New supersymmetric localizations from topological gravity

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Based on:
1510.00006 with J. Bae, S. J. Rey and C. Imbimbo
(see also 1411.6635 with C. Imbimbo)

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Main results of the last few years

Exact results for SUSY QFTs in different dimensions and for different manifolds

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Two main technical ingredients:

- Localization → Exact results [Pestun,...]
- Coupling to gravity → Manifolds with SUSY [Festuccia-Seiberg,...]

An algorithmical recipe

The two methods combined give a clear recipe:

1. Flat space SUSY → relevant SUGRA
2. SUSY backgrounds from SUGRA
3. Rigid SUSY → $\text{SUGRA}_{\text{freezed}}$ + matter
4. Localization → exact results

A couple of remarks

- Localization is deeply related to topological field theories

  Examples:
  1) Nekrasov instanton partition function
  2) Chern-Simons theory using SUSY
  3) 2d GLSM to compute GW invariants

- Localization relies on a cohomological formulation of SUSY

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Unfortunately SUGRA is NOT cohomological
Much more complicated!

A different perspective

Another, cohomological, gravity theory can be found

~~SUGRA~~ → Topological Gravity (TG)

coupled to some topological multiplets

Remark:

TG ≠ twisted SUGRA

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Many advantages:

1) Easier equations

2) Cohomological

3) 1 to 1 mapping to SUGRA

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3) 1 to 1 mapping to SUGRA

→ TG is the best choice for localization's users!

Plan of the talk

1) Our rigid topological starting point: 2d Yang-Mills

2d Yang-Mills coupled to TG

2) Classification of TG invariant backgrounds

3) From TG backgrounds to SUGRA backgrounds

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A full classification of 2d N=(2,2) SUGRA localizing backgrounds

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2d gauge inv. kills two photons ⟶ Topological theory!

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- Why Naïve? [Witten]

$$I_{\mathrm{YM}}(g,\epsilon) = \int_\Sigma \mathrm{Tr}\,\phi\, F + \epsilon \int_\Sigma \mathrm{d}^2x\,\sqrt{g}\,\tfrac{1}{2}\mathrm{Tr}\,\phi^2$$

Auxiliary scalar. Integrating it out we recover the standard action

2d Volume form

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2d Volume form

The action is not fully topological. It depends on the volume form

Making 2d Yang-Mills topological

- We want to control the metric-dependence

Replacement

Background 2-form
It replaces the metric

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Hödge decomposition

$$f^{(2)} = \Omega^{(2)} + \mathrm{d}\,\Omega^{(1)}$$

This part gives a spurious dependence:
We must remove it

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Let’s see how!

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1. We make the spurious part BRST-trivial

Top. gaugino

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This is the first actor of our new gravity background theory:

A topological abelian multiplet

The second actor: TG

Topological gauge theory for diffeomorphisms

Defined by the BRST rules

$$s\, g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu} + \psi_{\mu\nu}$$
$$s\, \psi_{\mu\nu} = -\mathcal{L}_\xi \psi_{\mu\nu} + \mathcal{L}_\gamma g_{\mu\nu}$$
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diffeomorphism ghost

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By trying to couple 2d YM to TG we found a consistent coupling between TG and the topological background abelian multiplet.
Let us call it ''equivariant topological gravity''

Equivariant topological gravity

- By coupling the two actors:

Equivariant topological gravity BRST operator

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Additional pieces due to the coupling between the two actors

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Additional pieces due to the coupling between the two actors

We will classify the bosonic BRST invariant backgrounds

1 to 1 correspondence with N = (2,2) SUGRA backgrounds

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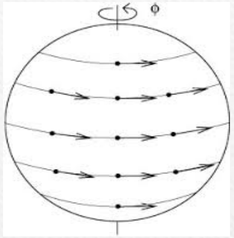
Well-known equation: Cartan equivariant cohomology

In this form the problem has been already solved!!

