20 November 2015 at ”8th Taiwan String Workshop”, National Tsing Hua University, Taiwan

Size scaling of self-gravitating polymers and strings

Shoichi Kawamoto
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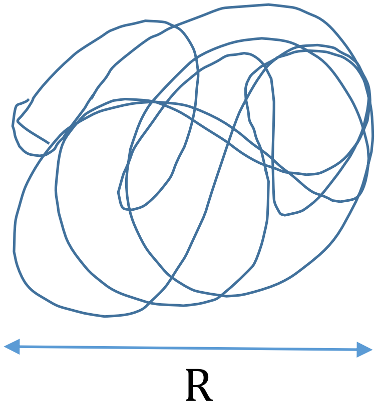
arXiv:1506.01160 [hep-th] (to appear in PTEP)

with Toshihiro Matsuo (NIT, Anan College, Japan)

Long Strings and Black Holes

a free string of level $\tilde{N}$

[Mitchell-Turok, Mañes]



$$\langle R^2 \rangle_{\tilde{N}} \sim \int d\sigma \langle : X^2(0,\sigma) : \rangle_{\tilde{N}} \sim \sqrt{\tilde{N}} \alpha' \sim \text{Length}$$

Free random walk of $N = \sqrt{\tilde{N}}$ steps

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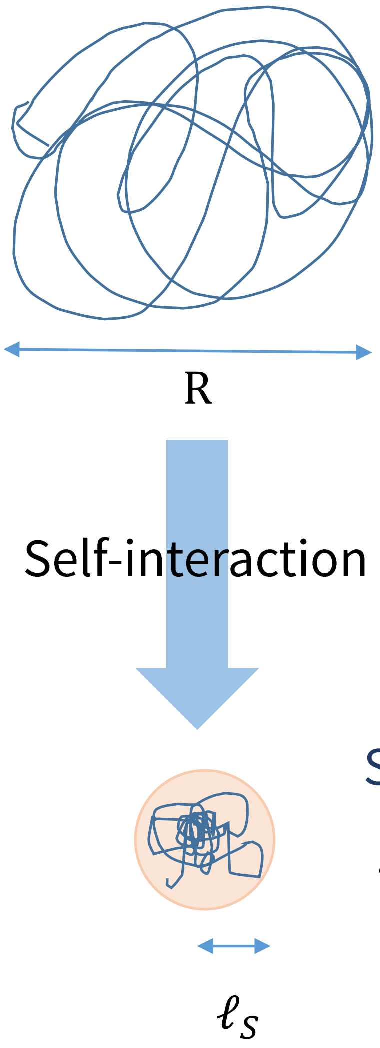

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Free random walk of $N = \sqrt{\widetilde{N}}$ steps

Self-interaction

String/black hole correspondence

Small black hole

$S_{\text{string}} \cong S_{\text{BH}}$

[Susskind, Horowitz-Polchinski]

$$R \cong \frac{\ell_s}{g_s^2\sqrt{\widetilde{N}}} \qquad (d = 3)$$

Random walks and Polymers

Typical configuration of long strings → (Interacting) random walks

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(Self-)interacting 1D objects

- Cosmic strings
- Vortex lines
- Polymer chains
- …

Random walks and Polymers

Typical configuration of long strings → (Interacting) random walks

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- $\cdots$

Coil-globule transition in polymer melt

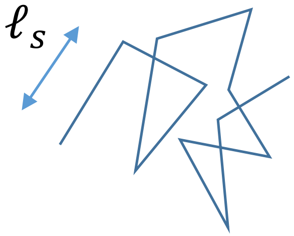
Interactions
(solvent, van der Waals, $\cdots$)

Coil$\sim N^{\frac{3}{5}}$

Θ-point

Globule$\sim N^{\frac{1}{3}}$

Strings as self-avoiding walks



Fundamental strings are modelled by free (Gaussian) random walks with the bond length ℓ_s.

$$R_0 \cong \ell\sqrt{N}$$

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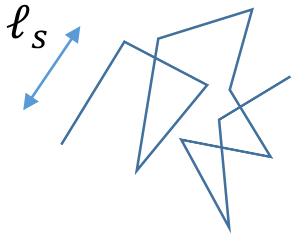

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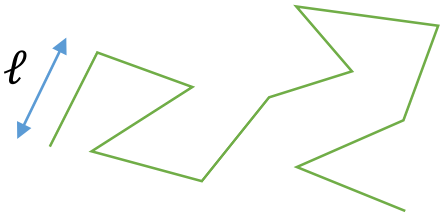
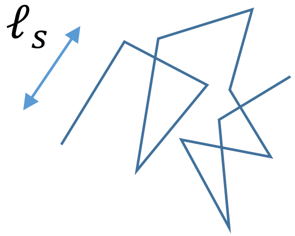


(Real) polymers are modelled by self-avoiding random walks with the (Kuhn) bond length ℓ.

$$R \cong \ell N^{\frac{3}{d+2}}$$

(Flory's exponent for real polymers)

Strings as self-avoiding walks



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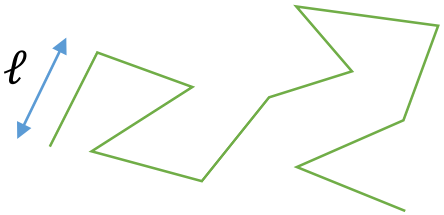

(Real) polymers are modelled by self-avoiding random walks with the (Kuhn) bond length ℓ.

$$\longrightarrow \quad R \cong \ell N^{\frac{3}{d+2}}$$

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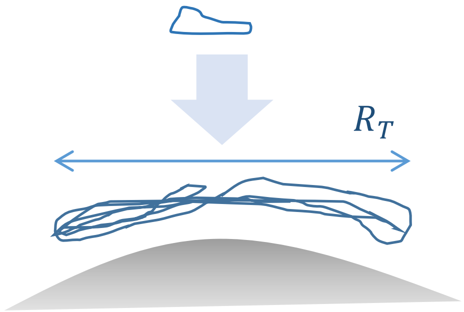
Note: Repulsive property may emerge in high-density regime nonperturbatively.

[Susskind-Lindesay, Ropotenko]



$$L \propto e^t$$

$$R_T \propto t \quad \longrightarrow \quad R_T \propto e^t$$

Effective Hamiltonian for Polymers

Edwards Hamiltonian

[Edwards-Muthukumar, Doi-Edwards]

$$\beta H = \frac{d}{2\ell^2}\int_0^N \left(\frac{\partial \boldsymbol{R}}{\partial \sigma}\right)^2 d\sigma \;+ \int_0^N d\sigma_1 \int_0^N d\sigma_2\, V\big(\boldsymbol{R}(\sigma_1), \boldsymbol{R}(\sigma_2)\big)$$

Interaction term

$$V = \underbrace{\frac{-g^2\,\ell^{d-2}}{|\boldsymbol{R}(\sigma_1) - \boldsymbol{R}(\sigma_2)|^{d-2}}}_{\text{Newton Interaction}} + \underbrace{u\,\ell^d \delta^{(d)}(\boldsymbol{R}(\sigma_1) - \boldsymbol{R}(\sigma_2))}_{\text{Repulsive force (Self-Avoiding effect)}}$$

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Evaluate the size (free case)

$$\langle \boldsymbol{R}^2 \rangle_{V=0} = \ell^2 N \equiv R_0^2$$

Free random walk: $R_0 = \ell\sqrt{N}$

Phenomenological Free Energy:

$$\beta F \sim -(d-1)\ln R + \frac{R^2}{N\ell^2} - \frac{g^2 \ell^{d-2} N^2}{R^{d-2}} + \frac{u \ell^d N^2}{R^d}$$

diffusion **elasticity** **gravity** **repulsive**

(excluded-volume effect)

Entropic force

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Balance determines the size: Free case $R_0 = \ell\sqrt{N}$

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Entropic force

Balance determines the size: Free case $R_0 = \ell\sqrt{N}$

Interaction becomes effective in free configurations:

$$\frac{g^2 \ell^{d-2} N^2}{R_0^{d-2}} \sim O(1) \qquad g_o \sim N^{\frac{d-6}{4}}$$

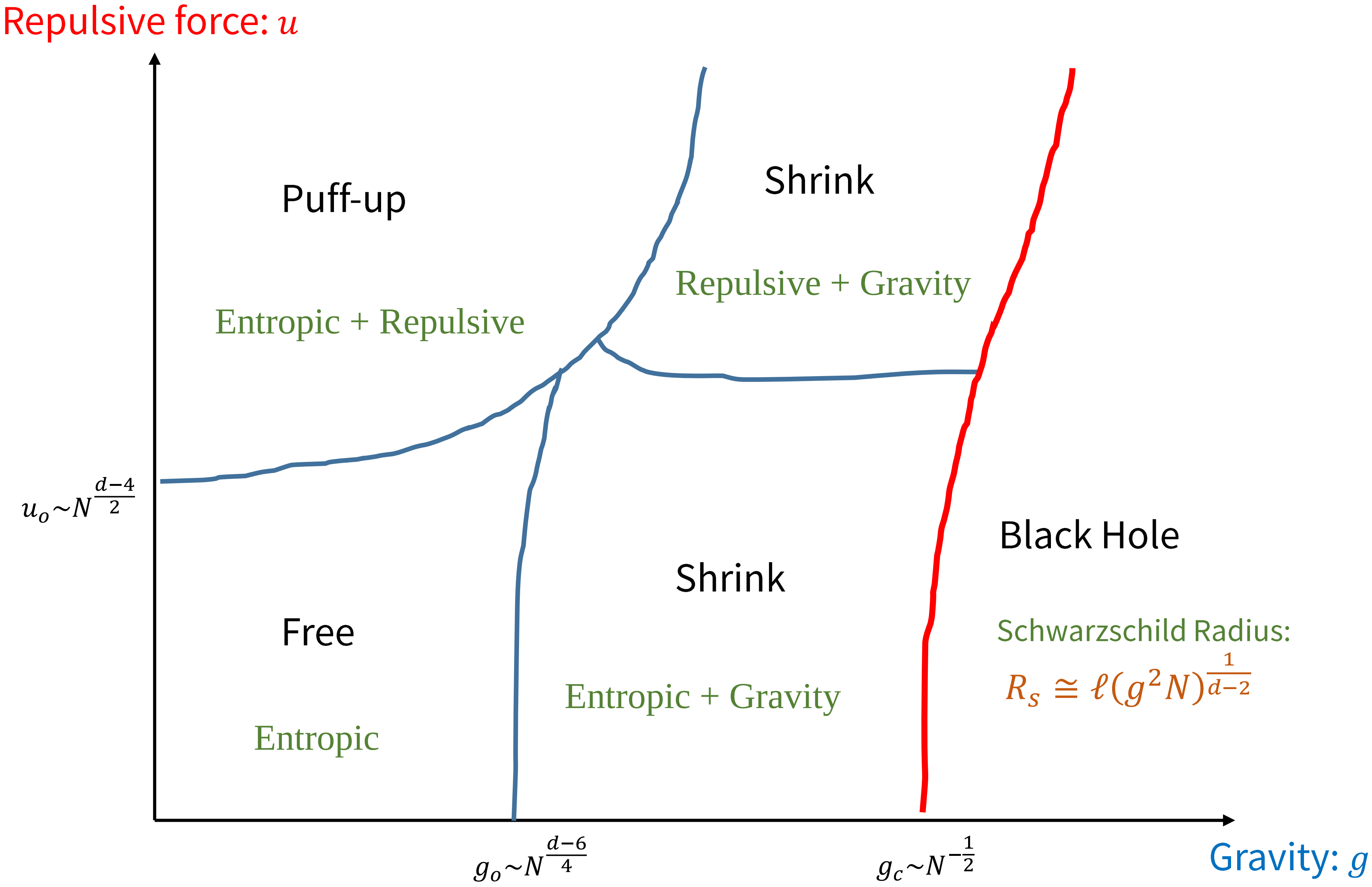
[Horowitz-Polchinski, Khuri]

⇒

$$\frac{u \ell^d N^2}{R_0^d} \sim O(1) \qquad u_o \sim N^{\frac{d-4}{2}}$$

($d = 4$ is critical dim.)

