On perturbative unitarization of non-renormalizable theories

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2015 Dec NCTS Theory Meeting

Preulde

Non-renormalizable theories:

$$\mathcal{L} = \mathcal{L}_{marginal} + \alpha^n \mathcal{L}_n$$

The arbitrariness of α

- The need for a UV completed theory
- No IR constraints other than data

Prelude

Not true: Analyticity imposes IR constraint without any detailed knowledge of UV Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi

The existence of UV completion $\rightarrow$ c_i of higher dimension operators must be Positive

- Euler-Hiesenberg

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{\Lambda^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{c_2}{\Lambda^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \dots$$

- DBI

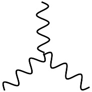
$$\mathcal{L} = -f^4\sqrt{1-(\partial y)^2} = f^4\Big[-1 + \frac{(\partial y)^2}{2} + \frac{(\partial y)^4}{8} + \dots\Big]$$

- String theory

$$\mathcal{M}^{\text{Regge}}(s, t\rightarrow 0) = -\psi_2(1)\, s^4 + \frac{-\psi_4(1)}{192}\, s^6 + \frac{-\psi_6(1)}{92160}\, s^8 + \dots$$

Prelude

What can we say about theories with massless three-point ?



($\lambda\phi^3$, YM, Gravity)$\leftarrow$ Focus of this talk, because All theories are EFT !

YM $D > 5$, $\lambda\phi^3$ $D > 6$, Gravity $D > 2$

Prelude

Are α_i as free as they appear?

$$\mathcal{L} = \int d^D x \sqrt{g}(R + \alpha_1 R^3 + \alpha_2 R^4 + \cdots)$$

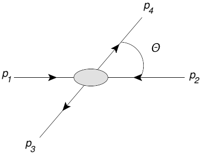
Instead of symmetries, consistency conditions?

Causality: if $\alpha_1 \neq 0$, a particle traveling pass a shock wave will experience time advancement Camanho, Edelstein, Maldacena, Zhiboedov

Can only be cured by an infinite $J > 2$ massive particles. (Assumes $\Lambda << M_p$)

Similar result from Unitarity and Analyticity Bellazzini, Cheung, Remmen

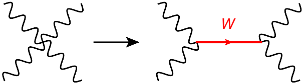
Preulde



The high energy behaviour of the four-point S-matrix must be bounded

- Fermi's four-fermion interaction

$$\mathcal{L}_{int} \sim (\bar{\psi}\psi)^2 \sim E^2$$



- Massive vector interactions Cornwall, Levin, Tiktopolous

$$\mathcal{L}_{int} \sim f_{abc} W^a_\mu W^b_\nu \partial^\nu W^{c\mu} \;\rightarrow\; f_{abc}f_{cde} + f_{adc}f_{ceb} + f_{aec}f_{cbd} = 0$$

- NLSM

$$\mathcal{L}_{int} \sim c_{abcd}\pi^a \partial^\mu \pi^b \pi^c \partial_\mu \pi^d \;\rightarrow\; c_{abcd} + c_{acdb} + c_{adbc} = 0$$

Prelude

What about the bad high energy behaviour ?

$$\mathcal{M}_4|_{s\to\infty} \sim \frac{s^2}{t} \sim E^2$$

- Assuming that the high energy behaviour is tamed while weakly coupled (true for all known examples in nature)
- Implies new degrees of freedom come in at tree-level
- Implies new degrees of freedom $M << M_{Plank}$

How constraining is the space of possible solutions?

Prelude

VERY

Tree Unitarity Constraint

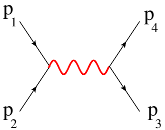
Fermi's four fermion theory

$$A(\phi_1, \phi_2, \bar{\phi}_3, \bar{\phi}_4) \sim -G_F s \sim E^2$$

Unitarity violation

$$s > \sqrt{\frac{16\pi}{G_F}} \sim 2.0\,TeV$$

There is a new particle!



$$A(\phi_1, \phi_2, \bar{\phi}_3, \bar{\phi}_4) \sim -G_F s \to G_F M^2 \frac{s}{s - M^2}$$

Unitarity then requires

$$\frac{G_F M^2}{8\pi} \leq \frac{1}{2} \to M < 1\,TeV$$

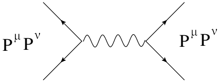
$M_W \sim 80\,GeV$

Unitarizing Gravity

Let's go back to gravity:

Consider scalar exchange via gravity coupling:

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle \sim \frac{(s^2+t^2+u^2)^2}{stu}$$