Classifying the solutions

- Take S^2 (round or squashed)



U(1) isometry

$$V^\mu = \partial_\phi$$

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implies

$$\gamma^\mu = \epsilon_\Omega V^\mu$$

equivariant parameter

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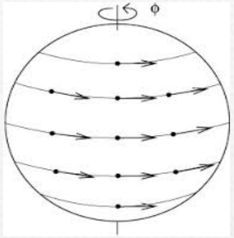

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The solutions are classified
Up to topological equivalences

integer for flux quantization

$$f^{(2)} = \mathrm{vol}_2|_g \cdot n$$

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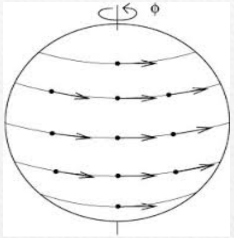

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Inequivalent solutions classified by three constants

$$(\epsilon_\Omega, n, A)$$

Killing spinor equations in SUGRA

- We need to solve the Killing spinor equation (KSE)

$$\delta\,\psi_\mu \equiv (D_\mu - i\mathcal{A}_\mu)\,\zeta + \tfrac{1}{2}\,H\,\Gamma_\mu\zeta - \tfrac{i}{2}\,G\,\Gamma_\mu\Gamma_3\zeta = 0$$

gravitino

U(1) R-symmetry
gauge field

graviphotons
field-strengths

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U(1) R-symmetry gauge field

graviphotons field-strengths

Solutions known in the literature
[Witten, Gomis et al., Benini and Cremonesi, Closset and Cremonesi]

- A-Twist $\quad A_\mu = \pm\tfrac{1}{2}\omega_\mu\ , \qquad H = G = 0$
- Ω-background $\quad A_\mu = \pm\tfrac{1}{2}\omega_\mu\ , \qquad H = G = \tfrac{\epsilon_\Omega}{2}\,\sqrt{g}\,\epsilon_{\mu\nu}\,D^\mu\,V^\nu$
- ‘’No twist’’ $\quad A_\mu = 0\ , \qquad H = -i\ , \qquad G = 0$

From spinors to forms: Fierzing

- A simple Fierz computation tells us:

$$\zeta_a(x)\,\zeta_b^\dagger(x) = c_0(x)\,\tfrac{1}{2}\mathrm{Id}_{ab} + c_\mu(x)\tfrac{1}{2}\Gamma^\mu_{ab} + \widetilde{c}_0(x)\,\tfrac{1}{2}\Gamma^3_{ab}$$

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- Not all independent

$$c^\mu c_\mu = c_0^2(x) - \widetilde{c}_0^2(x)$$

The KSE get rewritten:

$$D_\mu\, c_\nu + D_\nu\, c_\mu = 0$$

$$D_\mu\, \widetilde{c}_0 = -i\, H\, \sqrt{g}\, \epsilon_{\mu\nu}\, c^\nu$$

$$D_\mu\, c_\nu = \sqrt{g}\, \epsilon_{\mu\nu}\,(G\, c_0 + i\, H\, \widetilde{c}_0)$$

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These equations completely solve the KSE

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$$\mathcal{L}_\gamma g_{\mu\nu} = 0$$
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The topological system and the KSE are equivalent

with the identifications

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Remark

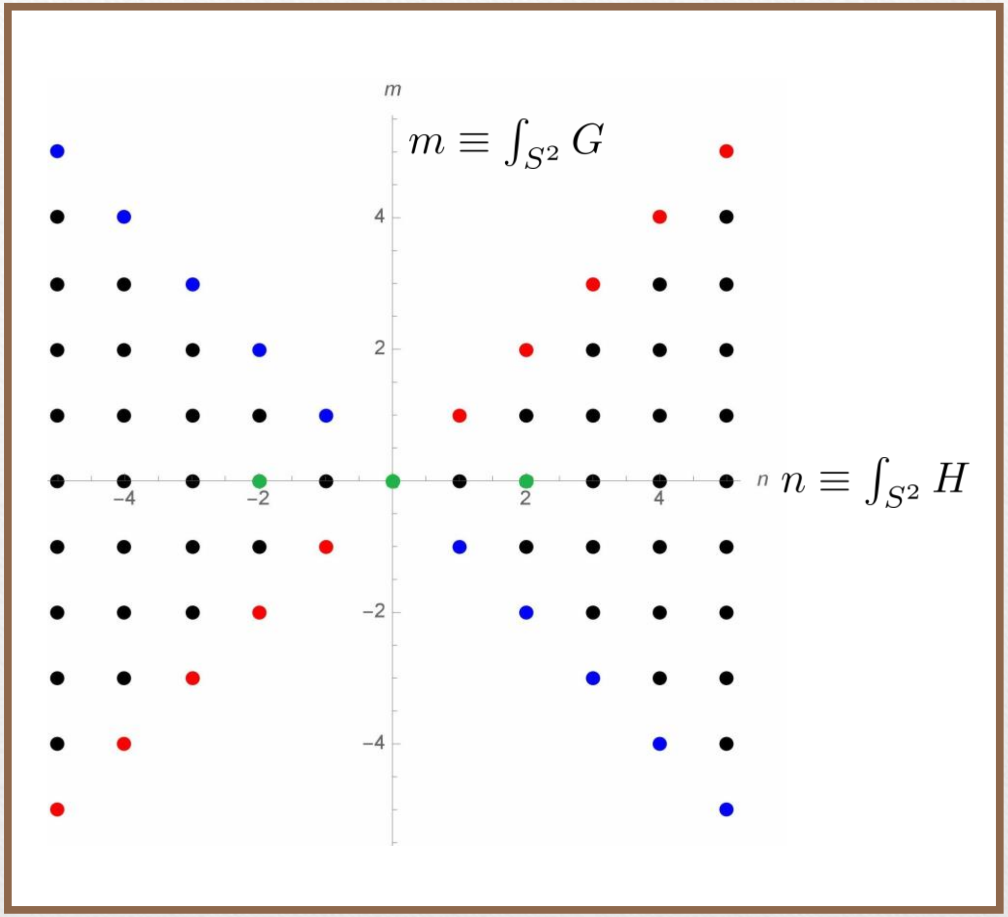
The topological approach has the notion of topological equivalence

