Entanglement Renormalization & Integral Geometry

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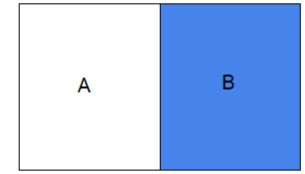
Motivation

- Explore the connection between a many-body quantum state and dual bulk geometry in the context of AdS/CFT correspondence. We need two ingredients:
- MERA (multi-scale entanglement renormalization ansatz) --- a quantum circuit picture of many-body ground state.
- Integral geometry --- replace the metric by the kinematic measure of geodesics.

Outline

- Holographic Entanglement
- MREA
- Integral geometry
- Holographic entanglement renormalization

Entanglement Entropy



- The way to characterize the entanglement is through the entanglement entropy (EE) of the reduced density matrix $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$.

$$S_A = -\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

- The leading contribution takes the form of area law, i.e., $S_A \sim (\frac{L}{\epsilon})^{d-1}$. This implies that most entanglement is short-ranged.
- The coefficient of the sub-leading terms could encode some information about long-ranged entanglement, e.g.,

$$S_{d=1,CFT} = \frac{c}{6} \ln \frac{L}{\epsilon}$$

$$S_{d=2,gapped} = \frac{L}{\epsilon} - \gamma$$

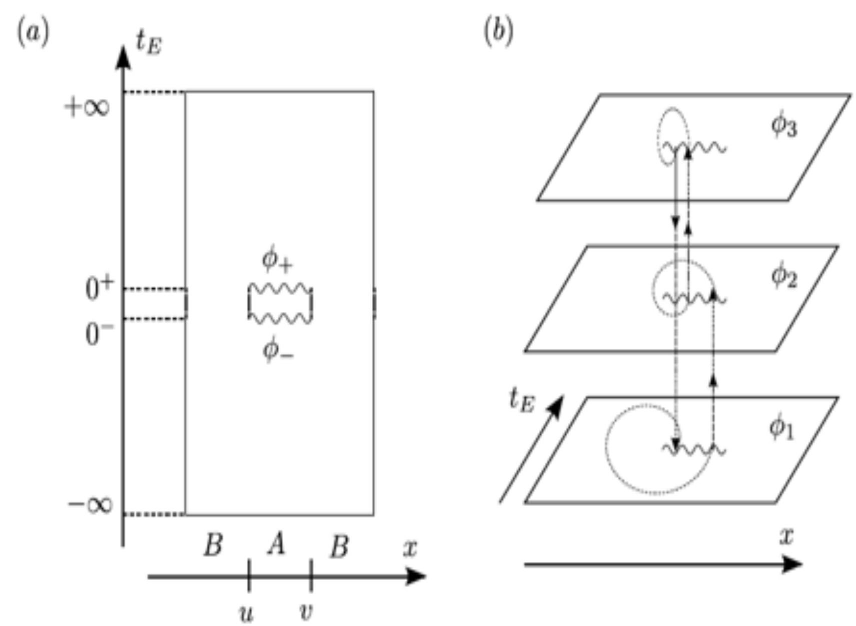
Replica trick

- EE is difficult to compute in field theory.
- In path integral one can evaluate it by the so-called replica trick.
- The cut is understood as an insertion of twist surface operator. (In this (1+1)D, it is a point operator)
- EE is derived from the partition function in the presence of surface operator.

$$S_A = -\frac{\partial}{\partial n}\mathrm{tr}_A\,\rho_A^n|_{n=1} = -\frac{\partial}{\partial n}\log\mathrm{tr}_A\;\rho_A^n|_{n=1}$$

$$[\rho_A]_{\phi_+\phi_-} = (Z_1)^{-1}\int_{t_E=-\infty}^{t_E=\infty} D\phi\; e^{-S(\phi)}\prod_{x\in A}\delta\left(\phi(+0,x)-\phi_+(x)\right)\cdot\delta\left(\phi(-0,x)-\phi_-(x)\right)$$

$$[\rho_A]_{\phi_{1+}\phi_{1-}}[\rho_A]_{\phi_{2+}\phi_{2-}}\cdots[\rho_A]_{\phi_{n+}\phi_{n-}}$$

