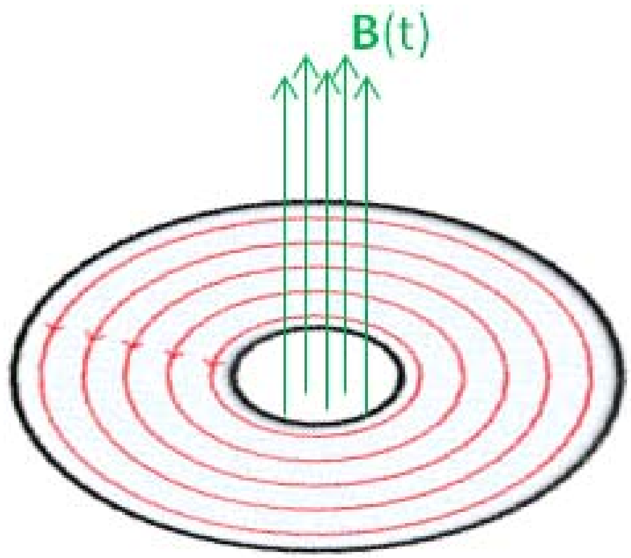
$$h(\boldsymbol{r}) = -\frac{1}{2\pi c \rho_T}\int_{\boldsymbol{r}'\in\text{flake}} \log\left|\boldsymbol{r}-\boldsymbol{r}'\right| \dot{B}_z\left(\boldsymbol{r}'\right). \tag{53}$$

$$\nabla \times \boldsymbol{B} - \dot{\boldsymbol{E}}/c = 4\pi \boldsymbol{J}/c. \tag{54}$$





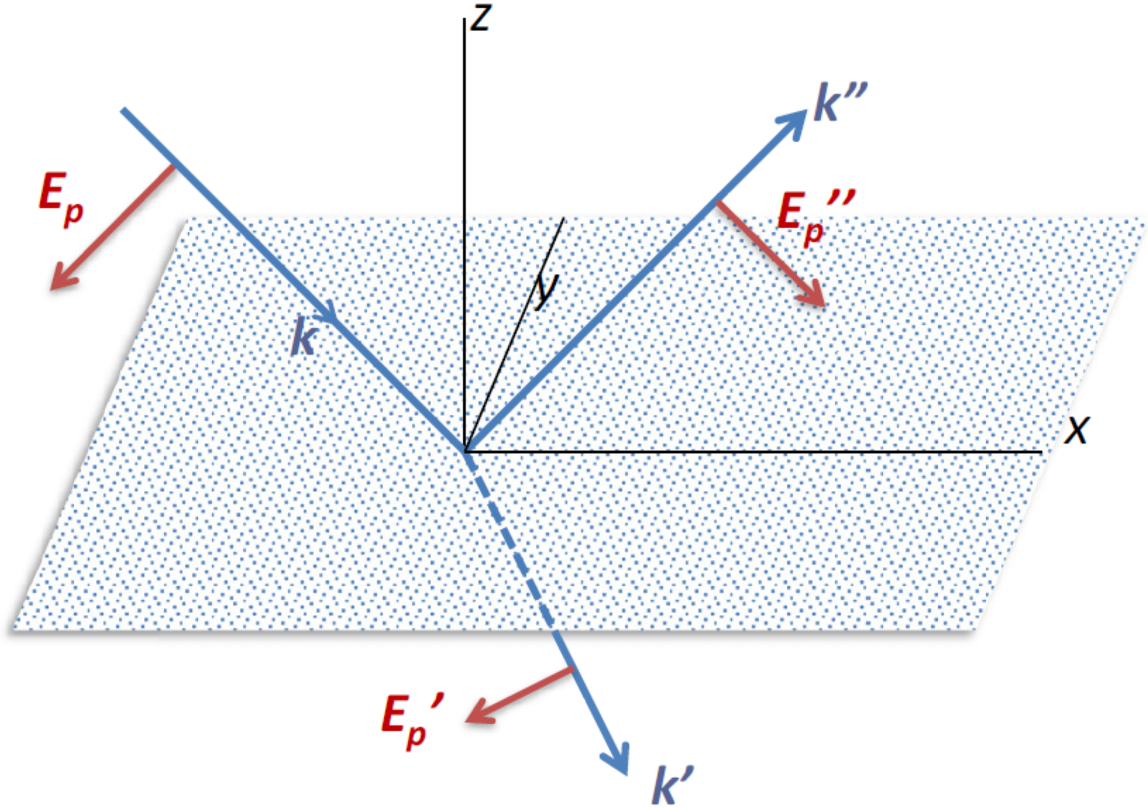




Corbino disk.