We have exported this notion also to the SUGRA side.

Complete classification also in SUGRA

All the SUGRA backgrounds on S^2

All the inequivalent backgrounds are parametrized by the fluxes of the graviphotons field-strengths



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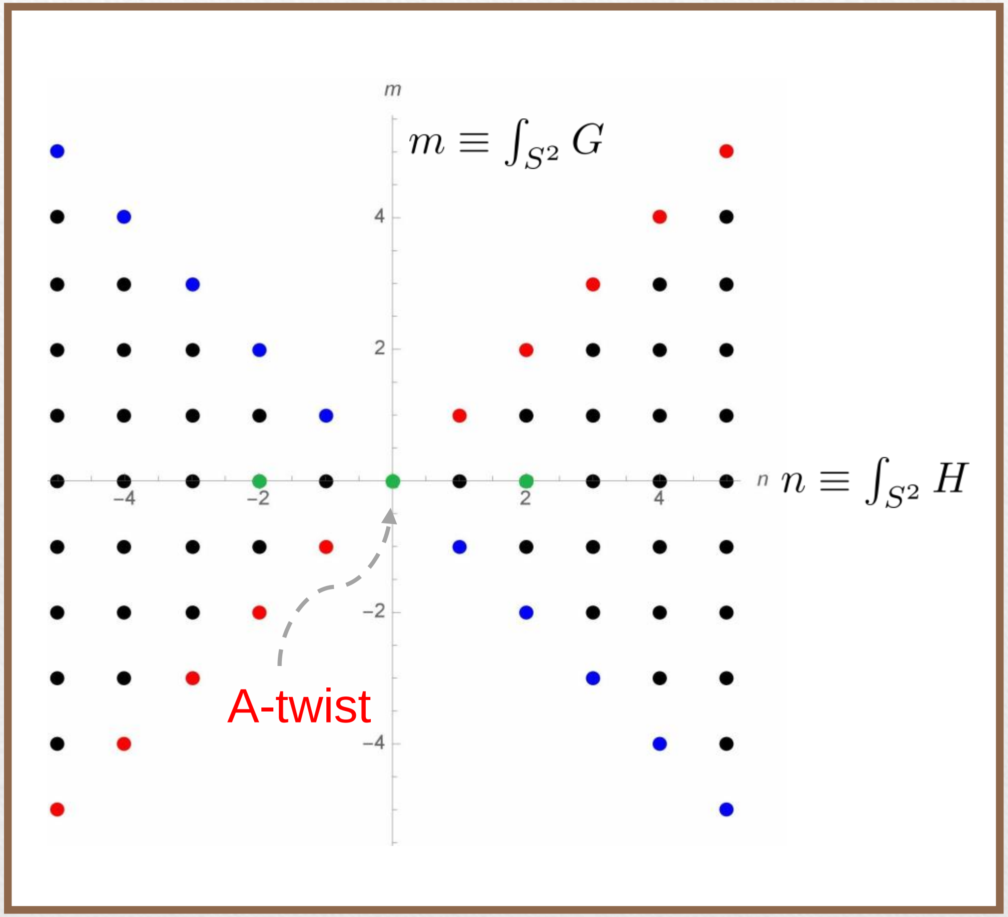