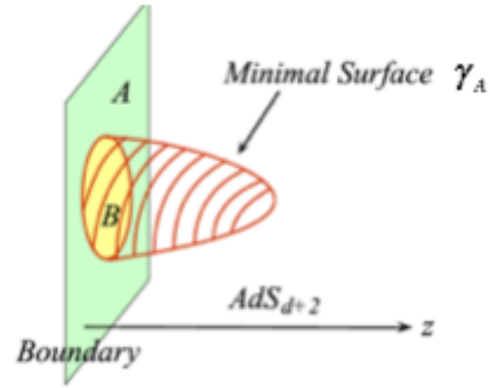


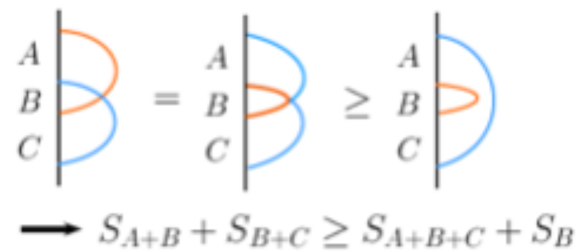
Holographic EE

- The AdS/CFT duality relates the strongly coupled CFT to the weak AdS gravity.
- In a sense, the CFT has a one-dim higher AdS gravity as its effective theory.
- Ryu & Takayanagi (2006) proposed the EE of dual CFT is given by the area of minimal surface covered the entangling region on the boundary.

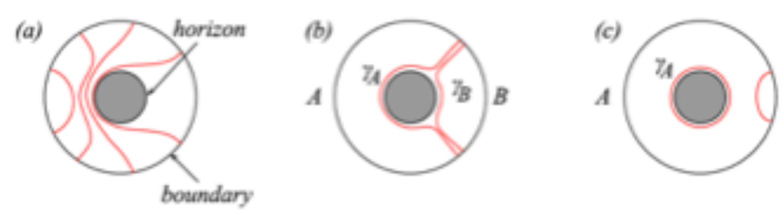
$$S_A = \frac{\text{Area of } \gamma_A}{4G_N^{(d+2)}} \qquad \text{(RT formula)}$$

- For spherical entangling surface, one can show that the **1st law of entanglement => Einstein eq**