Nonlocality effect in optics

Consider an incident plane EM wave with $(\omega, \boldsymbol{k})$:



Incident plane: $y = 0$.

Maxwell equations:

$$\nabla \times \boldsymbol{B} + \frac{i\omega}{c}\boldsymbol{E} = 4\pi \boldsymbol{J}/c,$$
$$\nabla \times \boldsymbol{E} - \frac{i\omega}{c}\boldsymbol{B} = 0,$$
$$\Rightarrow \frac{c}{i\omega}\nabla \times (\nabla \times \boldsymbol{E}) + \frac{i\omega}{c}\boldsymbol{E} = 4\pi \boldsymbol{J}/c. \tag{55}$$

$k_y = 0$, J_i non-vanishing only on the graphene sheet.

In components:

$$ic^2 k_x \partial_z E_z - c^2 \partial_z^2 E_x = \omega^2 E_x + 4\pi i\omega\sigma_L E_x \delta(z); \tag{56}$$
$$-c^2\left(\partial_z^2 - k_x^2\right) E_y = \omega^2 E_y + 4\pi i\omega\sigma_T E_y \delta(z); \tag{57}$$
$$c^2\left(k_x^2 E_z + ik_x \partial_z E_x\right) = \omega^2 E_z. \tag{58}$$

Horizontal polarization: $E_y = 0$, $E_z = \frac{ic^2 k_x}{\omega^2 - c^2 k_x^2} \partial_z E_x$.

$$-\frac{c^2\omega^2}{\omega^2 - c^2 k_x^2} \partial_z^2 E_x = \omega^2 E_x + 4\pi i \omega \sigma_L E_x \delta(z). \tag{59}$$

$$E_x = \Theta(z) \left(A_p e^{ik_z z} + B_p e^{-ik_z z} \right) + \Theta(-z) C_p e^{ik_z z}.$$

$$\frac{\omega}{k_z} B_p = 2\pi \sigma_L (A_p + B_p), \quad k_z = -k \cos\theta. \tag{60}$$

$$r_p(\omega) = \frac{B_p}{A_p} = \frac{-2\pi\sigma_L \cos\theta}{c + 2\pi\sigma_L \cos\theta}, \tag{61}$$

$$t_p(\omega) = 1 + r_p(\omega) = \frac{c}{c + 2\pi\sigma_L \cos\theta}, \tag{62}$$

$$1 - |t_p(\omega)|^2 - |r_p(\omega)|^2 = \frac{4\pi c \sigma_L \cos\theta}{(c + 2\pi\sigma_L \cos\theta)^2}. \tag{63}$$

Vertical polarization: $E_x = E_z = 0$.

$$-c^2\left(\partial_z^2 - k_x^2\right)E_y = \omega^2 E_y + 4\pi i\omega\sigma_T E_y \delta(z)\,. \tag{64}$$

$$E_y = \Theta(z)\left(A_s e^{ik_z z} + B_s e^{-ik_z z}\right) + \Theta(-z)\,C_s e^{ik_z z}.$$

$$\frac{c^2 k_z}{\omega} B_s = 2\pi\sigma_T\left(A_s + B_s\right). \tag{65}$$

$$r_s(\omega) = \frac{B_s}{A_s} = -\frac{2\pi\sigma_T}{c\cos\theta + 2\pi\sigma_T}, \tag{66}$$

$$t_s(\omega) = 1 + r_s(\omega) = \frac{c\cos\theta}{c\cos\theta + 2\pi\sigma_T}, \tag{67}$$

$$1 - |t_s(\omega)|^2 - |r_s(\omega)|^2 = \frac{4\pi c\sigma_T\cos\theta}{(c\cos\theta + 2\pi\sigma_T)^2}. \tag{68}$$

Nonlocality effect in WSM

The macroscopic Maxwell equations in a medium:

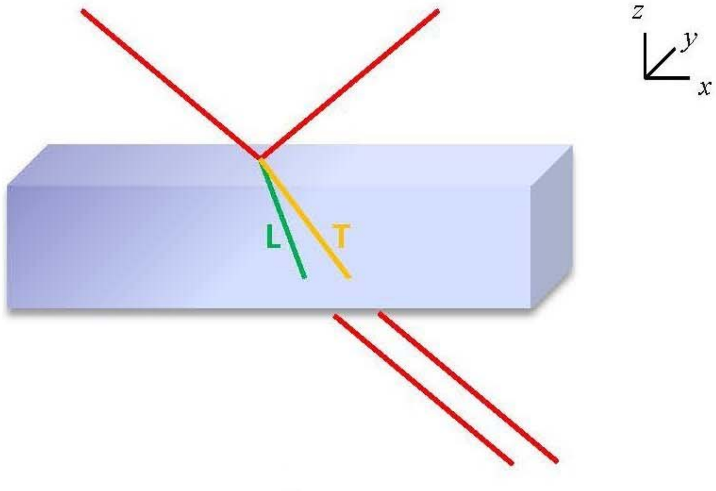
$$\nabla \cdot \mathbf{D} = 4\pi\rho_{\text{ext}}; \; \nabla \times \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = 0; \tag{69}$$

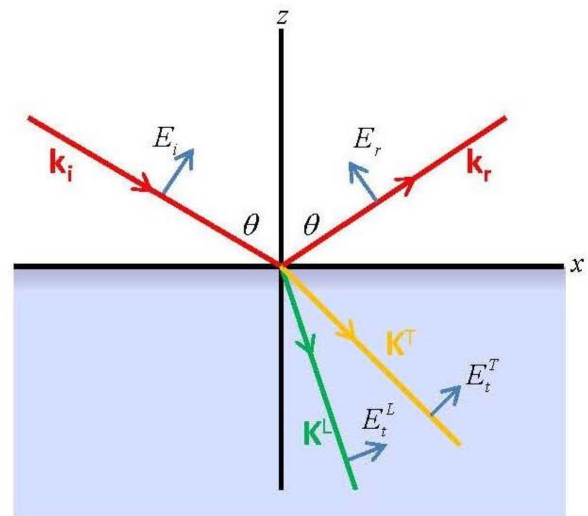
$$\nabla \cdot \mathbf{B} = 0; \qquad \nabla \times \mathbf{H} - \frac{1}{c}\dot{\mathbf{D}} = \frac{4\pi}{c}\mathbf{J}_{\text{ext}}.$$

$\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{H} = \mathbf{B}/\mu$.

Both the transversal and longitudinal modes exist in WSM's since $\sigma_{\text{T}} \neq \sigma_{\text{L}}$. This cannot occurs in local media.

Considering the reflection and transmission of an EM wave illuminating on a vacuum-WSM interface.





The horizontally-polarized light beam with $\mathbf{k}$ incident at angle θ splits in the WSM into transverse and longitudinal modes with $\mathbf{p}$ and $\mathbf{q}$.

Reflection and transmission on a vacuum-WSM interface.

For horizontally polarized incoming waves:

$$\frac{\mathbf{E}^{\mathsf{vac}}}{E_0} = \left\{\frac{k_z}{k}, 0, \frac{-k_x}{k}\right\} e^{i(k_x x + k_z z)} - r\left\{\frac{k_z}{k}, 0, \frac{k_x}{k}\right\} e^{i(k_x x - k_z z)}. \tag{70}$$

E_0 : the incoming amplitude, r : the reflecttion coefficient.

$k_x > 0$, $k_z < 0$, θ the incident angle.

In the WSM:

$$\frac{\mathbf{E}^{\mathsf{WSM}}}{E_0} = t_T\left\{\frac{p_z}{p}, 0, \frac{-k_x}{p}\right\} e^{i(k_x x + p_z z)} + t_L\left\{\frac{k_x}{q}, 0, \frac{q_z}{q}\right\} e^{i(k_x x + q_z z)}. \tag{71}$$

t_T, t_L : the transmitted transversal and longitudinal coefficients.

The boundary conditions are:

(i) $\mathbf{D}_\perp$: $E_z^{\mathsf{vac}} = \frac{4\pi i}{\omega}\sigma_{zx}E_x^{\mathsf{WSM}} + \left(1 + \frac{4\pi i}{\omega}\sigma_{zz}\right)E_z^{\mathsf{WSM}}$.