Today’s Goal



Repulsive force: u

Puff-up

Entropic + Repulsive

Shrink

Repulsive + Gravity

$u_o \sim N^{\frac{d-4}{2}}$

Free

Entropic

Shrink

Entropic + Gravity

Black Hole

Schwarzschild Radius:

$R_s \cong \ell(g^2 N)^{\frac{1}{d-2}}$

$g_o \sim N^{\frac{d-6}{4}}$

$g_c \sim N^{-\frac{1}{2}}$

Gravity: g

PLAN

- Introduction
- Two Approximation Methods and Size Evaluation
- Summary of Size Scaling
- Conclusion

[Doi-Edwards]

Variational Method

Harmonic Hamiltonian: $$\beta H_0 = \frac{d}{2\ell^2}\int_0^N d\sigma \left(\frac{\partial \boldsymbol{R}}{\partial \sigma}\right)^2 + \frac{q^2 d}{2\ell^2}\int_0^N d\sigma\, \boldsymbol{R}(\sigma)^2$$

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Convexity (Jensen’s Inequality): $\beta F \leq \beta F_0(q) + \langle \beta(H - H_0)\rangle_0$

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Tune q to minimize RHS

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Tune q to minimize RHS

$$\langle R^2 \rangle_0 = \frac{\ell^2}{q_0} \tanh q_0 N \begin{cases} \ell^2 N \quad \text{(free size)} & (q_0 N \ll 1) \\ \dfrac{\ell^2}{q_0} \quad \text{(contraction)} & (q_0 N \geq O(1)) \end{cases}$$

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$(2 < d < 4)$

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$$\beta F \leq qN - N^2 g^2 q^{\frac{d}{2}-1} + N^2 u q^{\frac{d}{2}}$$

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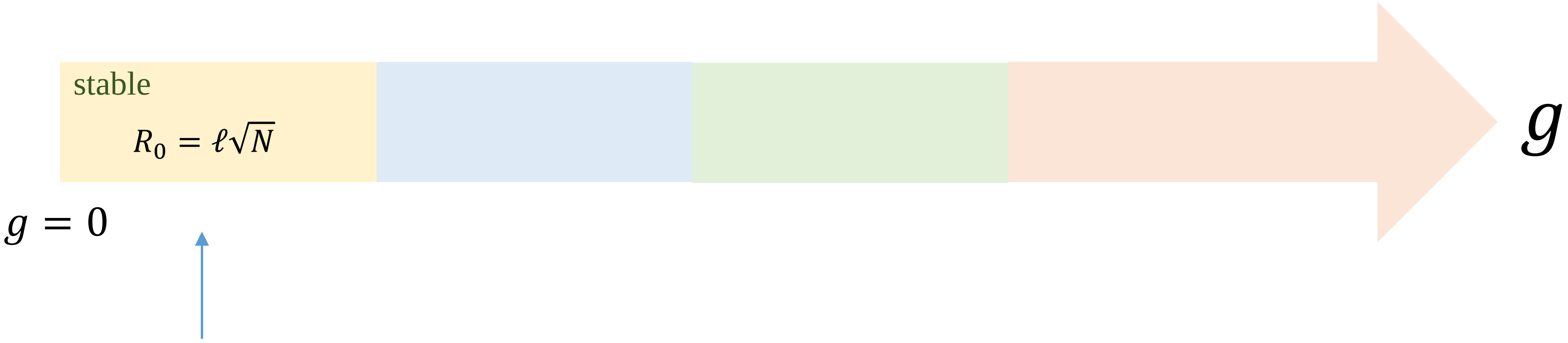

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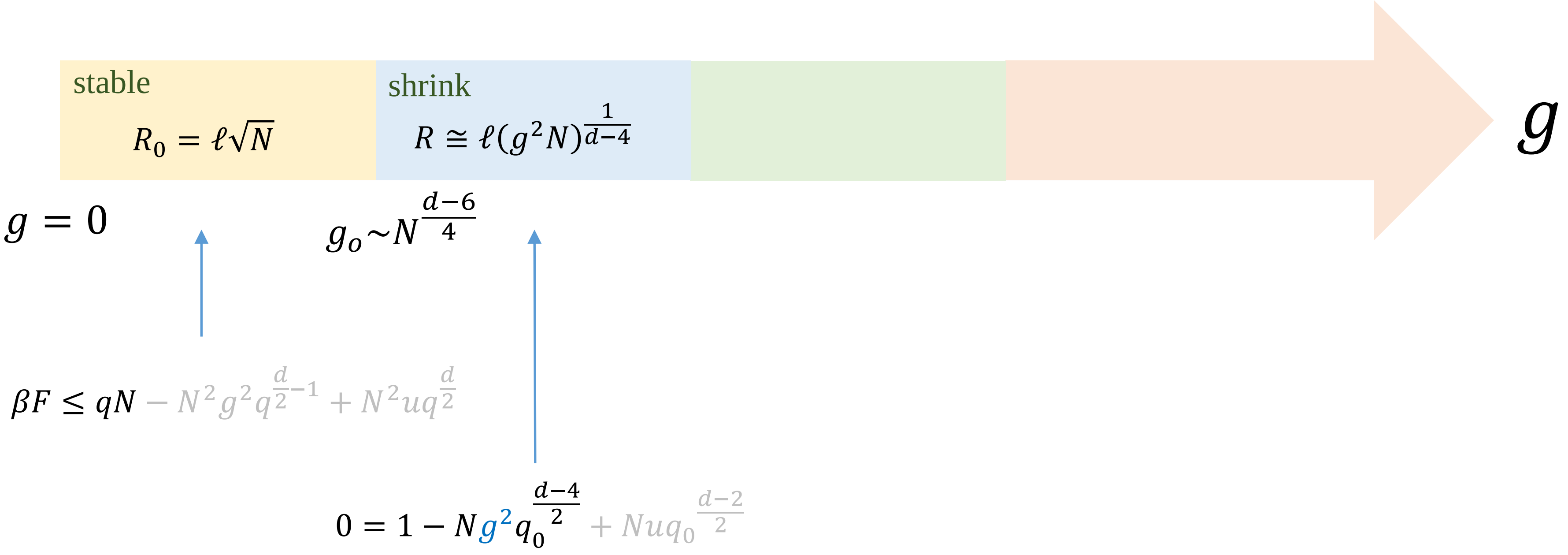

stable

$R_0 = \ell\sqrt{N}$

shrink

$R \cong \ell(g^2 N)^{\frac{1}{d-4}}$

g

$g = 0$

$g_o \sim N^{\frac{d-6}{4}}$

$\beta F \le qN - N^2 g^2 q^{\frac{d}{2}-1} + N^2 u q^{\frac{d}{2}}$

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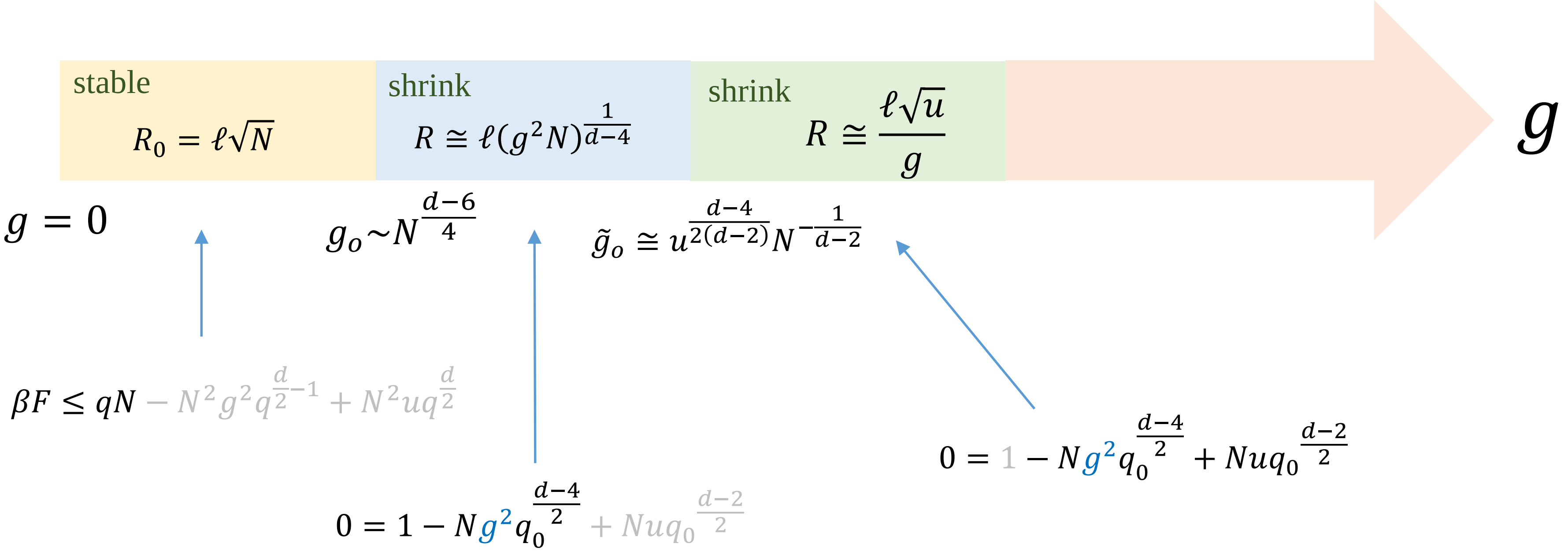