strong subadditivity

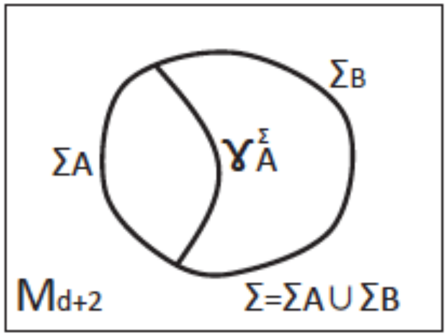


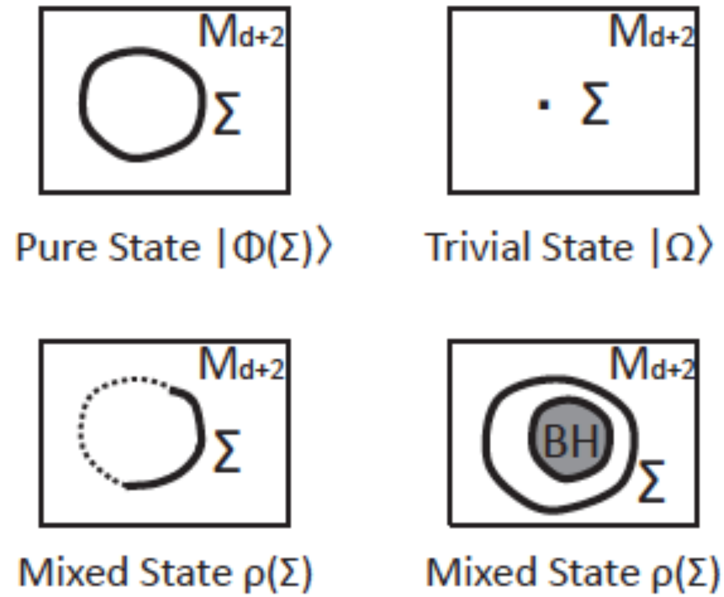
finite temperature

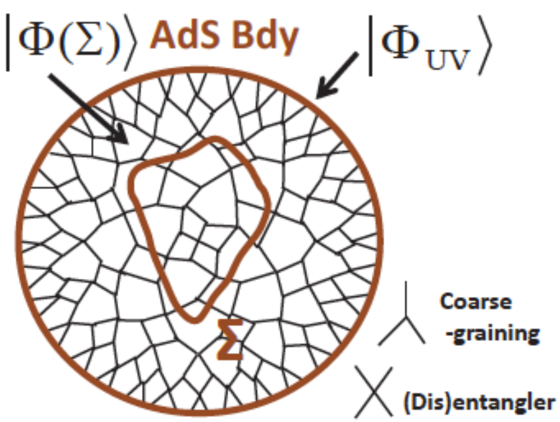
Entanglement & Geometry

- RT formula indicate a connection between entanglement and geometry.
- A more aggressive proposal: each bulk surface is dual to a CFT state, whose entanglement is encoded in the surface area.
- This is the so-called surface/state duality (Takayanagi et al, 2005).
- This proposal bases on a more refined duality between geometry and entanglement.