(ii) $\mathbf{E}_\parallel$: $E_x^{\mathsf{vac}} = E_x^{\mathsf{WSM}}$.

(iii) $\mathbf{B}_\perp$: $\partial_x E_y^{\mathsf{vac}} = \partial_x E_y^{\mathsf{WSM}}$.

(iv) $\mathbf{H}_\parallel$: $\partial_z E_x^{\mathsf{vac}} - \partial_x E_z^{\mathsf{vac}} = \partial_z E_x^{\mathsf{WSM}} - \partial_x E_z^{\mathsf{WSM}}$.

Together with the ABC (additional boundary condition), one finds the WSM becomes opaque at frequency-dependent incident angles.

From Ampère's and Faraday's law, we find two modes:

(i) $\mathbf{E}_T$: $1 + i\frac{4\pi}{\omega}\sigma_{\mathsf{T}}\left(\omega, \mathbf{p}\right) = \frac{c^2p^2}{\omega^2}$.

(ii) $\mathbf{E}_L$: $1 + i\frac{4\pi}{\omega}\sigma_{\mathsf{L}}\left(\omega, \mathbf{q}\right) = 0$.

For a "sharp" interface, no current may escape the WSM:

$$\sigma_{zx} E_x^{\mathsf{WSM}} + \sigma_{zz} E_z^{\mathsf{WSM}} = 0.$$

Combining all the boundary conditions:

$$\begin{aligned} r &= \frac{1-D}{1+D}; \quad t_{\mathsf{T}} = \frac{2k/p}{1+D}; \quad t_{\mathsf{L}} = -\frac{2k_x q\left(p^2-k^2\right)}{k q_z p^2 (1+D)} \ ; \\ D &= \frac{k^2 p_z q_z - \left(p^2-k^2\right) k_x^2}{k_z q_z p^2}. \end{aligned} \tag{72}$$

WSM becomes opaque at certain ω-dependent incident angle.

In contrast, the vertically polarized wave does not generate the longitudinal mode. Hence, the amplitudes are standard.

Dispersion relations for the transversal and longitudinal modes are

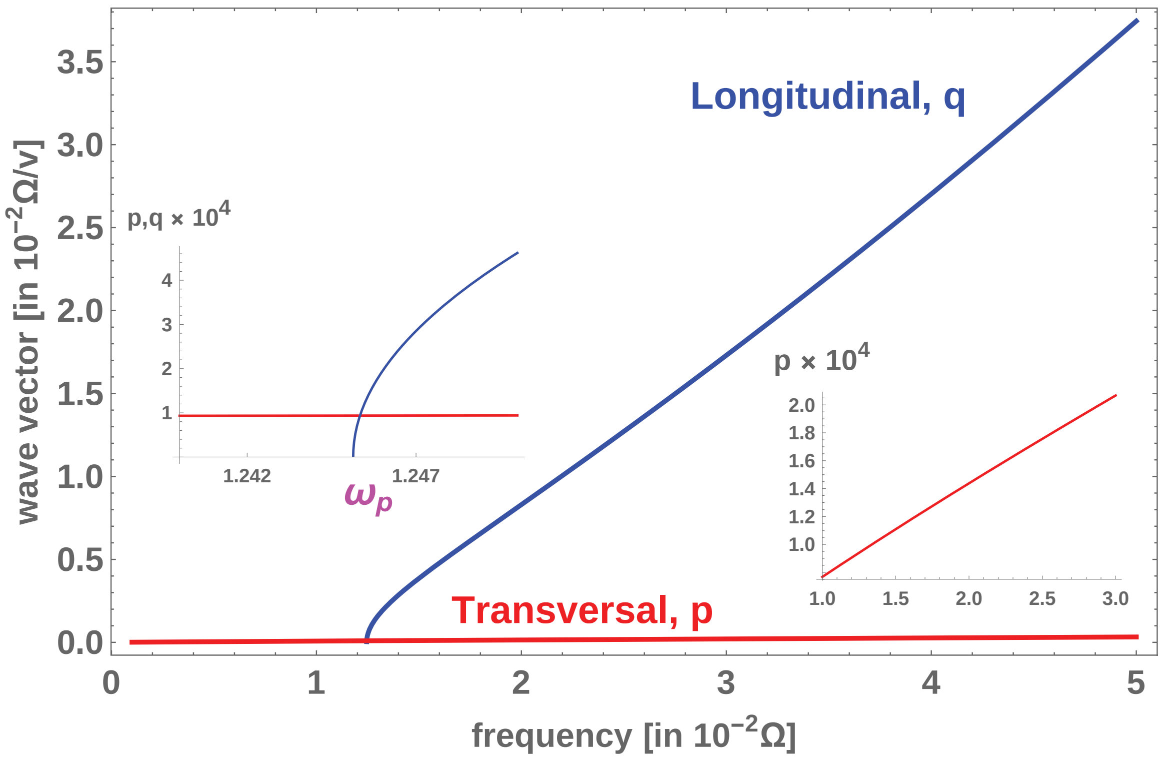
$$p = \frac{\omega}{c}\sqrt{\frac{Ne^2}{6\pi\hbar v}\log\frac{\Omega^2}{\omega^2}}; \qquad (73)$$
$$q = \frac{\omega}{v}\sqrt{1 - \frac{N\alpha}{4\pi}\log\frac{\Omega^2}{\omega^2}}.$$

