Yang-Baxter sigma models and Lax pairs arising from kappa-Poincare r-matrices

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References:

A. Borowiec, H. Kyono, J. Lukierski, J. Sakamoto and KY, arXiv: 1511.00404

T. Matsumoto, D. Orlando, S. Reffert, J. Sakamoto and KY, arXiv: 1505.04553

0. Introduction

What is Yang-Baxter sigma model?

The recent progress on it

Yang-Baxter sigma model

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A systematic way to study integrable deformations of 2D integrable non-linear sigma models, such as principal chiral model (PCM) and symmetric coset models.

Proposed by Klimcik in 2002, and the integrability was shown in 2008.

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The existence of Lax pair

Yang-Baxter-deformation of PCM

[Klimcik, 2002, 2008]

$$S = \int d^2x\, \eta^{\mu\nu} \mathrm{tr}(J_\mu J_\nu) \quad \longrightarrow \quad S^{(\eta)} = \int d^2x\, \eta^{\mu\nu} \mathrm{tr}\left(J_\mu \frac{1}{1-\eta R} J_\nu \right)$$

$$J_\mu = g^{-1}\partial_\mu g, \quad g \in G$$

deformation!

η : const.

Linear op. $R \,:\, \mathfrak{g} \longrightarrow \mathfrak{g}$ is related to classical r-matrix via

$$R(X) \equiv \langle r_{12}, 1 \otimes X \rangle = \sum_i a_i \langle b_i, X \rangle \qquad \text{for} \quad X, a_i, b_i \in \mathfrak{g}$$

where $r_{12} = \sum_i a_i \otimes b_i$ satisfies modified classical Yang-Baxter eq. (mCYBE)

NOTE

- An integrable deformation is specified by a classical r-matrix.
- Given a classical r-matrix, a Lax pair follows automatically.

Recent Progress

[Delduc-Magro-Vicedo, 1309.5850]

[Kawaguchi-Matsumoto-KY, 1401.4855]

Yang-Baxter deformations of the $AdS_5 \times S^5$ superstring

Classical r-matrices have been identified with a lot of gravity solutions, including well-known examples.

EX η-deformations of $AdS_5 \times S^5$ (New)

γ-deformations of S^5, gravity duals of NC gauge theories,

Schrödinger spacetimes, dipole backgrounds etc.

Motivated by this progress, as a generalization,

we have considered Yang-Baxter deformations of 4D Minkowski spacetime.

[Matsumoto-Orlando-Reffert-Sakamoto-KY, 1505.04553]

The content of my talk

1. Yang-Baxter deformations of Minkowski spacetime

   T. Matsumoto, D. Orlando, S. Reffert, J. Sakamoto and K.Y., arXiv: 1505.04553

2. Deformed geometries and Lax pairs
   arising from kappa-Poincare r-matrices

   A. Borowiec, H. Kyono, J. Lukierski, J. Sakamoto and K.Y., arXiv: 1511.00404

3. Summary & Discussion

1. Yang-Baxter deformations of Minkowski spacetime

T. Matsumoto, D. Orlando, S. Reffert, J. Sakamoto and K.Y., arXiv: 1505.04553

Question

Why do we study Yang-Baxter deformations of Minkowski spacetime?

FACT There are a lot of backgrounds for which string theories are exactly solved ⟷ The quantum spectrum is obtained like in flat space

EX Melvin backgrounds, pp-wave backgrounds, other solvable time-dependent backgrounds ⟷ Deformations of Minkowski spacetime

e.g. [Hashimoto-Sethi, hep-th/0208126]
[Spradlin-Takayanagi-Volovich, hep-th/0509036]

Motives

1) There is a possibility to unify the already-known solvable backgrounds as Yang-Baxter deformations of Minkowski spacetime.

2) By adopting the YB-deformations, new solvable backgrounds may be found.