Surface/State Duality



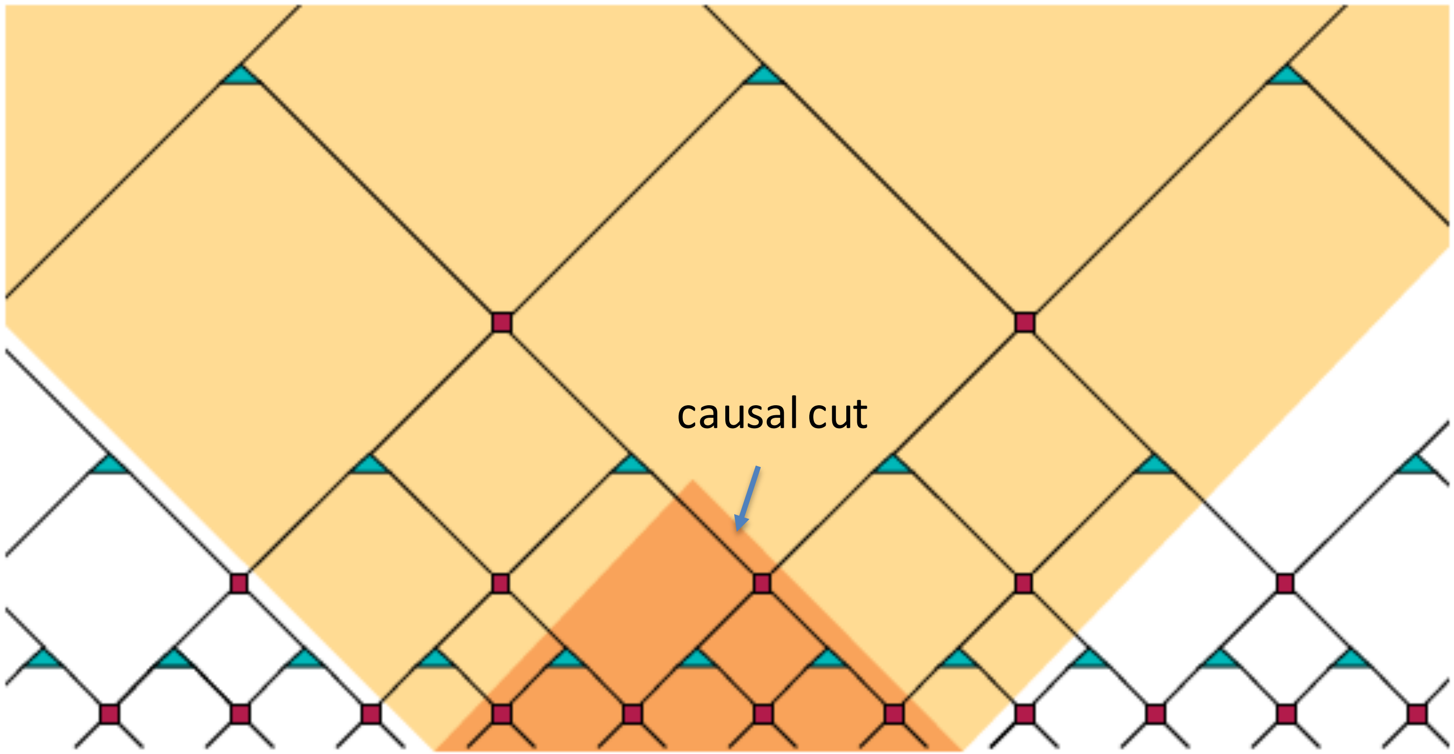




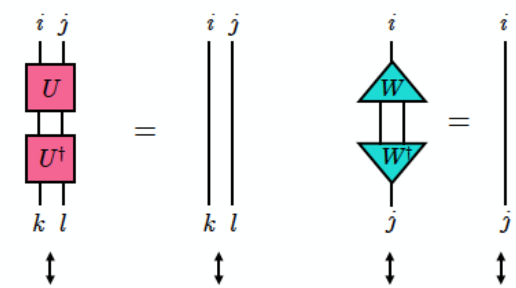
c.f. Miyaji & Takayanagi (1503.03542)

MERA (multi-scale Entanglement renormalization ansatz)

- MERA is a quantum circuit representation of the many-body quantum states.
- The circuit has layer structure. At each layer (scale), we try to disentangle some local d.o.f. by local unitary operations (LUs).
- Then we coarse grain the remaining entangled ones to go to next layer.
- In a sense, this is a RG like ansatz (unknown LUs) of many-body wavefunction with local entanglement removed.



disentanglers, isometries & causal cone

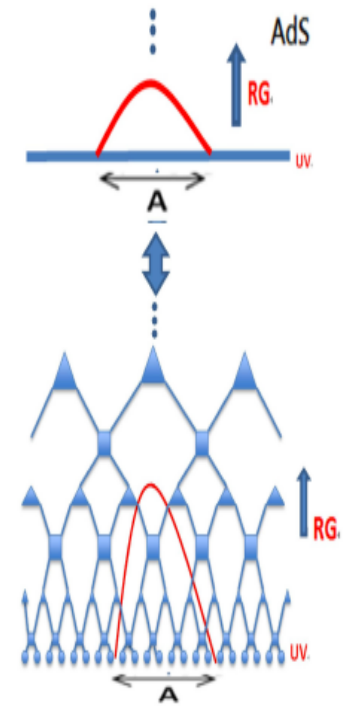


$$U_{ij}{}^{mn}U^{\dagger}{}_{mn}{}^{kl} = \delta_i^k \delta_j^l \qquad W_i^{mn}W^{\dagger}{}_{mn}{}^{j} = \delta_i^j$$

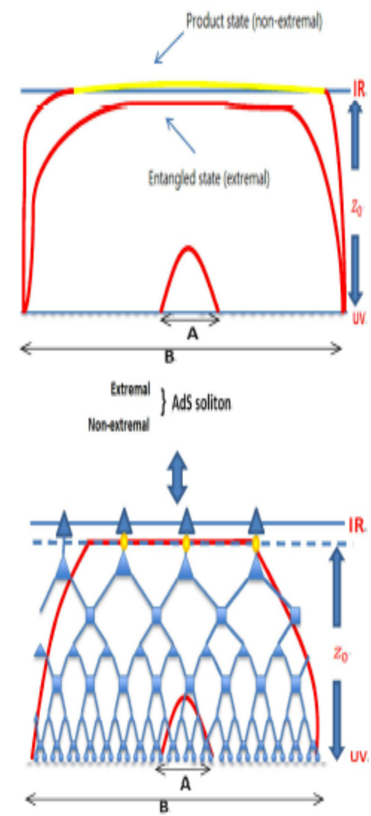
Taken from Czech et al (1512.12015)