As $s \to \infty$ s/t fixed, same violation of unitarity bound at Plank scale

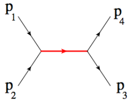
$$\sqrt{s} > 10^{19} GeV$$

Unitarizing Gravity

Let's consider modifying by adding one massive propagator

$$\langle \varphi_1\varphi_2\varphi_3\varphi_4\rangle \sim \frac{(s^2+t^2+u^2)^2}{stu}\frac{f(s,t,u)_{2,sym}}{(s-m^2)(t-m^2)(u-m^2)}$$

However, there are stringent unitarity constraints for $f(s,t,u)$



$$\rightarrow \quad A_3(\phi_1,\phi_2,h^\ell) \sim ic_\ell (p_1-p_2)^{\mu_1}(p_1-p_2)^{\mu_2}\cdots(p_1-p_2)^{\mu_\ell}\epsilon_{\mu_1\mu_2\cdots\mu_\ell}$$

The residue must take the simple form:

$$A_3(\phi_1,\phi_2,h^\ell)A_3(\phi_3,\phi_4,h^\ell) = -(t-u)^{2k}$$

with $u=-t-m^2$ definite negative function in t.

Unitarizing Gravity

There are also massless poles

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle \sim \frac{(s^2+t^2+u^2)^2}{stu} \frac{f(s,t,u)_{2,sym}}{(s-m^2)(t-m^2)(u-m^2)}$$

The residue for massless poles corresponds to interactions of EFT description

$$\mathcal{L} \sim R\phi^2 + (\nabla\phi)^2 + R^3 + R^2$$

Such interactions are limited, with unique:

$$A_3(\phi_1 h_2^- \phi_3) \sim \frac{\langle 12\rangle^2 \langle 32\rangle^2}{\langle 13\rangle^2}$$

Unitarizing Gravity

Let's consider unitizing

$$\langle\varphi_1\varphi_2\varphi_3\varphi_4\rangle \sim \frac{(s^2+t^2+u^2)^2}{stu}\frac{f(s,t,u)_{2,sym}}{(s-m^2)(t-m^2)(u-m^2)}$$

The rules of the game:

- At low energies, i.e. if $\sqrt{s}$ is smaller than the new mass scale M, then the S-matrix must reduce to the known amplitude.
- At intermediate scales, i.e. $0 \leq s \leq M^2$, the residues of the poles must be definite negative function of t:

$$n(t) = \sum_i a_i t^i, \quad a_i < 0$$

Unitarizing Gravity

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle \sim \frac{(s^2+t^2+u^2)^2}{stu} \frac{f(s,t,u)_{2,sym}}{(s-m^2)(t-m^2)(u-m^2)}$$

The rules of the game:

- The factorization pole of $s = 0$, Since for two scalars and one graviton, we only have $R\phi^2$, $(\nabla\phi)^2$, the residue must be the original tree residue
- On the massive pole the residue should be expanded on Gegenbauer basis with positive coefficient (Powerful)

Unitarizing Gravity

$$Ans_1 = -\frac{m^3(s^2+t^2+u^2)^2}{stu}\,\frac{a_1(t^2+u^2)+a_2tu+a_3s+1}{(s-1)(t-1)(u-1)}$$

Enforcing massless residues,

$$Res_{\mathrm{s=0}} : -4t^2, \quad Res_{\mathrm{t=0}} : \; -4s^2$$

This fixes

$$Ans_1 = -\frac{m^3(s^2+t^2+u^2)^2}{stu}\,\frac{1-(s^2+t^2+u^2)/2}{(s-1)(t-1)(u-1)}$$

However, consider the residue at $s = 1$

$$Res[Ans_1]_{s=1} = \frac{1}{(t-1)(t+2)}\,.$$

The residue changes sign when $t \sim 1$

Unitarizing Gravity

We can add another massive particle such that

$$Ans_2 = -\frac{m^3(s^2+t^2+u^2)^2}{stu}\,\frac{n}{(s-1)(t-1)(u-1)(s-b)(t-b)(u-b)}$$

where

$$\begin{aligned} n &= a_1(t^5+u^5)+a_2(t^4u+u^4t)+a_3(t^3u^2+u^3t^2)+a_4(t^4+u^4)+a_5(t^3u+u^3t)\\ &+ a_6t^2u^2+a_7(t^3+u^3)+a_8(tu^2+ut^2)+a_9(t^2+u^2)+a_{10}tu+a_{11}(t+u)-b^3 \end{aligned}$$

Consistency for massless as well as massive poles below the scale set by b yields

$$n = tu(a_2s^2-btu)+bs(s^2+\alpha_1st+\alpha_1t^2)+b(s^2+\alpha_2st+\alpha_2t^2)-b^3s-b^3$$

With $\alpha_1 = 1+a_2-b,\ \alpha_2 = 1+b^2.$

Unitarizing Gravity

However

$$Res[Ans_2]|_{s=b} = -\frac{1}{(b-t)(2b+t)}$$

On both residues, unitarity is again violated at the mass of the second new particle.

