On the scattering equations in massless theories.

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F. Cachazo, S. He, E. Yuan proposal for tree-level S-matrix for a wide class of massless theories in several dimensions ,

$$\mathcal{M}_n = \int \frac{d^n\sigma}{\mathrm{vol}\, SL(2,\mathbb{C})} \prod_a{}' \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{(k_a+k_b)^2}{\sigma_a-\sigma_b}\Big) \mathcal{I}_n(k,\epsilon,\tilde{\epsilon},\sigma)$$

$$\equiv \int d\mu_n \, \mathcal{I}_n(k,\epsilon,\tilde{\epsilon},\sigma)\,,$$

The integral is completely localized at sol. of scattering equations (SE):

$$\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{(k_a+k_b)^2}{\sigma_a-\sigma_b} = 0$$

CHY have made several proposals for the $\mathcal{I}_n(k,\epsilon,\tilde{\epsilon},\sigma)$, including: Gravity, YM, EM, EYM, DBI, NLSM YMS, ϕ^3, ϕ^4.

L. Dolan and Goddard show an equivalent form for the SE,

$$h_m = \frac{1}{m!} \sum_{a_1, a_2, \ldots, a_m} k^2_{1a_1a_2\ldots a_m} \sigma_{a_1}\sigma_{a_2}\ldots\sigma_{a_m} = 0\,,$$

where $k^2_{1a_1a_2\ldots a_m} = (k_1 + k_{a_2} + \ldots + k_{a_m})^2$.
By $SL(2,C)$ invariance, $\sigma_1 = \infty$, $\sigma_2 = 1$, $\sigma_n = 0$ and $m = 1, 2, \ldots (n-3)$.
Analytic solutions only up to $n = 5$. Our concern is too look for underlying mathematical structures in the SE which allows us to compute at general n.

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- Using Stickelberger theorem in conjunction with the two items above, we can encode the solutions of the SE in a single matrix, namely T_σ
- In general, after perform the integration over the punctures on the sphere, the tree level S-matrix can be expressed as a rational function. Schematically,

$$\mathcal{A}_n = \sum_{\sigma_{sol}} \frac{P({}_{j_1}\sigma_3,\ {}_{j_2}\sigma_4, \ldots,\ {}_{j_{(n-3)!}}\sigma_{(n-3)!})}{Q({}_{j_1}\sigma_3,\ {}_{j_2}\sigma_4, \cdots,\ {}_{j_{(n-3)!}}\sigma_{(n-3)!})} \equiv \mathrm{Tr}\left(\mathcal{P}(T_{\sigma_{n-1}})\right),$$

It is convenient to consider σ_{n-1} as a parameter and rewrite

$$g_m \equiv \sum_{i_j\in\{0,1\}} c_{i_3 i_4 \dots i_{n-2} m}\sigma_3^{i_3}\sigma_4^{i_4}\cdots\sigma_{n-2}^{i_{n-2}} = 0, \qquad m = 1, 2, \dots n-3,$$

where c now depends linearly on σ_{n-1}.
Computing the resultant is equivalent to impose conditions over the SE such that all the equations have a common solution
$\rightarrow$ polynomial in σ_{n-1} whose should be of order $(n-3)!$.

$$M_{1\,m} = g_m(\sigma_3, \ldots, \sigma_{n-2}),$$
$$M_{i\,m} = \frac{g_m(\tilde\sigma_3, \ldots, \tilde\sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{n-2}) - g_m(\tilde\sigma_3, \ldots, \tilde\sigma_i, \sigma_{i+1}, \ldots, \sigma_{n-2})}{\tilde\sigma_i - \sigma_i}\,.$$

$\det(M) \equiv B(\sigma, \tilde\sigma)$.
It can be proved that the monomials composing $B(\sigma, \tilde\sigma)$

$$\sigma_3^{\alpha_3} \cdots \sigma_{n-2}^{\alpha_{n-2}} \tilde\sigma_3^{\beta_3} \cdots \tilde\sigma_{n-2}^{\beta_{n-2}}\,,$$

satisfy $\alpha_i < i - 2, \quad \beta_i < (n-2) - i$. Hence,

$$B \in S(0, 1, \ldots, n-5) \otimes S(n-5, n-6, \ldots, 0)^*\,.$$

$B(\sigma, \tilde{\sigma})$ is interpreted as a linear map from the dual vector space $S(n-5, n-6, \ldots, 0)^*$ to $S(0, 1, \ldots, n-5) \to B_{(n-4)! \times (n-4)!}$,

$$B_{l\,m} = \left[\prod_{i=4}^{n-2} \frac{1}{\alpha_i!} \left(\partial_{\sigma_i}\right)^{\alpha_i} \prod_{j=3}^{n-3} \frac{1}{\beta_j!} \left(\partial_{\tilde{\sigma}_j}\right)^{\beta_j} \right] \det(M)|_{(\sigma,\tilde{\sigma})=0} \cdot$$

▸ Go back Notice each component $B_{l\,m}$ is a determinant.
$\det(B)$ is the resultant we were looking for. Size of M is $(n-3)$ and size of B is $(n-4)!$, hence the $\det(B)$ is a $(n-3) \times (n-4)! = (n-3)!$ order polynomial in σ_{n-1}.