Yang-Baxter sigma model for 4D Minkowski

[Matsumoto, Orlando, Reffert, Sakamoto and KY, 1505.04553]

$$S = -\frac{1}{2}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\, (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\, \mathrm{Tr}\left[A_\alpha P \circ \frac{1}{1 - 2\eta R_g \circ P}(A_\beta)\right]$$

$$A_\alpha = g^{-1}\partial_\alpha g, \qquad R_g(X) \equiv g^{-1}R(gXg^{-1})g \qquad \eta \;:\; \text{const.}$$

$$g = \exp\left[p_\mu x^\mu\right] \;\in\; \frac{ISO(1,3)}{SO(1,3)} \quad , \qquad p_\mu \equiv \frac{1}{2}\gamma_\mu - \frac{1}{4}[\gamma_\mu, \gamma_5]$$ (translation)

Projection: $P(x) = \frac{1}{4}\gamma^\mu\, \mathrm{Tr}(\gamma_\mu\, x)$

$\gamma_\mu \quad (\mu = 0, \ldots, 3)$: a basis of su(2,2)

→ conformal embedding

↳ picks up vielbeins, avoiding subtlety of the degenerate Killing form.

NOTE the general form of Lax pair has not been constructed yet.

A schematic list of classical r-matrices

$r = a \otimes b\,,\ \ [a,b]=0$

→ abelian

(a) $r =$ Poincaré $\otimes$ Poincaré

1. abelian e.g., $r \sim p_1 \wedge p_2\,,$ 2. non-abelian e.g., $r \sim p_1 \wedge n_{12}\,,$

(b) $r =$ Poincaré $\otimes$ non-Poincaré

1. abelian e.g., $r \sim n_{12} \wedge \hat{d}\,,$ 2. non-abelian e.g., $r \sim p_0 \wedge \hat{d}\,,$

(c) $r =$ non-Poincaré $\otimes$ non-Poincaré

1. abelian e.g., $r \sim k_1 \wedge k_2\,,$ 2. non-abelian e.g., $r \sim k_0 \wedge \hat{d}\,.$

The generators of so(2,4): $p_\mu,\ n_{\mu\nu},\ \hat{d},\ k_\mu$

$$[p_\mu, k_\nu] = 2(n_{\mu\nu} + \eta_{\mu\nu}\,\hat{d})\,,\quad [\hat{d}, p_\mu] = p_\mu\,,\quad [\hat{d}, k_\mu] = -k_\mu\,,$$

$$[p_\mu, n_{\nu\rho}] = \eta_{\mu\nu}\,p_\rho - \eta_{\mu\rho}\,p_\nu\,,\quad [k_\mu, n_{\nu\rho}] = \eta_{\mu\nu}\,k_\rho - \eta_{\mu\rho}\,k_\nu\,,$$

$$[n_{\mu\nu}, n_{\rho\sigma}] = \eta_{\mu\sigma}\,n_{\nu\rho} + \eta_{\nu\rho}\,n_{\mu\sigma} - \eta_{\mu\rho}\,n_{\nu\sigma} - \eta_{\nu\sigma}\,n_{\mu\rho}\,.$$

Classical r-matrices and the associated backgrounds (identified so far)

	r-matrix	Type of Twist	Background	
Exactly Solvable	$p_i \wedge p_j \ (i,j=1,2,3)$	Melvin Shift Twist	Seiberg-Witten	Class (a) 1
	$p_0 \wedge p_i$	Melvin Shift Twist	NCOS	
	$(p_0+p_i) \wedge p_j \ (i \neq j)$	Null Melvin Shift Twist	light-like NC	
	$\frac{1}{2} p_3 \wedge n_{12}$	Melvin Twist	T–dual Melvin	
	$\frac{1}{2\sqrt{2}} p_2 \wedge (n_{01}+n_{13})$	Melvin Null Twist	Hashimoto–Sethi	
	$\frac{1}{2} n_{12} \wedge n_{03}$	R Melvin R Twist	Spradlin–Takayanagi–Volovich	
	$\frac{1}{2} p_1 \wedge n_{03}$	Melvin Boost Twist	T–dual of Grant space	
	$\frac{1}{2\sqrt{2}}(p_0-p_3) \wedge n_{12}$	Null Melvin Twist	pp-wave	
	$\frac{1}{2\sqrt{2}}(\hat{d}-n_{03}) \wedge (p_0-p_3)$	Non-Twist	pp-wave	Class (b) 2
Integrable	$\frac{1}{2}\hat{d} \wedge p_0$	Non-Twist	T–dual of dS_4	
	$\frac{1}{2}\hat{d} \wedge p_1$	Non-Twist	T–dual of AdS_4	
Integrable?	DJ-type (mCYBE)	Non-Twist	q-deformation (?)	Class (b) 2