Going to three massive particles yield the same conclusion

Unitarizing Gravity

Summary:

- Introducing massive particles brings the unitarity disaster down to the scale of the heaviest particle.
- One could release bounds on the polynomial, and require high energy taming later $\rightarrow M_4 \sim E^8$ Unitarity violated at $10^{4.5}$GeV

Unitarizing Gravity

Non-polynomial high energy behavior incompatible with tree unitary for gravitational theories!

The root of the problem

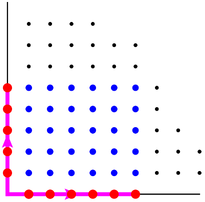
$$M \sim \frac{(s^2+t^2+u^2)^2}{stu}\frac{f(s,t,u)}{(s-m_1)(t-m_2)\cdots}\bigg|_{s=m_1} \rightarrow \frac{(s^2+t^2+u^2)^2}{stu}\frac{f(m_1,t,u)}{(t-m_2)\cdots}$$

Unitarity requires the function $f(m_1,t,u)$ to have a zero when $t=m_2$, and all other t-channel poles.

Unitarizing Gravity

The root of the problem

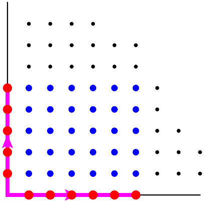
$$M \sim \frac{(s^2+t^2+u^2)^2}{stu}\,\frac{f(s,t,u)}{\color{red}{(s-m_1)}(t-m_2)\cdots}\Bigg|_{s=m_1} \to \frac{(s^2+t^2+u^2)^2}{stu}\,\frac{f(m_1,t,u)}{(t-m_2)\cdots}$$



$f(s,t,u)$ is a bounded polynomial function that has zero for each pair of $(s,t)=(m_i,m_j)$! There are more zeros than poles!

Unitarizing Gravity

For the polynomial to be bounded



The only way possible is if the zeros are shared by multiple double poles!

For single massive pole m_1^2, two zeros

$$(s=0, t=m_1^2) \to (u=-m_1^2), \quad (s=m_1^2, t=m_1^2) \to \color{red}{(u=-2m_1^2)}$$

Let $(u=-2m_1^2)$ be shared

$$(s=0, t=2m_1^2) \to (u=-2m_1^2), \quad (s=2m_1^2, t=0) \to (u=-2m_1^2)$$

m^2 is spread across integers

Unitarizing Gravity

In general, the solution is given as:

$$A(s,t)\frac{\prod_{i=1}^{\infty}(s+i)(t+i)(u+i)}{\prod_{i=0}^{\infty}(s-i)(t-i)(u-i)}$$

$A(s,t)$ can be fixed by considering residues of massless pole

$$(s^2+t^2+u^2)\frac{\Gamma[-\alpha' u]\Gamma[-\alpha' t]\Gamma[-\alpha' s]}{\Gamma[1+\alpha' t]\Gamma[1+\alpha' u]\Gamma[1+\alpha' s]}$$

Unitarizing Gravity

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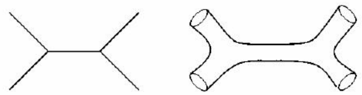


Fig.3: Particle scattering processes (left), string scattering processes (right).

Type II superstring

Unitarizing Gravity

How unique is the answer ?

We can consider

- $$\frac{\Gamma[f(s)]\Gamma[f(t)]\Gamma[f(u)]}{\Gamma[f(s)+f(t)+a]\Gamma[f(s)+f(u)+a]\Gamma[f(t)+f(u)+a]}$$

  Fixed by massless residuees.

- $$\frac{\Gamma[s]\Gamma[t]\Gamma[u]}{\Gamma[1-t]\Gamma[1-u]\Gamma[1-s]} \times (1+stu\Delta)$$