Difference between transveral and longitudinal components:

$$i\frac{4\pi}{\omega}\sigma_{\mathsf{nl}}(\omega, \mathbf{k}) = -\frac{c^2k^2}{\omega^2}. \qquad (74)$$

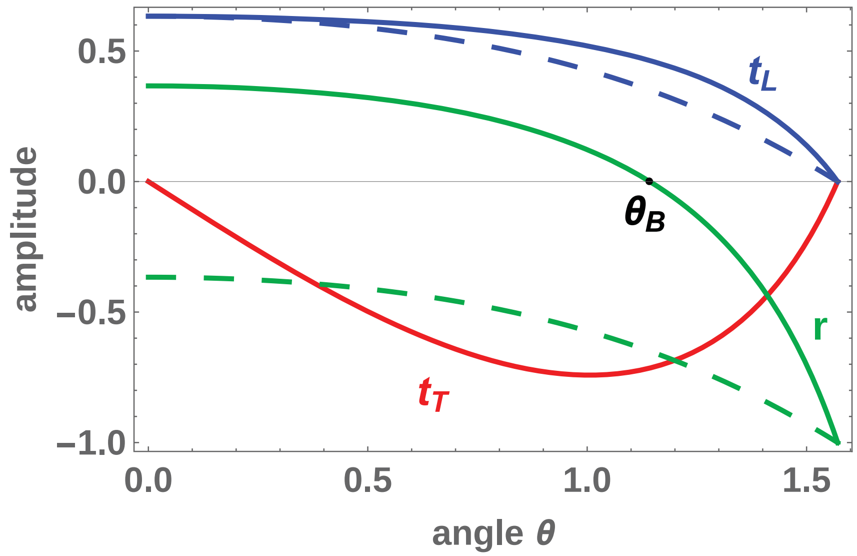
In local materials, this generally cannot be satisfied.

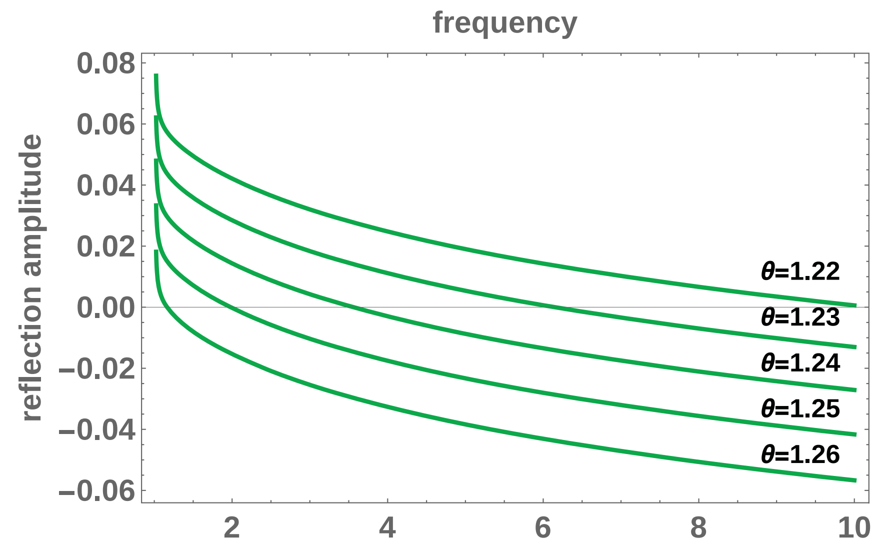
In a WSM, $\sigma_{\mathsf{nl}}(\omega)$ is finite and such solutions may exist.