- These results support the integrability of YB-deformed action.
- Would-be new backgrounds obtained (not mentioned above) ➡ Integrable?

2. Deformed geometries and Lax pairs arising from kappa-Poincare r-matrices

A. Borowiec, H. Kyono, J. Lukierski, J. Sakamoto and K.Y., arXiv: 1511.00404

Kappa-Poincare r-matrix

classical r-matrix: $r = a^\mu \, n_{\mu\nu} \wedge p^\nu$

(class (a) 2)

a^μ : a constant 4D vector

satisfies the modified classical Yang-Baxter eq.

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \frac{1}{2}(a_\mu a^\mu)\, p^\rho \wedge n_{\rho\sigma} \wedge p^\sigma$$

There are 3 special cases:

κ : a real positive const.

1. Standard deformation $a^\mu = (\frac{1}{\kappa}, 0, 0, 0)$: time-like vector
2. Tachyonic deformation $a^\mu = (0, \frac{1}{\kappa}, 0, 0)$: space-like vector
3. Light-cone deformation $a^\mu = (\frac{1}{\sqrt{2}\kappa}, 0, 0, -\frac{1}{\sqrt{2}\kappa})$: null vector

The deformed backgrounds for 3 special cases:

1. Standard deformation

(time-like)T-duality

$$ds^2 = \frac{-(dx^0)^2 + dr^2}{1 - \hat{\eta}^2 r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

4D de Sitter space

2. Tachyonic deformation

(space-like) T-duality
& double Wick rotation

$$ds^2 = \frac{dt^2 + (dx^1)^2}{1 + \hat{\eta}^2 t^2} + t^2 \cosh^2\phi d\theta^2 - t^2 d\phi^2$$

4D anti de Sitter space

3. Light-cone deformation

$$ds^2 = -2dx^+dx^- - \frac{2\hat{\eta}^2 r^2}{\cosh^2(\hat{\eta}x^+)}(dx^+)^2 + (dr)^2 + r^2 d\theta^2$$

4D time-dependent pp-wave

A list of classical r-matrices and backgrounds

r-matrix	Type of Twist	Background
$p_i \wedge p_j \ (i,j=1,2,3)$	Melvin Shift Twist	Seiberg-Witten
$p_0 \wedge p_i$	Melvin Shift Twist	NCOS
$(p_0+p_i) \wedge p_j \ (i \neq j)$	Null Melvin Shift Twist	light-like NC
$\frac{1}{2} p_3 \wedge n_{12}$	Melvin Twist	T–dual Melvin
$\frac{1}{2\sqrt{2}} p_2 \wedge (n_{01}+n_{13})$	Melvin Null Twist	Hashimoto–Sethi
$\frac{1}{2} n_{12} \wedge n_{03}$	R Melvin R Twist	Spradlin–Takayanagi–Volovich
$\frac{1}{2} p_1 \wedge n_{03}$	Melvin Boost Twist	T–dual of Grant space
Class (b) 2. $\frac{1}{2\sqrt{2}}(p_0-p_3) \wedge n_{12}$	Null Melvin Twist	pp-wave
$\frac{1}{2\sqrt{2}}(\hat{d}-n_{03}) \wedge (p_0-p_3)$	Non-Twist	pp-wave
$\frac{1}{2}\hat{d} \wedge p_0$	Non-Twist	T–dual of dS_4
$\frac{1}{2}\hat{d} \wedge p_1$	Non-Twist	T–dual of AdS_4
DJ-type (mCYBE)	Non-Twist	q-deformation (?)