stable $R_0 = \ell\sqrt{N}$

shrink $R \cong \ell (g^2 N)^{\frac{1}{d-4}}$

shrink $R \cong \frac{\ell\sqrt{u}}{g}$

g

$g = 0$

$g_o \sim N^{\frac{d-6}{4}}$

$\tilde{g}_o \cong u^{\frac{d-4}{2(d-2)}} N^{-\frac{1}{d-2}}$

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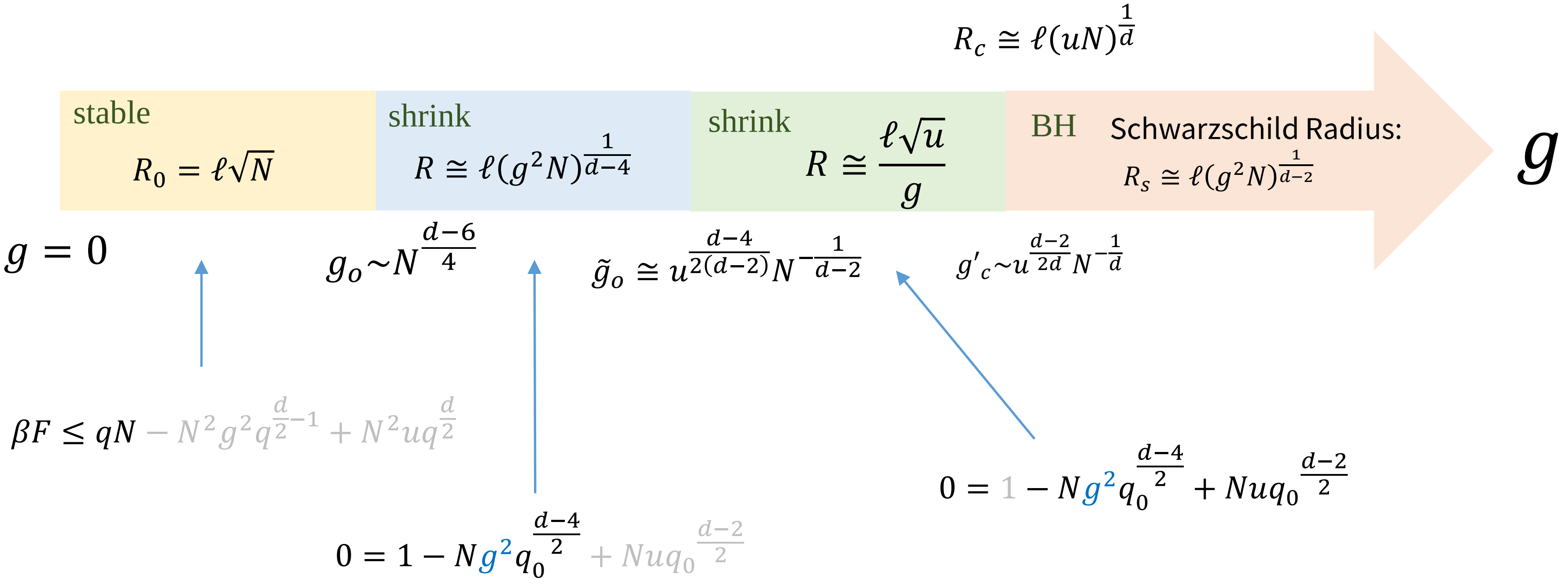

$R_c \cong \ell (uN)^{\frac{1}{d}}$

stable $R_0 = \ell\sqrt{N}$

shrink $R \cong \ell (g^2 N)^{\frac{1}{d-4}}$

shrink $R \cong \dfrac{\ell\sqrt{u}}{g}$

BH Schwarzschild Radius: $R_s \cong \ell (g^2 N)^{\frac{1}{d-2}}$

g

$g = 0$

$g_o \sim N^{\frac{d-6}{4}}$

$\tilde{g}_o \cong u^{\frac{d-4}{2(d-2)}} N^{-\frac{1}{d-2}}$

$g'_c \sim u^{\frac{d-2}{2d}} N^{-\frac{1}{d}}$

$$\beta F \le qN - N^2 g^2 q^{\frac{d}{2}-1} + N^2 u q^{\frac{d}{2}}$$

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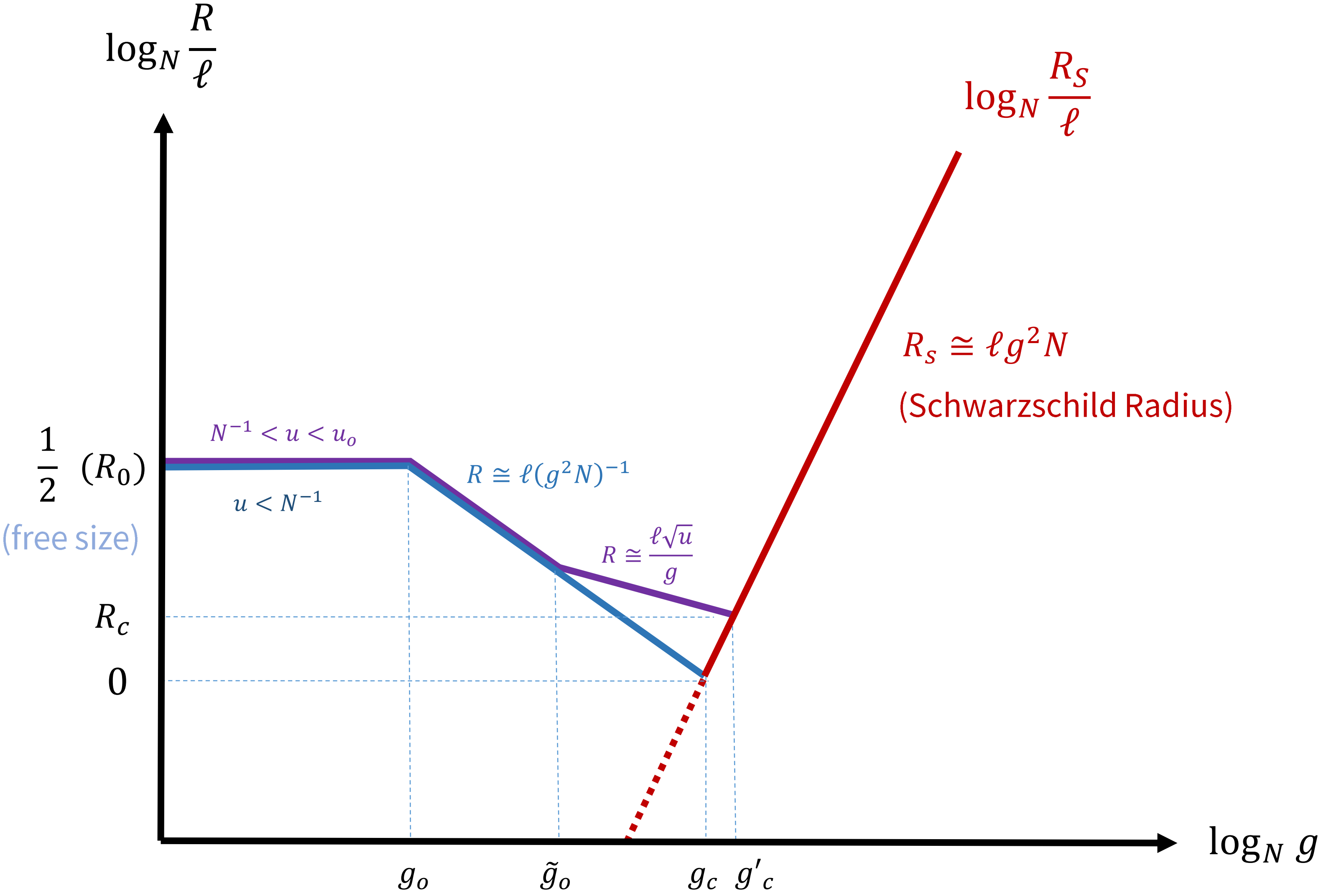
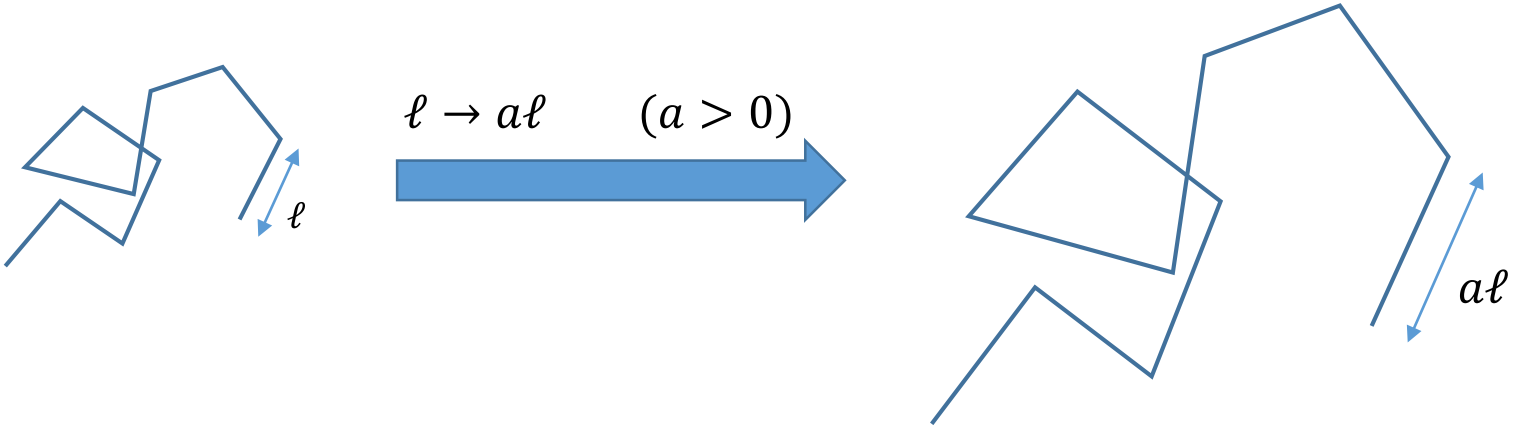
Cf. [Horowitz-Polchinski, Khuri]

Size change ($d = 3$)