The transverse p and longitudinal q are increasing functions of ω.

They intersect at (q_m, ω_m).





The horizontal polarization amplitudes (solid): reflected (green), transmitted transversal (red) and longitudinal (blue).

$\theta_B(\omega)$: "Brewster" angle of the WSM.

The vertical polarization amplitudes(dashed).

IV. Conclusion

- In the linear response regime.
  1. dc and ac conductivity $\sigma_0 = \frac{1}{4}\frac{e^2}{\hbar}$.
  2. Exp. optical conductivity in good agreement with σ_0.

- $\sigma_L^{(2)}(\omega) = C_L \sigma_0 \alpha_g,\ \sigma_T^{(2)}(\omega) = C_T \sigma_0 \alpha_g.$

  $C_L = 19/12 - \pi/2 \approx 0.01,\ C_T = 31/12 - \pi/2 \approx 1.01,$

  $\sigma_{\text{nl}} = \sigma_L - \sigma_T = -\sigma_0 \alpha_g;\ \sigma_0 = e^2/(4\hbar),\ \alpha_g = e^2/(\epsilon \hbar v_g).$

- $E_i = \rho_L J_i^{\text{irrot}} + \rho_T J_i^{\text{sol}} = \rho_L \partial_i u + \rho_T \varepsilon_{ij} \partial_j h.$

- $r_p(\omega) = -\dfrac{2\pi\sigma_L \cos\theta}{c + 2\pi\sigma_L \cos\theta}, \quad r_s(\omega) = -\dfrac{2\pi\sigma_T}{c\cos\theta + 2\pi\sigma_T}.$

- Nonlocal effect in 3D WSM.

$$\bar{\sigma}_{ii}^{(2)}(\omega) = \frac{e^2\omega}{432\pi^3}\Bigg\{3\left[\ln\left(\frac{4\Lambda^2}{\omega^2}\right)\right]^2 - 4(8-3\ln 2)\ln\left(\frac{4\Lambda^2}{\omega^2}\right) - 175/3 - 24\ln 2 + 12(\ln 2)^2 + 63\zeta(3)\Bigg\}.$$

$$\sigma_L^{(2)}(\omega) = \frac{e^2\omega}{432\pi^3}\Bigg\{6\left[\ln\left(\frac{4\Lambda^2}{\omega^2}\right)\right]^2 - 4(5-3\ln 2)\ln\left(\frac{4\Lambda^2}{\omega^2}\right) - 133/3 - 24\ln 2 + 12(\ln 2)^2 + 63\zeta(3)\Bigg\}.$$

$$\sigma_{\text{nl}} = \sigma_L - \sigma_T = \frac{e^2\omega}{144\pi^3}\Bigg\{\frac{3}{2}\left[\ln\left(\frac{4\Lambda^2}{\omega^2}\right)\right]^2 + 6\ln\left(\frac{4\Lambda^2}{\omega^2}\right) + 7\Bigg\}.$$

- $r = \frac{1-D}{1+D}$; $t_{\mathsf{T}} = \frac{2k/p}{1+D}$; $t_{\mathsf{L}} = -\frac{2k_x q \left(p^2 - k^2\right)}{k q_z p^2 (1+D)}$;

  $D = \frac{k^2 p_z q_z - \left(p^2 - k^2\right) k_x^2}{k_z q_z p^2}$.

  WSM has no reflection for horizontally-polarized EM wave at certain Brewster angle.

- Chiral magnetic effect: the generation of current along an external B field induced by chirality imbalance, $\sim$ chiral anomaly.

  The chiral magnetic current is non-dissipative, because it is topologically protected.

  Q. Li et al, Nature Physics (2016).

- Anomalous Hall effect: Hall effect in ferromagnetic materials.

It involves concepts based on topology and geometry.