  Ruled out via Gegenbauer analysis

Gegenbaueroligy

There is much stronger constraint for the massive residues

$$A_3(\phi_1, \phi_2, h^\ell) A_3(\phi_3, \phi_4, h^\ell) = -(t-u)^{2k}$$

$$n(t) = \sum_\ell a_\ell t^\ell, \qquad a_\ell < 0$$

Project the residue into irreducible representation

$$n(t) = n\left(-\frac{s(1-\cos\theta)}{2}\right) = \sum_\ell c_\ell C_\ell^D(\cos\theta)$$

with

$$c_\ell \geq 0$$

Gegenbaueroligy

Schoenberg (1938) Theorem

$$f(x) = \sum_{\ell} c_\ell C_\ell^D(x), \quad c_\ell \geq 0$$

If and only if $f(x)$ is a positive function in the sense that for points v_i on S^{D-1}

$$\sum_{i,j} f(\langle v_i, v_j \rangle) c_i c_j > 0$$

for $c_i, c_j \in R$ and $\langle v_i, v_j \rangle$ is the spherical geodesic.
For two points this implies

$$|f(x)| \leq f(1)$$

Gegenbaueroligy

$$|f(x)| \leq f(1)$$

But

$$stu = \frac{s^3}{4}(1 - x^2)$$

So anything with an *stu* factor cannot be positive!!!!

$$\frac{\Gamma[s]\Gamma[t]\Gamma[u]}{\Gamma[1-t]\Gamma[1-u]\Gamma[1-s]} \times (1 + stu\Delta)$$

At large $s = n$ $n \to \infty$ the residue will be dominated by $stu\Delta$

No deformations possible!

Gegenbaueroligy

One exception: unitarize each massless pole separately:

$$\frac{(s^2+t^2+u^2)^2}{stu} = (s^2+t^2+u^2)\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)$$

The absence of massless poles in other channels leads to the absence of zeros

$$(s^2+t^2+u^2)\frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]}{\Gamma[1+s]\Gamma[1+t]\Gamma[1+u]}\left(\frac{tu}{1+s}+\frac{su}{1+t}+\frac{st}{1+u}\right)$$

Heterotic superstring

Gegenbaueroligy

Extensions to other mass-less three-point theories

For YM

$$(s^2+t^2+u^2)\left(\frac{\Gamma[-s]\Gamma[-t]}{\Gamma[1+u]}\right)$$

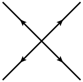
For $\lambda\phi^3$

$$(s^2+t^2+u^2)\frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]}{\Gamma[1+s]\Gamma[1+t]\Gamma[1+u]}$$

$\lambda\phi^3 - \lambda\phi^4$

$$\frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]}{\Gamma[1+s]\Gamma[1+t]\Gamma[1+u]}\left(\frac{tu}{1+s}+\frac{su}{1+t}+\frac{st}{1+u}\right)$$

Gegenbaueroligy

Perturbative completions of four-point contact terms 

$$(\partial\phi)^2\Box^n\phi^2$$

We cannot remove the massless pole by $\times stu$

let them cancel $$\left(\frac{\Gamma[-s]\Gamma[-t]}{\Gamma[1+u]}+\frac{\Gamma[-u]\Gamma[-t]}{\Gamma[1+s]}+\frac{\Gamma[-s]\Gamma[-u]}{\Gamma[1+t]}\right)$$

A poitive function multiply by a positive function yields positive function

$$C_m(x)C_n(x)=a_1C_{m+n}(x)+\cdots a_pC_{|m-n|}(x),\qquad a_p>0$$

$$(s^2+t^2+u^2)^n\quad\left(\frac{\Gamma[-s]\Gamma[-t]}{\Gamma[1+u]}+\frac{\Gamma[-u]\Gamma[-t]}{\Gamma[1+s]}+\frac{\Gamma[-s]\Gamma[-u]}{\Gamma[1+t]}\right)$$

Extended Lessons

$$(s^2+t^2+u^2)^n \quad \left(\frac{\Gamma[-s]\Gamma[-t]}{\Gamma[1+u]}+\frac{\Gamma[-u]\Gamma[-t]}{\Gamma[1+s]}+\frac{\Gamma[-s]\Gamma[-u]}{\Gamma[1+t]}\right)$$

At low energy corresponds to

$$A_4 \sim s^2,\ s^4,\ s^6\ \cdots$$

(s^2) DBI, (s^4) Galileons DGP (Dvali-Gabadadze-Porrati) Each has a unique perturbative completion, but

$$\text{In the forward limit} \quad \rightarrow \frac{A_4}{s} tan\pi s$$

Violates the Froissart bound $s \log s$ for s^n $n > 2$

Conclusions

- Assuming tree-unitarization while gravity is weakly coupled, and infinite tower of massive spin is necessary
- Their mass^2 must have integer spacing.
- Positivity of Gegenbauer expansion renders the solution almost unique
- EFT without massless three-points also admit unique unitarization
- The basic building block for positive residue: (Open string)

$$n(x) = \Pi_{i=1}^{2\ell}\left(x - \frac{2\ell+1-2i}{2\ell+1}\right), \quad x = \cos(\theta)$$

Is this the answer to another question ?