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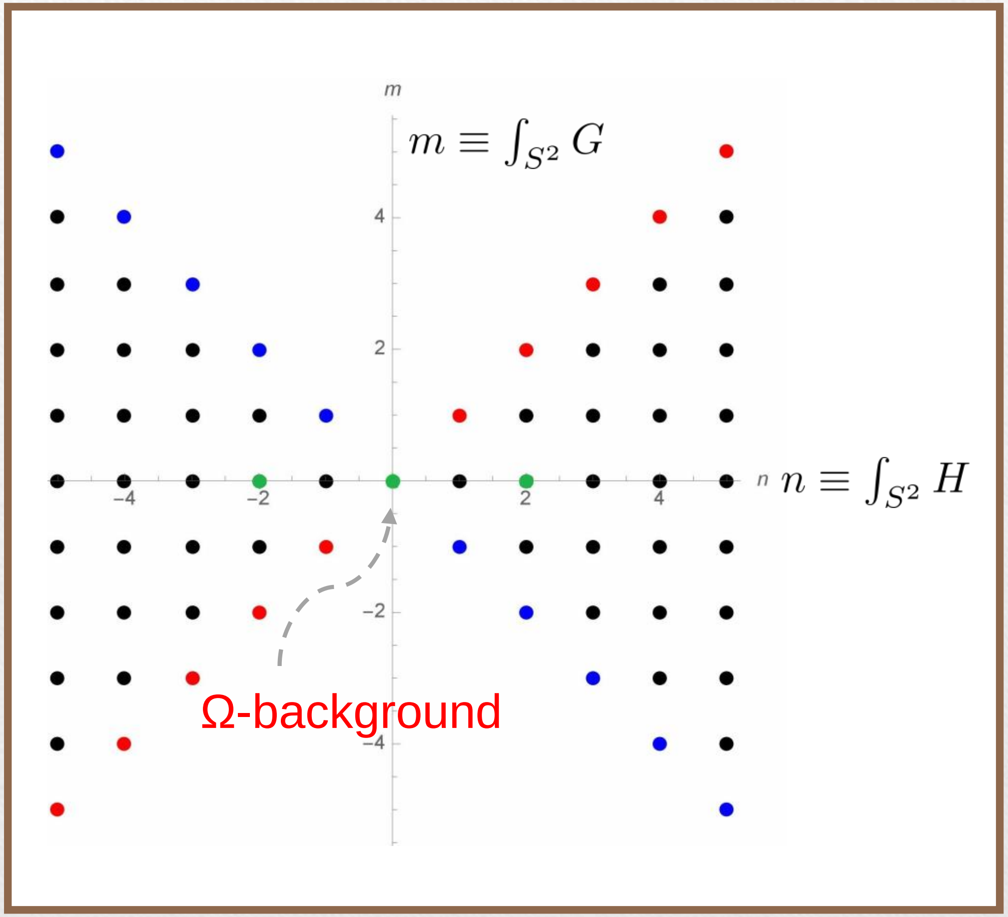

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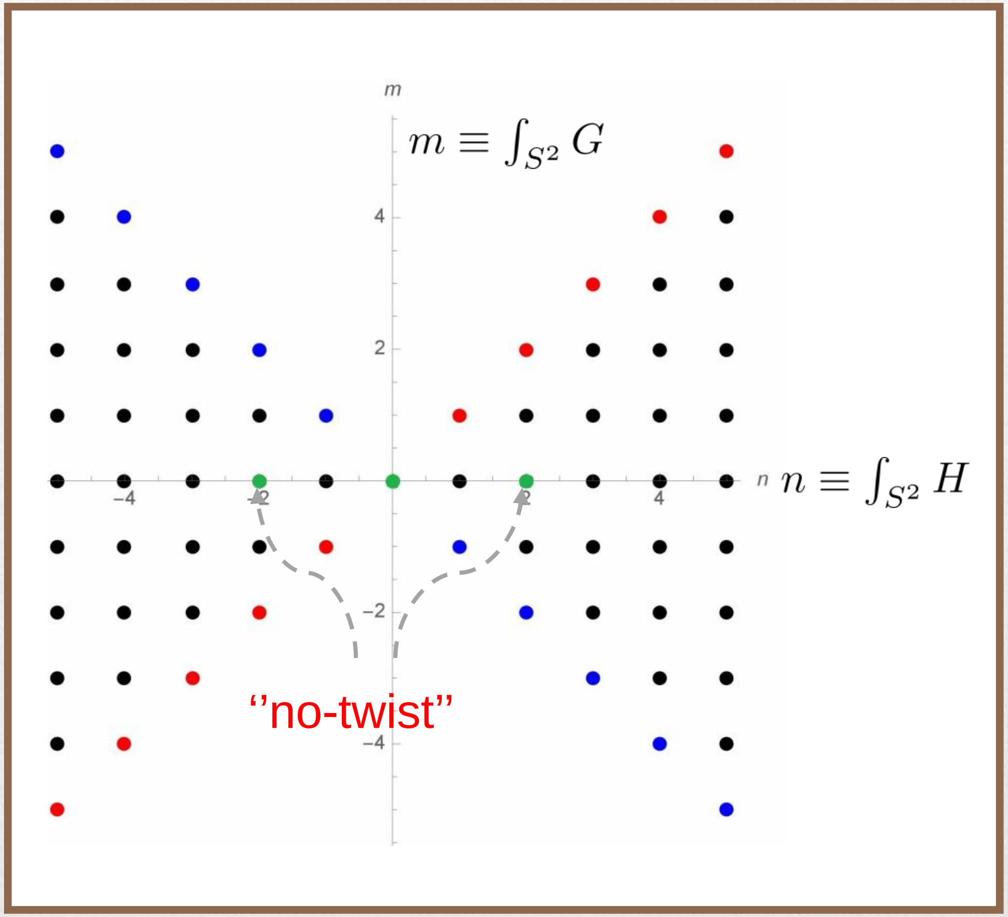