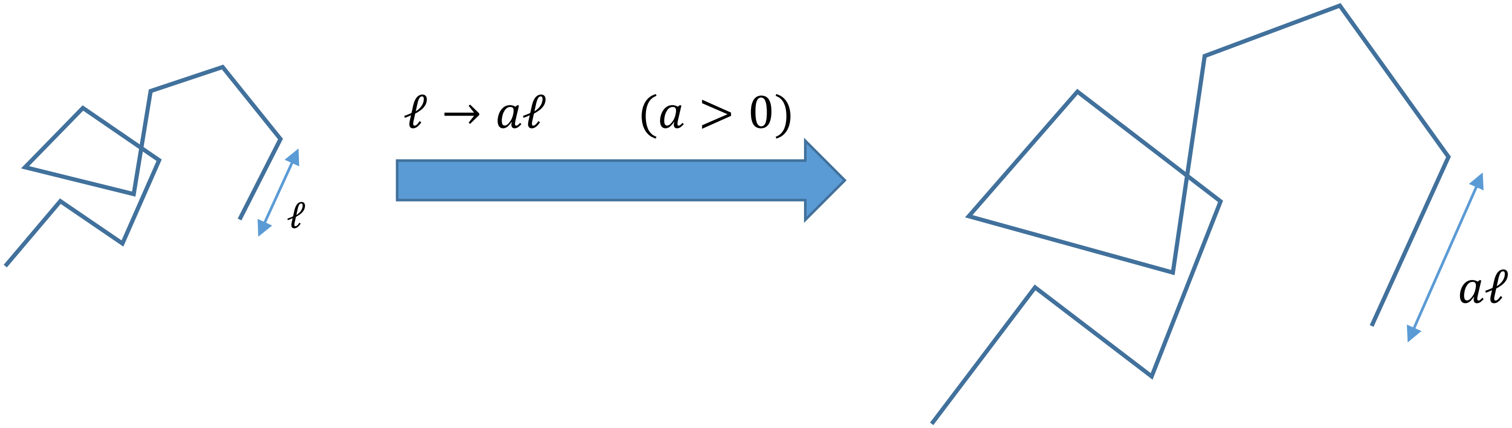


Uniform Expansion Model (UEM)



- Bond length is rescaled
- The configuration remains free walk one.

Uniform Expansion Model (UEM)



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$$R = aR_0 = a\ell\sqrt{N}$$

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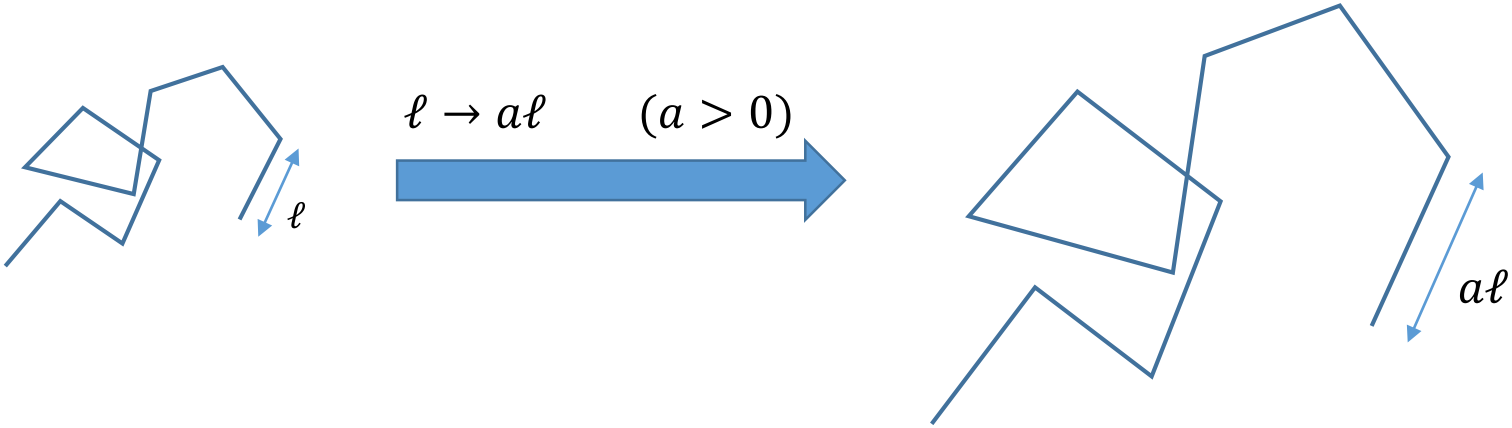

- Bond length is rescaled
- The configuration remains free walk one.

$$\boxed{R = aR_0 = a\ell\sqrt{N}}$$

Free (Gaussian) Hamiltonian with the bond length $a\ell$

$$\beta H' = \frac{d}{2a^2\ell^2}\int_0^N d\sigma \left(\frac{\partial \boldsymbol{R}}{\partial \sigma}\right)^2$$

$$\langle A\rangle' \equiv \frac{1}{Z'}\int A e^{-\beta H'}$$

Evaluate the mean-size-squared:

$$\langle \boldsymbol{R}^2\rangle = \frac{\int \left(\boldsymbol{R}(N) - \boldsymbol{R}(0)\right)^2 e^{-\beta H}}{\int e^{-\beta H}} = \frac{\left\langle e^{-\beta(H-H')}\left(\boldsymbol{R}(N) - \boldsymbol{R}(0)\right)^2\right\rangle'}{\left\langle e^{-\beta(H-H')}\right\rangle'}$$

$$\langle \boldsymbol{R}^2 \rangle = \frac{\int \left(\boldsymbol{R}(N) - \boldsymbol{R}(0)\right)^2 e^{-\beta H}}{\int e^{-\beta H}} = \frac{\left\langle e^{-\beta\left(H-H'\right)}\left(\boldsymbol{R}(N) - \boldsymbol{R}(0)\right)^2\right\rangle'}{\left\langle e^{-\beta(H-H')}\right\rangle'}$$

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$$\cong \left\langle \left(\boldsymbol{R}(N) - \boldsymbol{R}(0)\right)^2 \right\rangle' \ (1 + \langle \beta(H - H') \rangle') - \left\langle \beta(H - H')\left(\boldsymbol{R}(N) - \boldsymbol{R}(0)\right)^2 \right\rangle'$$

$$+O([\beta(H - H')]^2)$$

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$$+O([\beta(H-H')]^2)$$

$$\cong \underbrace{Na^2\ell^2}_{\text{Required Size (free walk config.)}} + \underbrace{\left[a^d(1-a^2) + C_1 u N^{\frac{4-d}{2}} - C_2 g^2 N^{\frac{6-d}{2}} a^2\right]}_{=0} N\ell^2 a^{2-d}$$

C_1, C_2: Positive N independent constants

$$\langle \boldsymbol{R}^2 \rangle = \frac{\int \left(\boldsymbol{R}(N) - \boldsymbol{R}(0)\right)^2 e^{-\beta H}}{\int e^{-\beta H}} = \frac{\left\langle e^{-\beta(H-H')}\left(\boldsymbol{R}(N) - \boldsymbol{R}(0)\right)^2 \right\rangle'}{\left\langle e^{-\beta(H-H')} \right\rangle'}$$

$$\cong \left\langle \left(\boldsymbol{R}(N) - \boldsymbol{R}(0)\right)^2 \right\rangle' \left(1 + \langle \beta(H-H') \rangle'\right) - \left\langle \beta(H-H')\left(\boldsymbol{R}(N) - \boldsymbol{R}(0)\right)^2 \right\rangle'$$

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Required Size
(free walk config.)

C_1, C_2: Positive N independent constants

Consistency condition: $a^d - a^{d+2} + u N^{\frac{4-d}{2}} - g^2 N^{\frac{6-d}{2}} a^2 = 0$

Size: $R = \ell\, a \sqrt{N}$

$$a^d - a^{d+2} + u N^{\frac{4-d}{2}} - g^2 N^{\frac{6-d}{2}} a^2 = 0$$

$$R = \ell \, a \sqrt{N}$$

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$$R = \ell\, a\sqrt{N}$$

1. Pure repulsive: $g^2 = 0, \quad u > 0$

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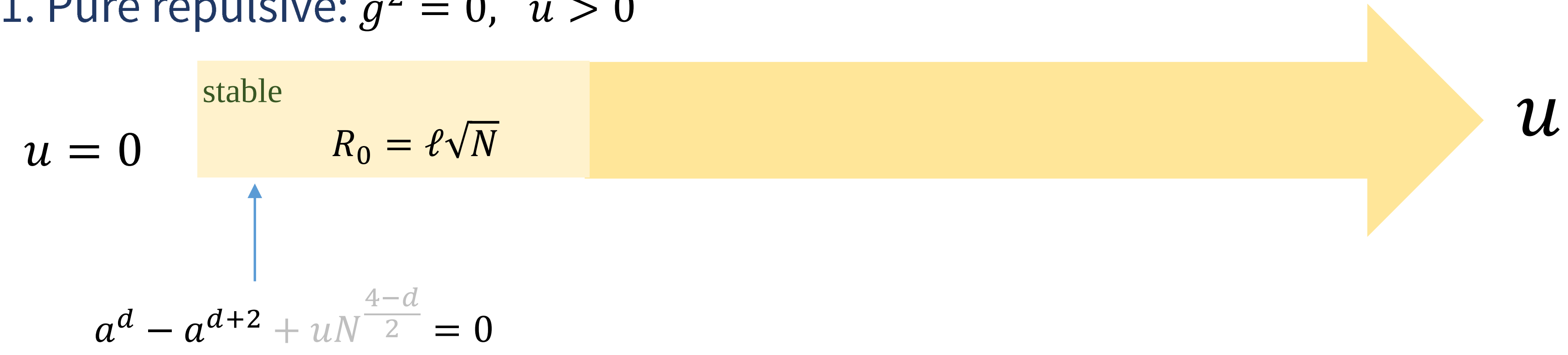
$$R = \ell\, a\sqrt{N}$$

1. Pure repulsive: $g^2 = 0, \quad u > 0$



$u = 0$

stable

$R_0 = \ell\sqrt{N}$

u

$$a^d - a^{d+2} + uN^{\frac{4-d}{2}} = 0$$

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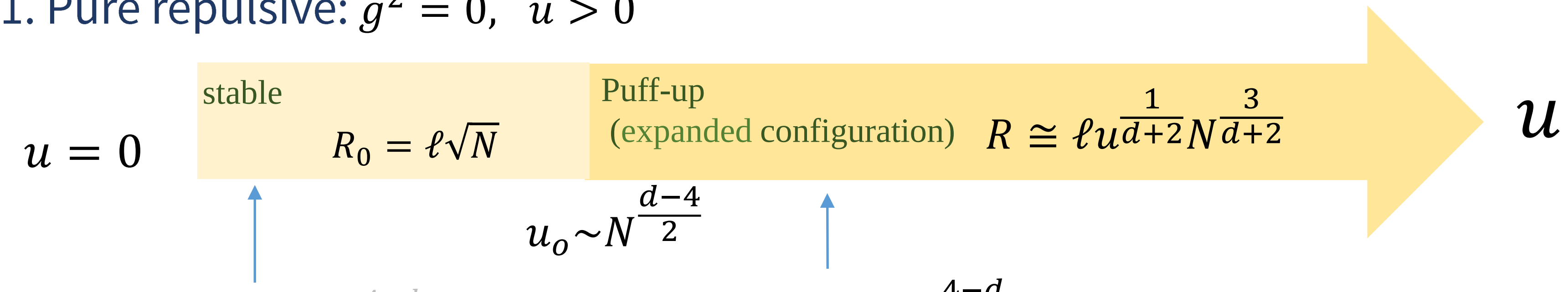