AdS/MERA

- The entanglement entropy of an “UV” interval counts the number of bonds on the causal cut, which is the boundary of the causal cones of the interval and its complement.
- This reminds the RT formula for the holographic EE, and motivates AdS/MERA duality.

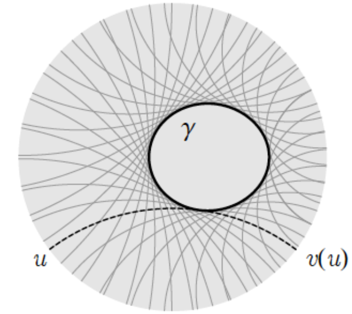


MERA for CFT
Minimal surface for Entanglement entropy for region A
(III)



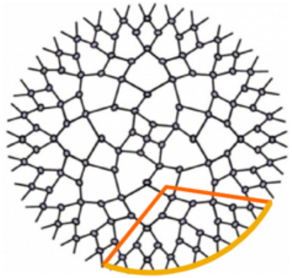
MERA for gapped system
Minimal surface for chosen region A and B
(IV)

Integral Geometry

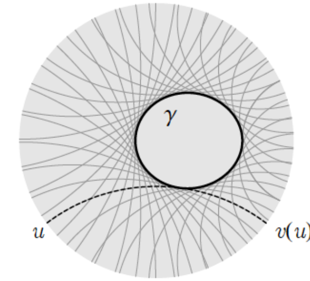


- AdS/MERA motivates one to find a more refined picture for RT formula.
- Instead, one can evaluate the area of a bulk surface cutting it with geodesics ending on boundaries.
- This was first proposed by Czech et al (2015) to evaluate CFT's differential entropy associated with a closed bulk surface.

$$E = \sum_{k=1}^{n} [S_{EE}(I_k) - S_{EE}(I_{k+1} \cap I_k)]$$



Crofton’s formula



- For AdS3, the area of a bulk curve is computed by

$$\frac{\sigma(\gamma)}{4G} = \frac{1}{4}\int_{\gamma\cap\Gamma\neq\emptyset} N(\gamma\cap\Gamma)\,\epsilon_{\mathcal{K}} = \frac{1}{4}\int_{\tilde{p}_A\Delta\tilde{p}_B} \epsilon_{\mathcal{K}} \qquad \tilde{p}_A\Delta\tilde{p}_B \equiv (\tilde{p}_A\cup\tilde{p}_B) - (\tilde{p}_A\cap\tilde{p}_B)\,.$$

- The kinematic space measure (of geodesics) is

$$\epsilon_{\mathcal{K}}(u,v) = \frac{\partial^2 S(u,v)}{\partial u \partial v} \mathrm{d}u \wedge \mathrm{d}v$$

- In this case, the kinematic space is de Sitter.
- Crofton’s formula can be generalized to higher D.