As a result, this part has been reproduced by different classical r-matrices!

(class (a) 2.)

What is the reasoning behind this coincidence?

Lax pair for general kappa-deformations

Lax pair

$$\mathcal{L}_\pm = P_0(J_\pm) + \lambda^{\pm 1}\,[P(J_\pm) + P'(J_\pm)]$$
$$-a_\mu a^\mu\,\eta^2\,\lambda^{\pm 1}\,[P(J_\pm) - P'(J_\pm)]$$

λ : spectral parameter

Here we have introduced a deformed current

$$J_\pm \equiv \frac{1}{1 \mp 2\eta R_g \circ P} A_\pm$$

and 2 extra projections: (P is utilized in the classical action)

$$P'(x) \equiv \sum_{\mu=0}^{3} n_{\mu 5}\,\frac{\mathrm{Tr}(n_{\mu 5} x)}{\mathrm{Tr}(n_{\mu 5} n_{\mu 5})}\,, \qquad P_0(x) \equiv \frac{1}{2}\sum_{\mu,\nu=0}^{3} n_{\mu\nu}\,\frac{\mathrm{Tr}(n_{\mu\nu} x)}{\mathrm{Tr}(n_{\mu\nu} n_{\mu\nu})}$$

NOTE: the associated ∞-dim. symmetry has not been clarified yet.

NOTE2: the Lax pair can be constructed for the class (a). [Kyono-Sakamoto-KY, to appear]

3. Summary & Discussion

Summary

Yang-Baxter deformations of 4D Minkowski spacetime

A lot of well-known backgrounds have been reproduced.

[Matsumoto, Orlando, Reffert, Sakamoto and KY, 1505.04553]

New results

[Borowiec, Kyono, Lukierski, Sakamoto and KY, 1510.03083]

κ -Poincare r-matrix: $r = a^{\mu}\, n_{\mu\nu} \wedge p^{\nu}$

- 3 special cases ➡ (T-duals of) dS_4 & AdS_4, a pp-wave
- Constructed Lax pairs for general κ-deformations

Discussion

Infinite-dim. algebras? SUSY? New backgrounds? Integrability?

Future perspectives:

Yang-Baxter deformations of other 4D geometries

EX Nappi-Witten model → Yang-Baxter invariance

2D Poincare alg. with a center

[Kyono-KY, 1511.00404]

Question: Are pp-wave backgrounds YB-invariant in general?

If YES,

The low-energy behavior of the gauge-theory duals of YB-deformed $AdS_5 \times S^5$ may be determined universally.

→ Low-energy excited states of spin chain are universal?

Back up

Coset construction of Poincare AdS_5

AdS_5 is represented by a coset: $$\mathrm{AdS}_5 = SO(2,4)/SO(1,4)$$

Here we will use the following notation:

4D gamma matrices:

$$\gamma_\mu \ (\mu = 0,1,2,3), \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$$

$$n_{\mu\nu} \equiv \frac{1}{4}[\gamma_\mu, \gamma_\nu], \quad n_{\mu 5} \equiv \frac{1}{4}[\gamma_\mu, \gamma_5]$$

Lie algebras

Lie algebra of *SO*(2,4): $$\mathfrak{so}(2,4) = \mathrm{span}_{\mathbb{R}}\{\ \gamma_\mu\ , \gamma_5\ , n_{\mu\nu}\ , n_{\mu 5}\ \}$$

Lie algebra of *SO*(1,4): $$\mathfrak{so}(1,4) = \mathrm{span}_{\mathbb{R}}\{\ n_{\mu\nu}\ , n_{\mu 5}\ \}$$

The group element:

$$g = \exp\left[p_\mu x^\mu\right] \exp\left[\gamma_5 \frac{1}{2} \log z\right]$$