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Puff-up

(expanded configuration) $R \cong \ell u^{\frac{1}{d+2}} N^{\frac{3}{d+2}}$

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$u_o \sim N^{\frac{d-4}{2}}$

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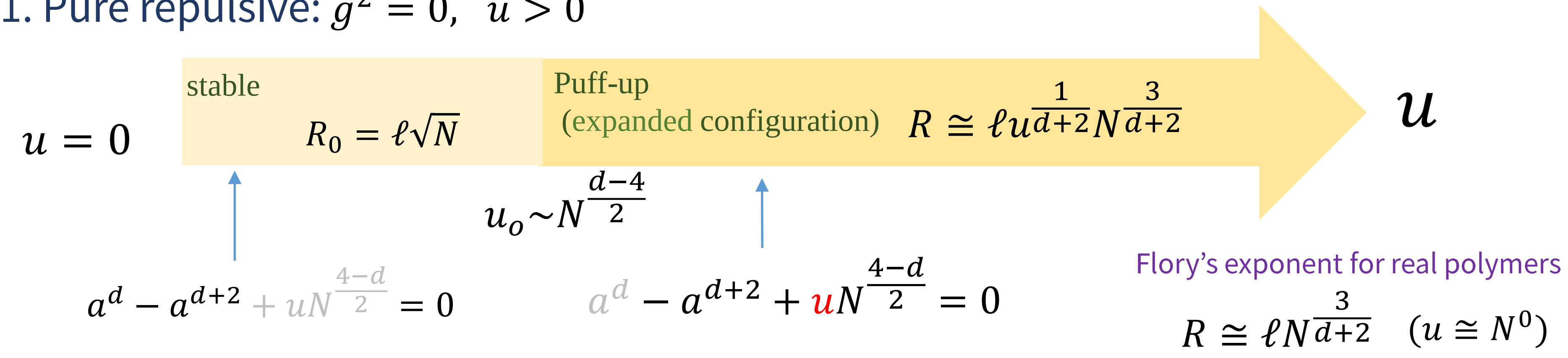

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Flory's exponent for real polymers

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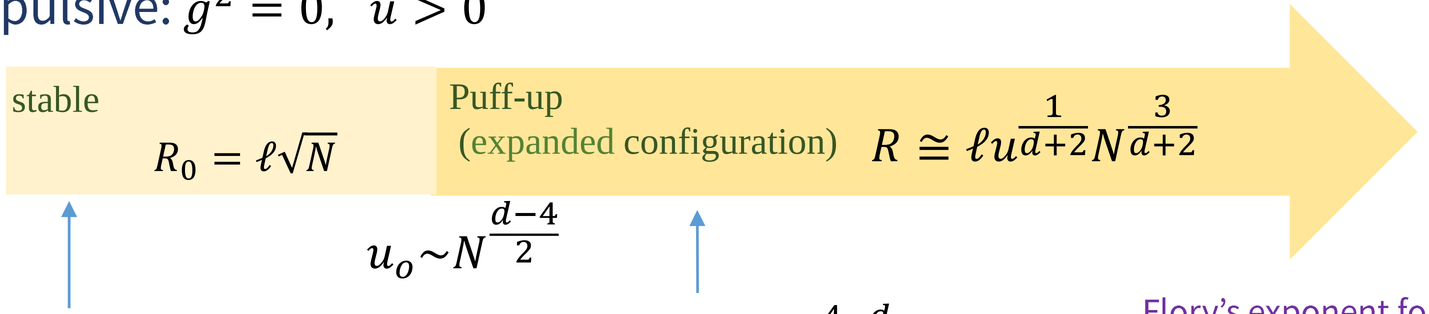

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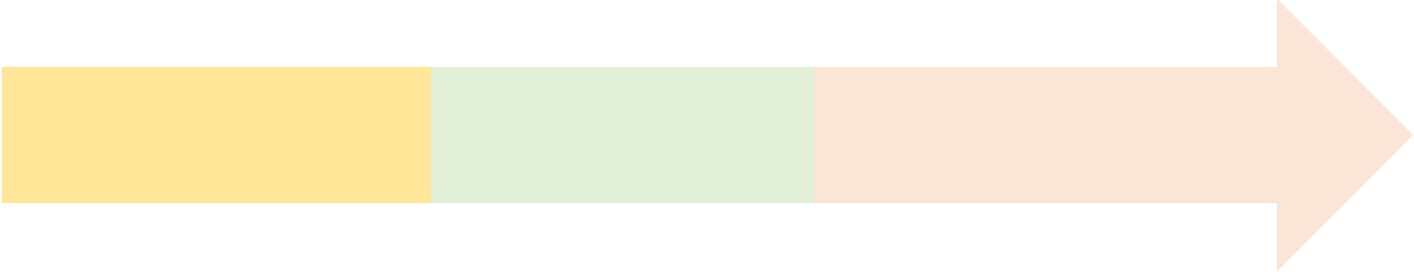

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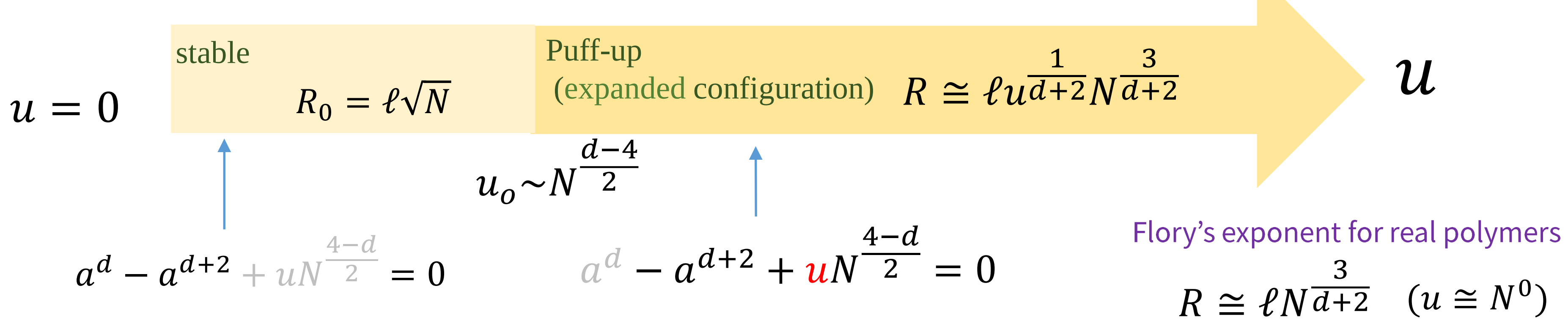

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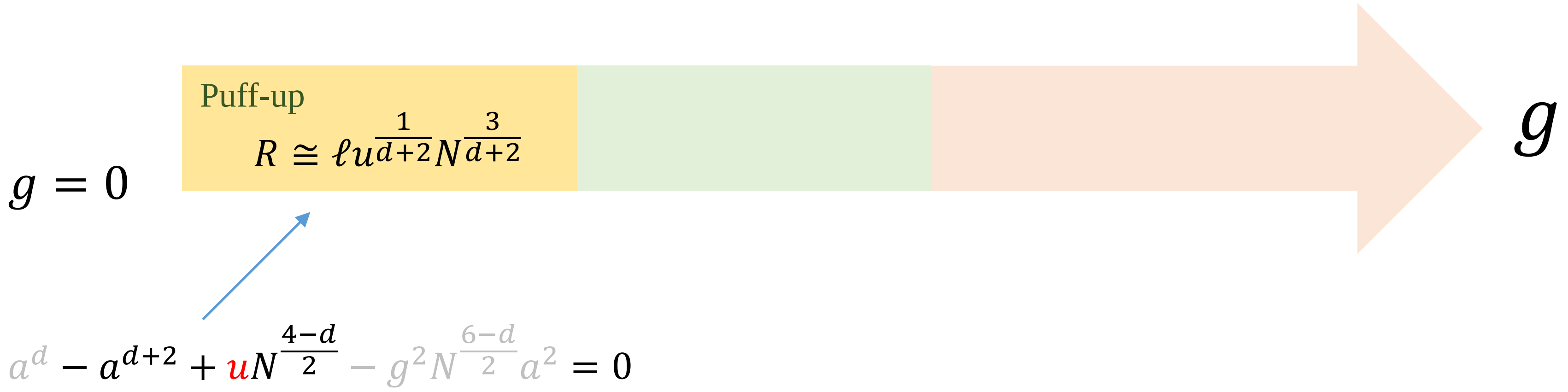

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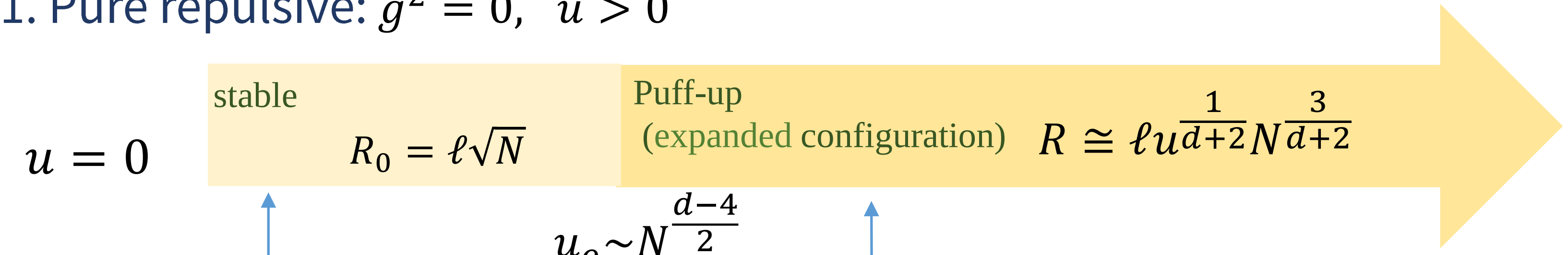