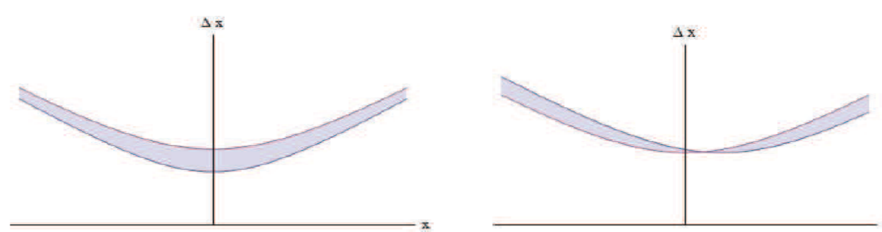
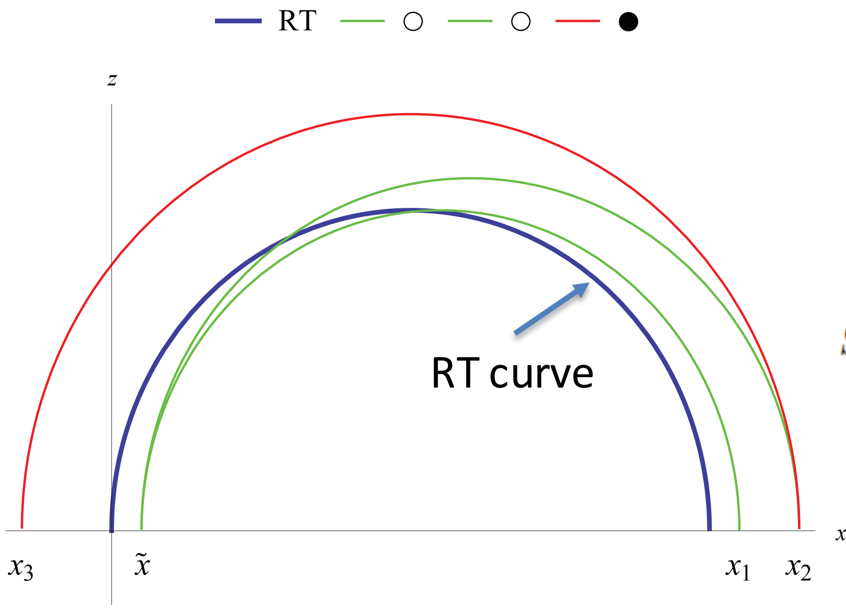


Figure 1. Examples of point curves. Coordinates of x (midpoint) and Δx (size) instead of end points u, v are used to parameterize a geodesic. On the left $A = (\tilde{x}, \tilde{z})$ and $B = (\tilde{x}, \tilde{z} + d\tilde{z})$ and on the right $A = (\tilde{x}, \tilde{z})$ and $B = (\tilde{x} + d\tilde{x}, \tilde{z})$. In either case, $\tilde{p}_A\Delta\tilde{p}_B$ is indicated by the shaded region.

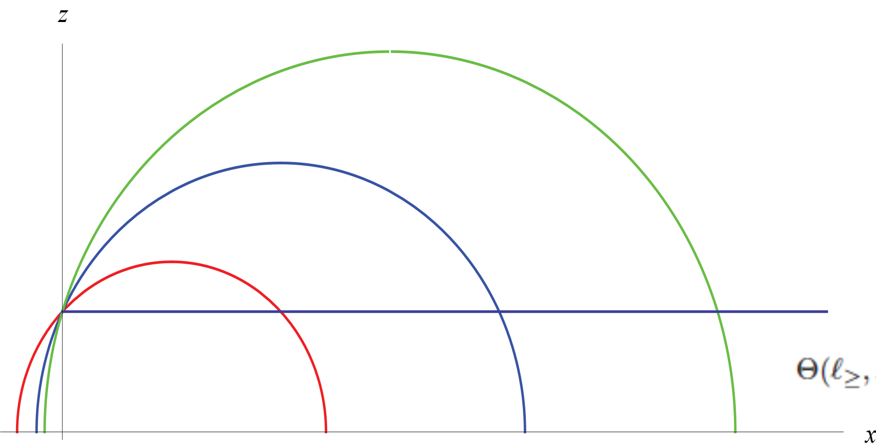
Entanglement Contour

- Integral geometry picture of RT formula resembles the MERA network.
- Besides, the kinematic space can be thought as the entanglement contour, which is the additive entanglement density.
- Naively, one can think the end-points of each geodesic is associated with a Bell’s pair contributing to the total EE.