Radial direction of AdS5

$$p_\mu \equiv \frac{1}{2}(\gamma_\mu - 2n_{\mu 5})$$

$$[p_\mu, p_\nu] = 0$$

Commute each other

Then, by using the left-invariant 1-form: $A \equiv g^{-1} dg$,

one can compute the metric:

$$ds^2 = g_{MN} dx^M dx^N = \mathrm{Tr}(A\overline{P}(A))$$

where the projection is defined as

(respecting the grading property)

$$\overline{P}(x) \equiv \frac{1}{4}\left[-\gamma_0 \,\mathrm{Tr}(\gamma_0 x) + \sum_{i=1}^{3} \gamma_i \,\mathrm{Tr}(\gamma_i x) + \gamma_5 \,\mathrm{Tr}(\gamma_5 x)\right]$$

Finally, we obtain the usual expression:

$$ds^2 = \frac{-(dx^0)^2 + \sum_{i=1}^{3}(dx^i)^2 + dz^2}{z^2}$$

Coset construction of 4D Minkowski spacetime

The group element: $g = \exp\left[p_\mu x^\mu\right]$ e.g., a slice of AdS_5 with $z = 1$

Instead of $\overline{P}$, we will use the following projection:

$$P(x) \equiv \frac{1}{4}\left[-\gamma_0 \,\mathrm{Tr}(\gamma_0 x) + \sum_{i=1}^{3} \gamma_i \,\mathrm{Tr}(\gamma_i x)\right]$$

γ_5 -dependence is dropped off.

This *P* may be seen as a map from p_μ to a ``dual'' basis γ_μ 's $(\because)\ \ \mathrm{Tr}(p_\mu \gamma_\nu) = \eta_{\mu\nu}$

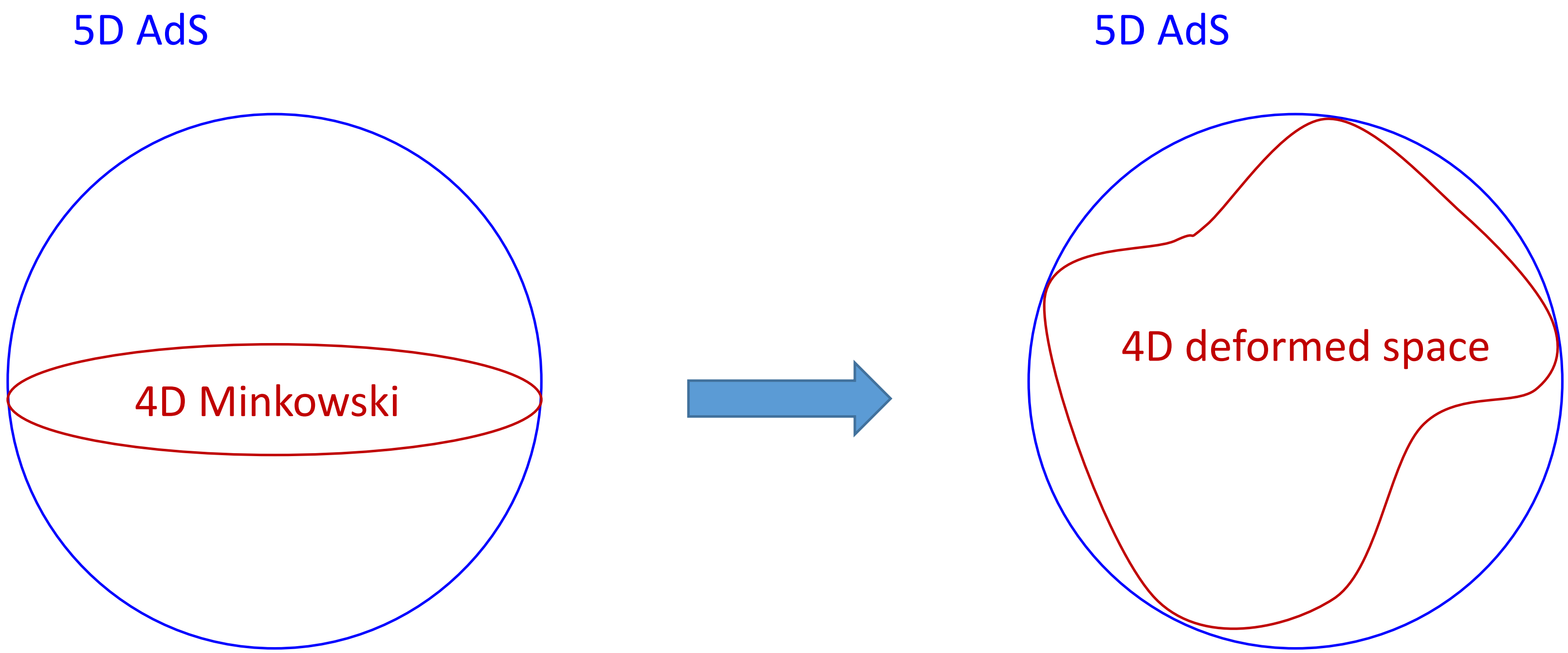
$$\mathfrak{iso}(1,3) = \mathrm{span}_{\mathbb{R}}\{p_\mu, n_{\mu\nu}\} \iff \mathfrak{iso}^*(1,3) = \mathrm{span}_{\mathbb{R}}\{\gamma_\mu, n_{\mu\nu}\}$$