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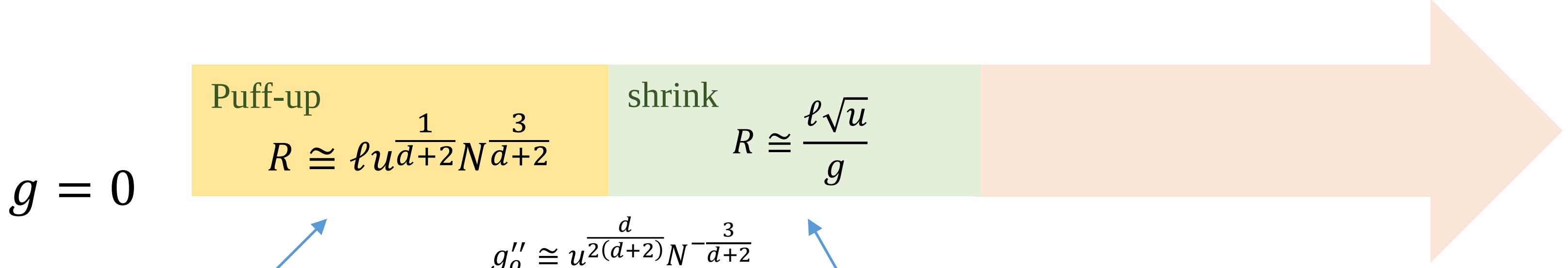

$g = 0$

Puff-up

$R \cong \ell u^{\frac{1}{d+2}} N^{\frac{3}{d+2}}$

shrink

$R \cong \frac{\ell\sqrt{u}}{g}$

g

$g_o'' \cong u^{\frac{d}{2(d+2)}} N^{-\frac{3}{d+2}}$

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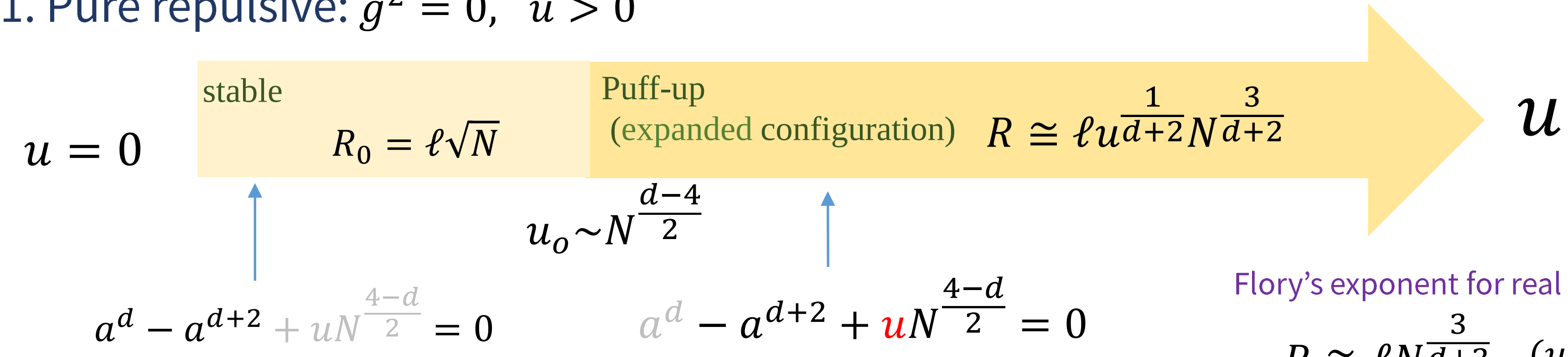

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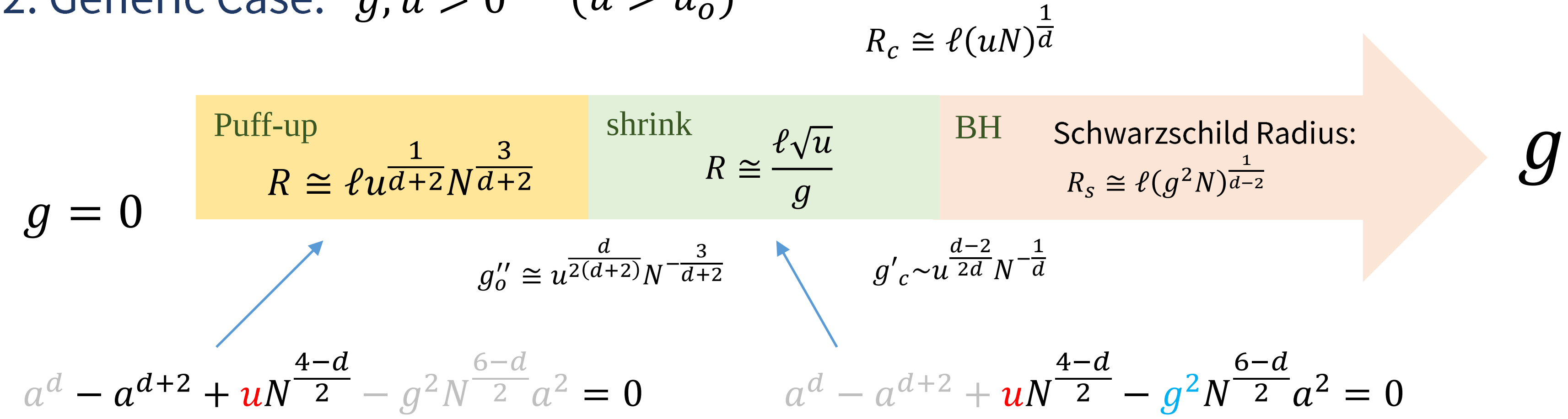

$R_c \cong \ell (uN)^{\frac{1}{d}}$

$g = 0$

Puff-up

$R \cong \ell u^{\frac{1}{d+2}} N^{\frac{3}{d+2}}$

shrink

$R \cong \frac{\ell\sqrt{u}}{g}$

BH

Schwarzschild Radius:

$R_s \cong \ell (g^2 N)^{\frac{1}{d-2}}$

g

$g''_o \cong u^{\frac{d}{2(d+2)}} N^{-\frac{3}{d+2}}$

$g'_c \sim u^{\frac{d-2}{2d}} N^{-\frac{1}{d}}$

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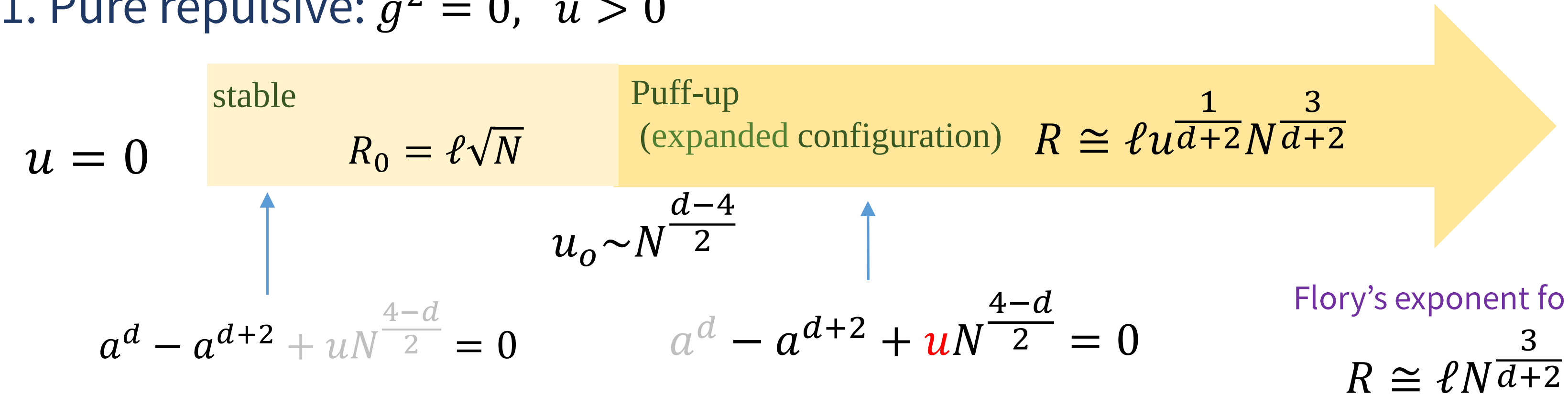

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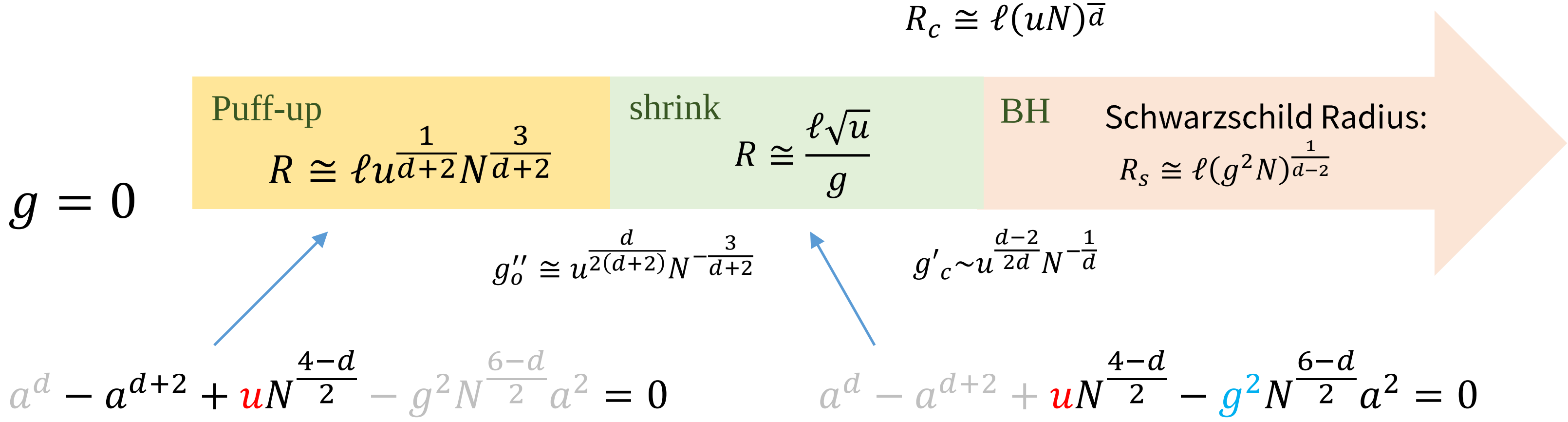