EE of an interval

$$S_{EE}(\ell;\tilde{z}) = \int_{-\infty}^{\tilde{x}} \mathrm{d}u \int_{\tilde{x}}^{\tilde{x}+l} \mathrm{d}v \frac{\partial^2 S(u,v;\tilde{z})}{\partial u \partial v} + \int_{\tilde{x}+\ell}^{\infty} \mathrm{d}v \int_{\tilde{x}}^{\tilde{x}+l} \mathrm{d}u \frac{\partial^2 S(u,v;\tilde{z})}{\partial u \partial v} .$$



EE of a point

$$S_{EE}(\mathrm{d}x;\tilde{z}) := \int_0^{\infty} \bar{s}(r,\tilde{x};\tilde{z})\mathrm{d}r = \mathrm{d}x \left(\int_{-\infty}^{\tilde{x}} \frac{\partial^2 S(u,v;\tilde{z})}{\partial u \partial v}\Big|_{v=\tilde{x}} \mathrm{d}u + \int_{\tilde{x}}^{\infty} \frac{\partial^2 S(u,v;\tilde{z})}{\partial u \partial v}\Big|_{u=\tilde{x}} \mathrm{d}v \right)$$

Long-distance entanglement contour

$$\Theta(\ell_{\geq},\tilde{x};\tilde{z})\mathrm{d}x := \int_{\ell_{\geq}}^{\infty} \bar{s}(r,\tilde{x};\tilde{z})\mathrm{d}r = \frac{1}{2}\mathrm{d}x \left(\frac{\partial S(\tilde{x}-\ell_{\geq},v;\tilde{z})}{\partial v}\Big|_{v=\tilde{x}} - \frac{\partial S(u,\tilde{x}+\ell_{\geq};\tilde{z})}{\partial u}\Big|_{u=\tilde{x}} \right)$$

Entanglement RG

- Based on Surface/State duality, we can consider the holographic picture of entanglement renormalization:
- As we push the surface (RG running), the end-points of the kinematic geodesic changes. This change can be holographically thought as the LUs of MERA:
- Short-distance pairs are removed (disentangler) and long-distance ones are reshuffled (isometry).