A more illuminating way to compute the resultant is as follows. We consider the map

$$\begin{aligned} S(0,1,\dots,n-5)^{n-3} \quad &\to S(1,2,\dots,n-4), \\ \Phi:\ (f_1,f_2,\dots,f_{n-3}) \quad &\mapsto \sum_{i=1}^{n-3} f_i g_i. \end{aligned}$$

The dimension of the linear map Φ is $(n-3)! \times (n-3)!$. We introduce the notation

$$A^{(2)}_{j_1 j_2 \dots j_{n-5}} = \begin{pmatrix} c_{0 j_1 j_2 \dots j_{n-5} 1} & c_{0 j_1 j_2 \dots j_{n-5} 2} & \cdots & c_{0 j_1 j_2 \dots j_{n-5} (n-3)} \\ c_{1 j_1 j_2 \dots j_{n-5} 1} & c_{1 j_1 j_2 \dots j_{n-5} 2} & \cdots & c_{1 j_1 j_2 \dots j_{n-5} (n-3)} \end{pmatrix}$$

$$A^{(m)}_{j_1 j_2 \dots j_{n-m-3}} = \underbrace{\begin{pmatrix} A^{(m-1)}_{0 j_1 j_2 \dots j_{n-m-3}} & 0 & \cdots & 0 \\ A^{(m-1)}_{1 j_1 j_2 \dots j_{n-m-3}} & A^{(m-1)}_{0 j_1 j_2 \dots j_{n-m-3}} & \cdots & 0 \\ 0 & A^{(m-1)}_{1 j_1 j_2 \dots j_{n-m-3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{(m-1)}_{1 j_1 j_2 \dots j_{n-m-3}} \end{pmatrix}}_{m-1},$$

$m \geq 2$. The resultant of the system is given by,

$$\det A^{(n-3)} = 0.$$

The linear map Φ is $(n-3)! \times (n-3)!$ dimensional

$\Rightarrow$ We can solve for $(n-3)!$ monomials as functions of the parameters $c_{j_1 j_2 \dots j_{n-5} i}$ i.e, as a functions of σ_{n-1}. Particularly for the subset of monomials $(\sigma_3, \sigma_4, \dots, \sigma_{n-2})$.

It can be written in terms of subdeterminants of $A^{(n-3)}$, but we do not have a closed general form (i.e as fancy as $A^{(n-3)}$ in terms of recursions), although is straightforward case by case.

For a given polynomial f in a single variable σ_i, we define yet another matrix by the linear transformation,

$$T_{\sigma_i} : \ f \ \rightarrow \ \sigma_i f \, .$$

Go back From Stickelberger's theorem, the roots of f are the eigenvalues of the companion matrix T_{σ_i}.

By knowing the relation between the remaining variables σ_i and σ_{n-1} we can compute the remaining companion matrices.

$$\mathcal{A}_n = \sum_{\sigma_{sol}} \mathcal{F}(s_{ij}, \sigma_{ij}, \epsilon)\,.$$

This expression can always be rewritten as a rational function of polynomials as

$$\mathcal{A}_n = \sum_{\sigma_{sol}} \frac{P({}_{j_1}\sigma_3,\ {}_{j_2}\sigma_4, \cdots,\ {}_{j_{(n-3)!}}\sigma_{(n-3)!})}{Q({}_{j_1}\sigma_3,\ {}_{j_2}\sigma_4, \cdots,\ {}_{j_{(n-3)!}}\sigma_{(n-3)!})}\,,$$

we replace the solutions ${}_j\sigma_i$ by their corresponding companion matrices which are written in terms of $T_{\sigma_{n-1}}$. We can rewrite the amplitude schematically as,

$$\mathcal{A}_n = \mathrm{Tr}\left(\tilde{P}(T_{\sigma_{n-1}})\tilde{Q}^{-1}(T_{\sigma_{n-1}})\right) \equiv \mathrm{Tr}\left(\mathcal{P}(T_{\sigma_{n-1}})\right)\,.$$

$$g_i = \quad c_{00i} + c_{10i}\sigma_3 + c_{01i}\sigma_4 c_{11i}\sigma_3\sigma_4, \qquad i = 1, 2, 3.$$

bezoutian

$$M_6 = \begin{pmatrix} c_{00i} + c_{01i}\sigma_4 \\ c_{10i} + c_{11i}\sigma_4 \\ c_{01i} + c_{11i}\tilde{\sigma}_3 \end{pmatrix} .$$

The determinant of M is given by a polynomial in $S(0,1) \otimes S(1,0)^*$ and the B_6 in the basis $\{1, \sigma_4\} \otimes \{1, \tilde{\sigma}_3\}$ can be written as ,

$$B_6 = \begin{pmatrix} [00, 10, 01] & [00, 10, 11] \\ [00, 01, 11] & [10, 01, 11] \end{pmatrix} ,$$

$\det(B_6)_{2\times 2} = \det(A)_{6\times 6}$ $\rightarrow$ Poly. of order 6 whose coefficients define the elements of matrix companion

$$T_{\sigma_4} = \begin{pmatrix} 0 & 1 \\ -\dfrac{s_{12}s_{45}}{s_{14}s_{25}} & -\dfrac{s_{12}s_{25} - s_{13}s_{35} + s_{14}s_{45}}{s_{14}s_{25}} \end{pmatrix}.$$

Solving σ_3 in terms of σ_4

$$\sigma_3 = -\frac{s_{14}}{s_{13}}\sigma_4 - \frac{s_{12}}{s_{13}} \quad \Rightarrow T_{\sigma_3} = -\frac{s_{14}}{s_{13}}T_{\sigma_4} - \frac{s_{12}}{s_{13}}.$$

- We have developed a method for the evaluation of the S-matrix in theories satisfying the SE's in terms of the trace of a matrix.
- The data needed for the construction of such a matrix, is encoded in the determinant of another matrix, that can be computed straightforwardly from the coefficients of the SE's polynomials.
- In developing the method we have found nice mathematical structures hidden in the SE's, such as recursion relations satisfied by the resultants of such polynomials.
- Those resultants, can also be written in terms of brackets, which at the same time are the Chow form for the coordinates of projective spaces.

Many Thanks!