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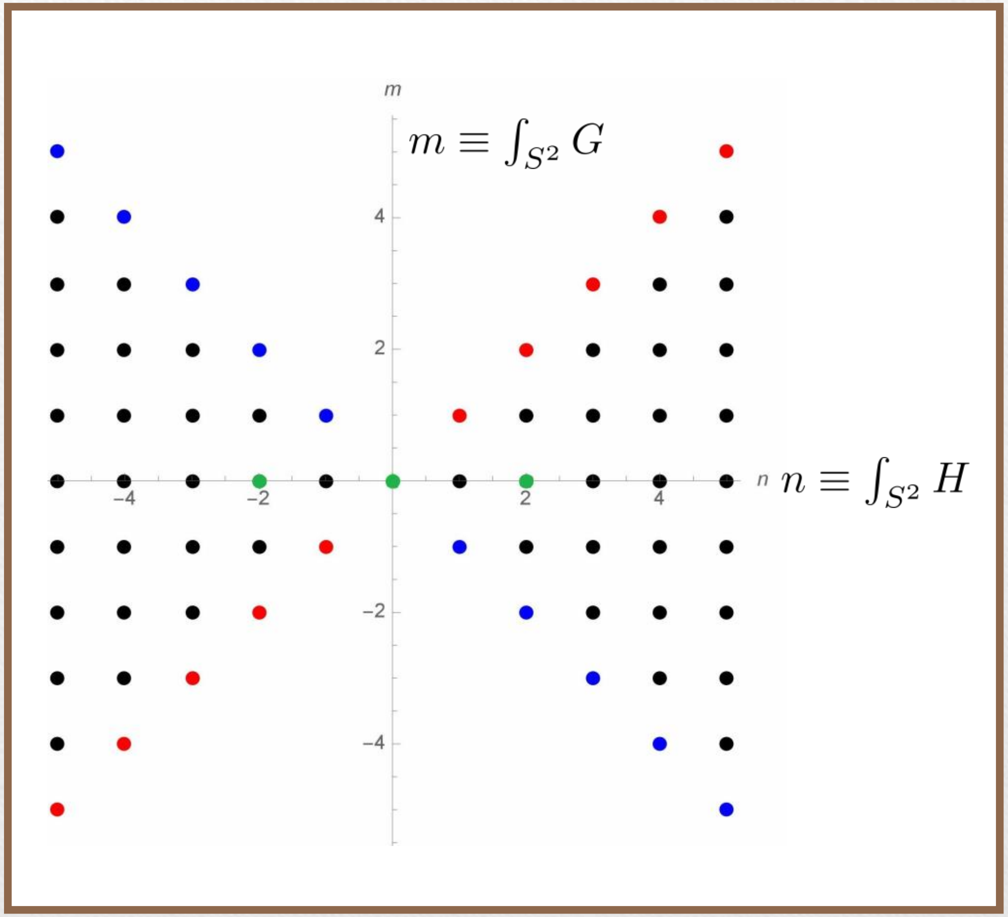

Infinite new solutions!
Inequivalent localizations!

Conclusions

- A new gravity theory for finding localizing backgrounds

  ⇒ cohomological classification

- Infinite new localizing backgrounds in two dimensions

  ⇒ great extension of the known results

- An intrinsic topological reformulation of localization