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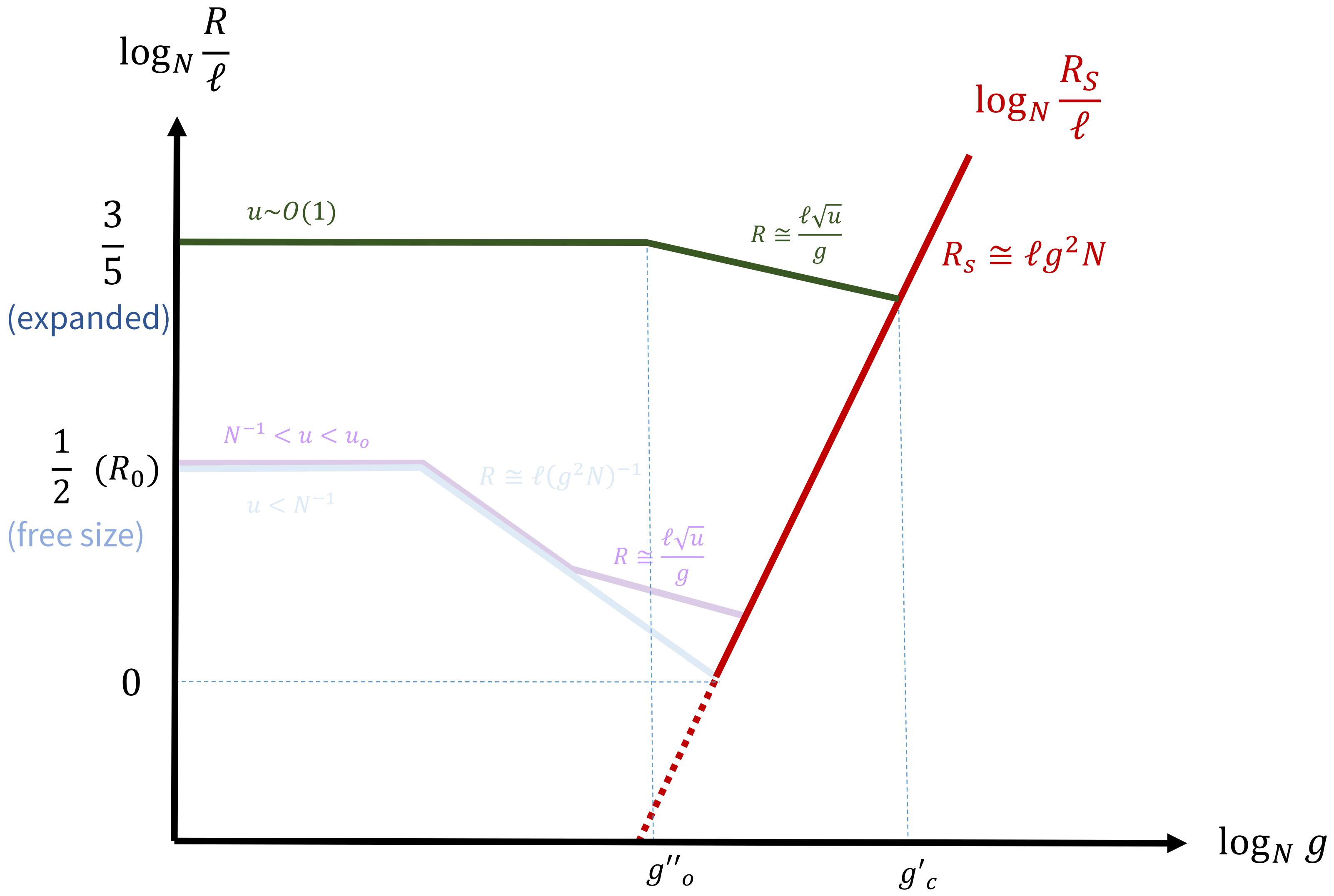
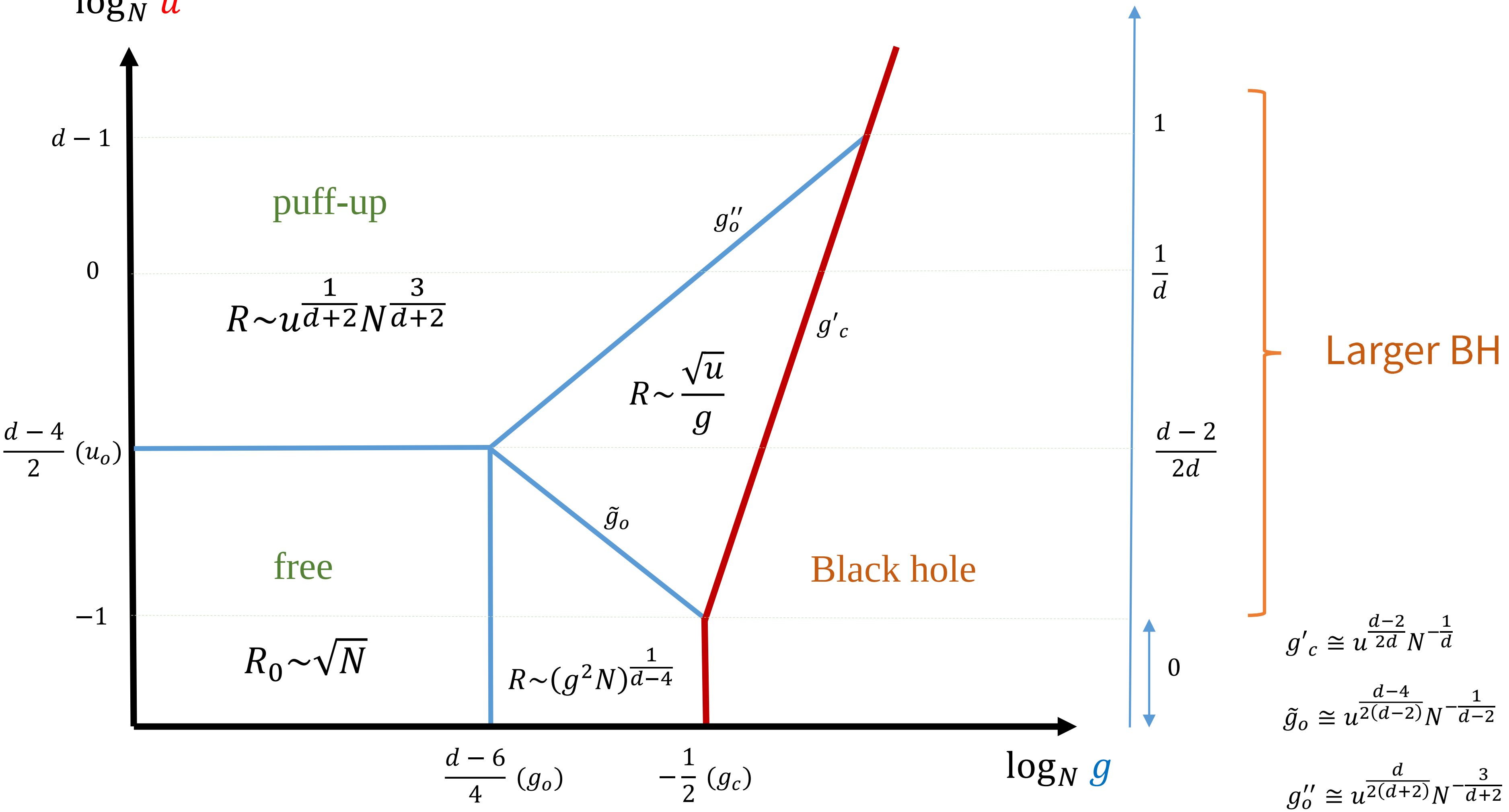
(Note: if $u < u_0$, no solution for small g)

Size change ($d = 3$)



The Size Scaling ($2 < d < 4$)

$\ell = 1$



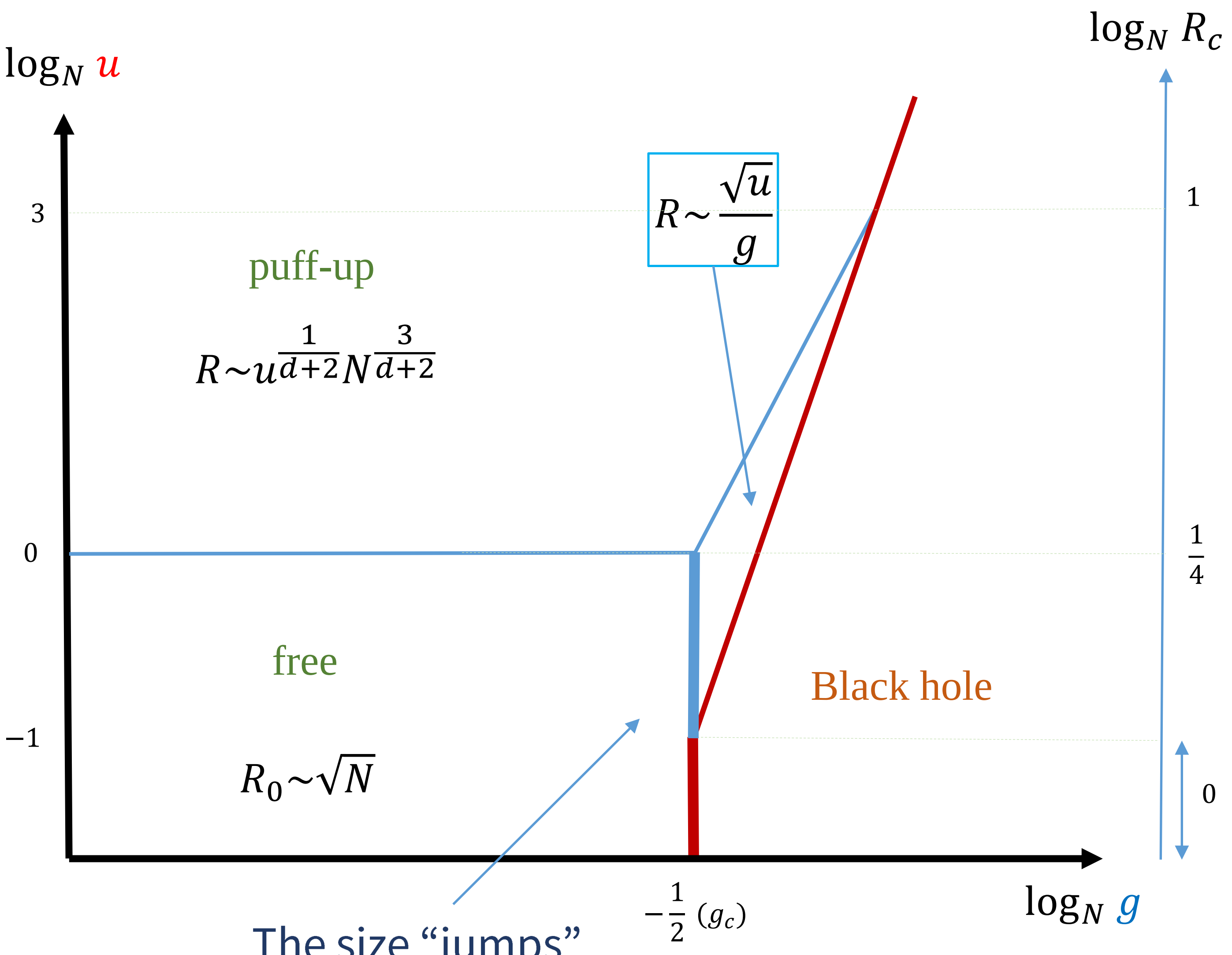
$$g'_c \cong u^{\frac{d-2}{2d}} N^{-\frac{1}{d}}$$

$$\tilde{g}_o \cong u^{\frac{d-4}{2(d-2)}} N^{-\frac{1}{d-2}}$$

$$g''_o \cong u^{\frac{d}{2(d+2)}} N^{-\frac{3}{d+2}}$$

The Size Scaling ($d = 4$)