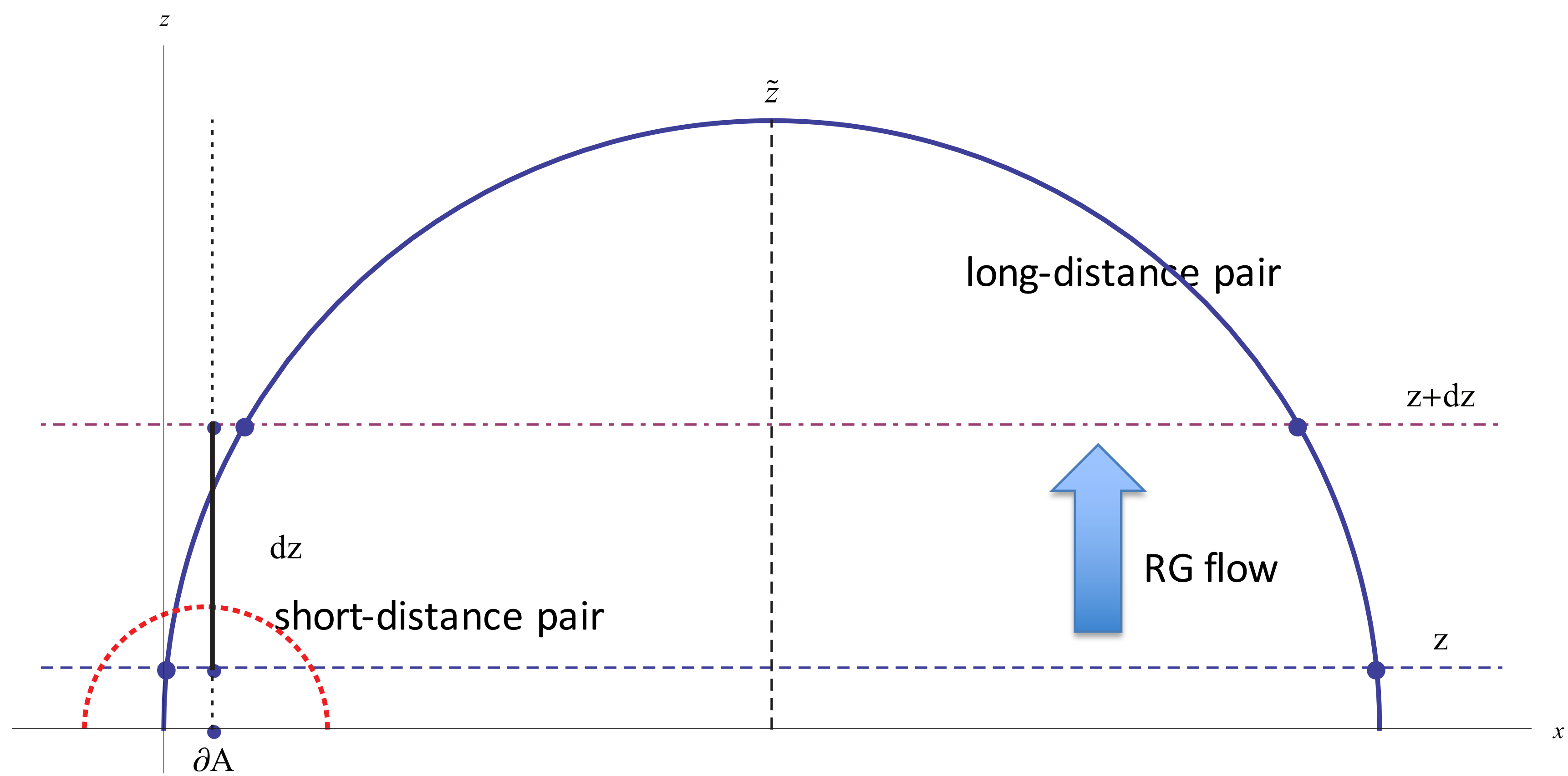


Figure 6. The long-distance entanglement is invariant under RG flow but it is reshuffled to shorter scale. Two situations are shown here. The first is the blue geodesic which intersects both surface states at z and $z + dz$. This implies that the entanglement contour for the intersecting pair of points is just reshuffled but not removed. However, the entanglement entropy is reduced (and turned into line element dz) as the left end point is dragged across the entangling surface (the intersection points of the vertical bar with the horizontal lines). The second situation is the red-dotted geodesic for which the intersecting points shrink to none as the surface state is pushed up from z to $z + dz$. The first situation is similar to isometry operation in MERA, while the second is to disentangler.

Entanglement RGE

- An RGE of long-distance entanglement (for transl. inv. system) can be derived from the fact:
- Kinematic space measure will not change under the change of cutoff lines from z=z1 to z=z2 as long as z1,z2< scale $\tilde{z}$ defining the pair.
- It then yields

$$\Theta(2\sqrt{\tilde{z}^2 - z^2}; z) = \Theta(0; \tilde{z}) \,.$$

Differential
Entropy of surface at
for surface/state at z. $\tilde{z}$

length of the surface at $\tilde{z}$

$$\frac{d}{dz}\Theta(2\sqrt{\tilde{z}^2 - z^2}; z) = 0 \,.$$

Higher D story

- Though there are higher D Crofton's formulas to evaluate the bulk surface area, there are some ambiguity as the kinematic object could be the ones with co-dimension less than 2.
- Using geodesic as the kinematic objects, we can reproduce RT formulas. However, the length of geodesic is no longer the holographic EE in higher D.
- Despite that, the idea of entanglement contour still holds.

Conclusion

- There are intimate relation between quantum entanglement and geometry in the holographic context.
- We show that the integral geometry can yield the intuitive refined picture for the AdS/MERA duality.
- An RGE for long-distance entanglement is obtained, which may encode bulk dynamics.
- Higher D story is more nontrivial, but the picture of holographic entanglement contour should still hold.