NOTE: This step is quite important because $\mathrm{Tr}(p_\mu p_\nu) = 0$!

Finally, the 4D Minkowski metric is reproduced as

$$ds^2 = \mathrm{Tr}(AP(A)) = -(dx^0)^2 + \sum_{i=1}^{3}(dx^i)^2$$

A schematic picture for a classical r-matrix of the 2nd class



4D Minkowski spacetime is embedded into 5D AdS space.

The 4D slice is deformed by a classical r-matrix

Example 1: A Melvin background

classical r-matrix: $r = \frac{1}{2} p_3 \wedge n_{12}$

Class (a) 1.

With a coordinate transformation, $x^1 = r\cos\theta\,, \quad x^2 = r\sin\theta$

the metric and B-field are obtained as

$$\begin{aligned} ds^2 &= -(dx^0)^2 + dr^2 + \frac{r^2 d\theta^2 + (dx^3)^2}{1+\eta^2 r^2}\,, \\ B &= \frac{\eta r^2}{1+\eta^2 r^2} d\theta \wedge dx^3 \end{aligned}$$

This result nicely agrees with the known background.

[Gibbons-Maeda, 1988]
[Tseytlin, 1994]
[Hashimoto-Thomas, 2004]

Note: the dilaton is fixed by imposing that the 1-loop β function vanishes,

$$\Phi = -\frac{1}{2}\log(1+\eta^2 r^2)$$

Example 2: T-dual of 4D de Sitter space

classical r-matrix: $r = \frac{1}{2}\hat{d} \wedge p_0$

Class (b) 2.

Then the metric and B-field are obtained as

$$ds^2 = \frac{-(dx^0)^2 + dr^2}{1-\eta^2 r^2} + r^2 \sin^2\theta d\phi^2 + r^2 d\theta^2 \,,$$

$$B = \frac{\eta r}{1-\eta^2 r^2} dx^0 \wedge dr$$

→ Total derivative

where we have performed a coordinate transformation

$$x^1 = r\cos\phi\,\sin\theta\,, \quad x^2 = r\sin\phi\sin\theta\,, \quad x^3 = r\cos\theta$$

Then by performing a T-duality for the x^0-direction, the metric is rewritten as

$$ds^2 = (dr + \eta r\, dx^0)^2 - (dx^0)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

By performing a further coordinate transformation

$$x^0 = t - \frac{1}{2\eta}\log(\eta^2 r^2 - 1)\,,$$

we obtain the metric of 4D de Sitter space

$$ds^2 = -(1-\eta^2 r^2)dt^2 + \frac{dr^2}{1-\eta^2 r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

NOTE: There exists a cosmological horizon at $r = 1/\eta$.

Comments:

1) The Lax pair has not been constructed for the general form of the deformed action, but this case should be classically integrable.

2) The B-field vanishes and the RR-flux should be turned on, but it would be complex.