$\ell = 1$



puff-up

$R \sim u^{\frac{1}{d+2}} N^{\frac{3}{d+2}}$

$R \sim \frac{\sqrt{u}}{g}$

free

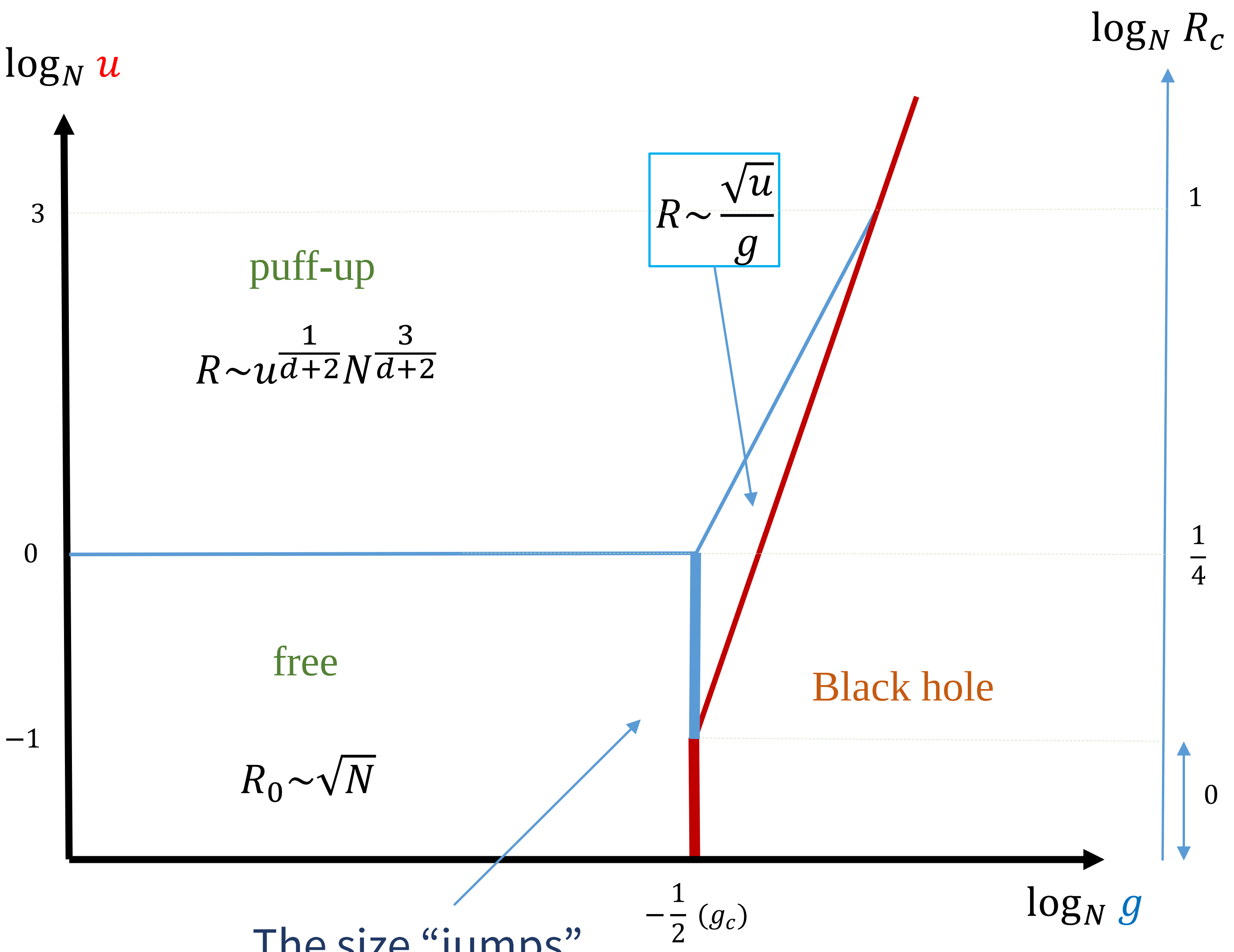
$R_0 \sim \sqrt{N}$

Black hole

The size “jumps”

The Size Scaling ($d = 4$)

$\ell = 1$



(Note: $d > 4$ exhibits a pathological behavior)

Summary

- Self-gravitating polymers (self-avoiding walks)

→ Collapse to a larger size black hole

- Interesting size scaling behaviors are observed.

Next:

- Density distribution, elasticity (pressure), detailed gravitational collapse
  (Need GR ? -- Tolman-Oppenheimer-Volkoff eq.)
- Possible source of self-avoiding property
  Fermionic walk? Higher form field exchange?
- Beyond a mean field calculation. RG analysis
- Entropy? Corresponding point?

Variational Method (details)

Harmonic Hamiltonian: $\beta H_0 = \frac{d}{2\ell^2}\int_0^N d\sigma \left(\frac{\partial \boldsymbol{R}}{\partial \sigma}\right)^2 + \frac{q^2 d}{2\ell^2}\int_0^N d\sigma\, \boldsymbol{R}(\sigma)^2$

Convexity: $e^{-\beta F} = \int\, d\boldsymbol{R}\, e^{-\beta H} = \int\, d\boldsymbol{R}\, e^{-\beta H_0} e^{-\beta(H-H_0)} \geq \mathrm{e}^{\langle -\beta(H-H_0)\rangle_0}\mathrm{e}^{-\beta F_0}$

$$\beta F \leq \beta F_0(q) + \langle \beta(H - H_0)\rangle_0$$

Tune q to minimize RHS

Propagator:

$$G_0(\sigma,\sigma') = \left(\frac{qd}{2\pi\ell^2\ \sinh q|\sigma-\sigma'|}\right)^{\frac{\mathrm{d}}{2}} \exp\left(-\frac{qd\ [\boldsymbol{R}(\sigma)^2 + \boldsymbol{R}(\sigma')^2]\cosh q|\sigma-\sigma'| - 2\boldsymbol{R}(\sigma)\cdot\boldsymbol{R}(\sigma')}{2\ell^2 \sinh q|\sigma-\sigma'|}\right)$$

Evaluate $\beta F_0 = -\log Z_0$ and $\langle\beta(H-H_0)\rangle_0 = \left\langle \int_0^N d\sigma \int_0^N d\sigma'\, V\ - \frac{q^2 d}{2\ell^2}\int_0^N d\sigma\, \boldsymbol{R}(\sigma)^2 \right\rangle$

$$\beta F \leq \frac{d}{2}\ln(\cosh qN) - \frac{qdN}{4}\tanh qN$$

$$-2\int_0^N d\sigma' \int_0^{\sigma'} d\sigma \left[\frac{g^2}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{qd}{2F_1(\sigma,\sigma';q)}\right)^{\frac{d-2}{2}} - u\left(\frac{qd}{2F_2(\sigma,\sigma';q)}\right)^{\frac{d}{2}}\right]$$

$$F_1(\sigma,\sigma';q) = \frac{\sinh q\sigma \cosh q(N-\sigma) + \sinh q\sigma' \cosh q(N-\sigma') - 2\sinh q\sigma \cosh q(N-\sigma')}{\cosh qN}$$

$$F_2(\sigma,\sigma';q) = \frac{\sinh q\sigma \sinh q(\sigma'-\sigma)}{\sinh q\sigma'} + \frac{\cosh q(N-\sigma')\sinh q\sigma'}{\cosh qN}\left(1 - \frac{\sinh q\sigma}{\sinh q\sigma'}\right)^2$$

Simplification (an approximation): $$\begin{matrix} e^{-qN} & e^{-q(N-\sigma')} & e^{-q(N-\sigma)} \\ e^{-q\sigma} & e^{-q\sigma'} & e^{-q(\sigma'-\sigma)} \end{matrix} \ll 1$$

Bound: $$\beta F \leq qN - N^2 g^2 q^{\frac{d}{2}-1} + N^2 u q^{\frac{d}{2}}$$

Stationary Cond.: $$0 = 1 - N g^2 q_0^{\frac{d-4}{2}} + N u q_0^{\frac{d-2}{2}}$$

Omit (positive N-independent) numerical factors

$$\langle \boldsymbol{R}^2 \rangle_0 = \frac{\ell^2}{q_0} \tanh q_0 N \begin{cases} \ell^2 N & (q_0 N \ll 1) \\ \dfrac{\ell^2}{q_0} & (q_0 N \geq O(1)) \end{cases}$$

(Free walk size)

(shrink)

Uniform Expansion Model (details)

Free (Gaussian) Hamiltonian with the bond length $a\ell$

$$\beta H' = \frac{d}{2a^2\ell^2}\int_0^N d\sigma \left(\frac{\partial \boldsymbol{R}}{\partial \sigma}\right)^2$$

Propagator: $$G'(\sigma,\sigma') = \left(\frac{d}{2\pi a^2\ell^2\,|\sigma-\sigma'|}\right)^{\frac{\mathrm{d}}{2}} \exp\left(-\frac{d}{2a^2\ell^2|\sigma-\sigma'|}\big(\boldsymbol{R}(\sigma)-\boldsymbol{R}(\sigma')\big)^2\right)$$

Calculate the mean-size-squared:

$$\langle A\rangle' \equiv \frac{1}{Z'}\int A e^{-\beta H'}$$

$$\langle \boldsymbol{R}^2\rangle = \frac{\int \big(\boldsymbol{R}(N)-\boldsymbol{R}(0)\big)^2 e^{-\beta H}}{\int e^{-\beta H}} = \frac{\left\langle e^{-\beta(H-H')}\big(\boldsymbol{R}(N)-\boldsymbol{R}(0)\big)^2\right\rangle'}{\left\langle e^{-\beta(H-H')}\right\rangle'}$$

$$\cong \left\langle\big(\boldsymbol{R}(N)-\boldsymbol{R}(0)\big)^2\right\rangle' \left(1+\langle\beta(H-H')\rangle'\right) - \left\langle \beta(H-H')\big(\boldsymbol{R}(N)-\boldsymbol{R}(0)\big)^2\right\rangle' + O([\beta(H-H')]^2)$$

$$\cong \underbrace{Na^2\ell^2}_{\text{Required Size}} + \underbrace{\left[a^d(1-a^2) + C_1 u N^{\frac{4-d}{2}} - C_2 g^2 N^{\frac{6-d}{2}} a^2\right]}_{=\,0} N\ell^2 a^{2-d}$$

C_1, C_2: Positive N independent constants